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Tito M. Tonietti

And Yet It Is Heard

Musical, Multilingual and Multicultural History of the Mathematical Sciences — Volume 1

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The history of the sciences is a grand fugue, in which the voices of various peoples chime in, each in their turn.

It is as if an eternal harmony conversed with itself as it may have done in the bosom of God, before the creation of the world

Wolfgang Goethe

Foreword

Ἡ μουσικὴ μηδὲν ἐστὶν ἕτερον, ἢ πάντων τάξιν εἰδέναι.

Musica nihil aliud est, quam omnium ordinem scire.

Music is nothing but to know the order of all things.

Trismegistus in Asclepius, cited by Athanasius Kircher, *Musurgia universalis*, Rome 1650, vol. II, title page

Tito Tonietti has certainly written a very ambitious, extraordinary book in many respects. Its subtitle precisely describes his scientific aims and objectives. His goal here is to present a musical, multilingual, and multicultural history of the mathematical sciences, since ancient times up to the twentieth century. To the best of my knowledge, this is the first serious, comprehensive attempt to do justice to the essential role music played in the development of these sciences.

This musical aspect is usually ignored or dramatically underestimated in descriptions of the evolution of sciences. Tonietti stresses this issue continually. He states his conviction at the very beginning of the book: Music was one of the primeval mathematical models for natural sciences in the West. "By means of music, it is easier to understand how many and what kinds of obstacles the Greek and Roman natural philosophers had created between mathematical sciences and the world of senses."

Yet, also in China, it is possible to narrate the mathematical sciences by means of music, as Tonietti demonstrates in Chap. 3. Even in India, certain ideas would seem to connect music with mathematics. Narrating history through music remains his principle and style when he speaks about the Arabic culture.

Tonietti emphasizes throughout the role of languages and the existence of cultural differences and various scientific traditions, thus explicitly extending the famous Sapir-Whorf hypothesis to the mathematical sciences. He emphatically rejects Eurocentric prejudices and pleads for the acceptance of cultural variety. Every culture generates its own science so that there are independent inventions in different contexts. For him, even the texts of mathematicians acquire sense only if they are set in their context: "The Indian brahwana and the Greek philosophers developed their mathematical cultures in a relative autonomy, maintaining their own characteristics."

To mention another of Tonietti's examples: The Greek and Latin scientific cultures, the Chinese scientific culture cannot be reduced to some general characteristics. Chinese books offered different proofs from those of Euclid. He draws a crucial conclusion: Such differences should not be transformed into inferiority or exclusion.

For the Chinese, as well as for the Indians, the Pythagorean distinction between integers – or ratios between them – and other, especially irrational numbers does not seem to make sense. The Chinese mathematical theory of music was invented through solid pipes.

Tonietti does not conceal another matter of fact: In his perspective of history, harmony is not only the daughter of Venus, but also of a father like Mars. For good reasons he dedicates a long chapter to Kepler's world harmony, which indeed deserves more attention. He disagrees with the many modern historians of science who transformed Kepler's diversity into inferiority "with the aggravating circumstances of those intolerable nationalistic veins from which we particularly desire to stay at a good distance."

Tonietti's original approach enables him to gain many essential new insights: The true achievements of Aristoxenus, Vincenzo Galilei, Stevin (equable temperament), Lucretius's contributions to the history of science overlooked up to now, the reasons the prohibition of irrational numbers was eclipsed during the seventeenth century, and the understanding of the reappearance of mathematics as the language essential to express the new science in this century, to mention some of them. Or, as he puts it: "The question has become rather how to interpret the musical language of the spheres and not whether it came from God."

Tonietti emphatically refuses corruptions, discriminations, distortions, simplifications, anachronisms, nationalisms of authors, and cultures trying to show that "even the mathematical sciences are neither neutral nor universal nor eternal and depend on the historical and cultural contexts that invent them." He places music in the foreground, he has not written a history of music with just hints to acoustic theories.

In spite of all his efforts and the more than thousand pages of his book, Tonietti calls his attempt a modest proposal, a beginning. It is certainly a provocative book that is worth diligently studying and continuing even if not every modern scholar will accept all of its statements and conclusions.

Berlin, Germany
February 2014

Eberhard Knobloch

Contents for Volume I

Contents for Volume II

Chapters 3, 6 and 8–12 have been respectively obtained by re-elaborating, completing or simplifying the following talks, articles and books:

Paper presented at Hong Kong in 2001, "The Mathematics of Music During the 16th Century: The Cases of Francesco Maurolico, Simon Stevin, Cheng Dawei, Zhu Zaiyu", *Ziran kexueshi yanjiu [Studies in the History of Natural Sciences]*, (Beijing), 2003, **22**, n. 3, 223–244.

Le matematiche del Tao, Roma 2006, Aracne, pp. 266.

"Tra armonia e conflitto: da Kepler a Kauffman", in *La matematizzazione della biologia*, Urbino 1999, Quattro venti, 213–228.

"Disegnare la natura (I modelli matematici di Piero, Leonardo da Vinci e Galileo Galilei, per tacer di Luca)", *Punti critici*, 2004, n. 10/11, 73–102.

"The Mathematical Contributions of Francesco Maurolico to the Theory of Music of the 16th Century (The Problems of a Manuscript)", *Centaurus*, **48**, (2006), 149–200.

Paper presented at Naples in 1995, "Verso la matematica nelle scienze: armonia e matematica nei modelli del cosmo tra seicento e settecento", in *La costruzione dell'immagine scientifica del mondo*, Marco Mamone Capria ed., Napoli 1999, La Città del Sole, 155–219.

Paper presented at Perugia in 1996, "Newton, credeva nella musica delle sfere?", in *La scienza e i vortici del dubbio*, Lino Conti and Marco Mamone Capria eds., Napoli 1999, Edizioni scientifiche italiane, 127–135. Also "Does Newton's Musical Model of Gravitation Work?", *Centaurus*, **42**, (2000), 135–149.

Paper presented at Arcidosso in 1999, "Is Music Relevant for the History of Science?", in *The Applications of Mathematics to the Sciences of Nature: Critical Moments and Aspects*, P. Cerrai, P. Freguglia, C. Pellegrino (eds.), New York 2002, Kluwer, 281–291.

"Albert Einstein and Arnold Schoenberg Correspondence", *NTM - Naturwissenschaften Technik und Medizin*, **5** (1997) H. 1, 1–22. Also *Nuvole in silenzio (Arnold Schoenberg svelato)* , Pisa 2004, Edizioni Plus, ch. 58.

“Il pacifismo problematico di Albert Einstein”, in *Armi ed intenzioni di guerra*, Pisa 2005, Edizioni Plus, 287–309.

Chapters 2, 4 and 5 are completely new. In the meantime, thanks to the help of Michele Barontini, a part of Chap. 5 has become “Umar al-Khayyam’s Contributions to the Arabic Mathematical Theory of Music”, *Arabic Science and Philosophy* v. 20 (2010), pp. 255–279. The problems of Chap. 4 produced, in collaboration with Giacomo Benedetti, “Sulle antiche teorie indiane della musica. Un problema a confronto con altre culture”, *Rivista di studi sudasiatici*, v. 4 (2010), pp. 75–109; also, “Toward a Cross-cultural History of Mathematics. Between the Chinese, and the Arabic Mathematical Theories of Music: the Puzzle of the Indian Case”, in *History of the Mathematical Sciences II*, eds. B.S. Yadav & S.L. Singh, Cambridge 2010, Cambridge Scientific Publishers, 185–203.

In the meantime, a part of Chaps. 11 and 12 has been published as “Music between Hearing and Counting (A Historical Case Chosen within Continuous Long-Lasting Conflict)”, in *Mathematics and Computation in Music*, Carlos Agon *et al.* eds., *Lectures Notes in Artificial Intelligence* 6726, Berlin 2011, Springer Verlag, 285–296.

Appendix C is the translation of the edition for Maurolico’s *Musica*, edited by the author for the relative *Opera Mathematica* in www.maurolico.unipi.it, subsequently also Pisa-Roma, Fabrizio Serra editore, to be published, perhaps.

Chapter 1
Introduction

The history of the sciences can be (and has been) told in many ways. In general, however, treatments display systematic, recurring partialities. Many of the characters who contributed to them also wrote about music, and sure, the best approximation would be to say that all of them did. And yet the musical aspect, though present on a relatively continuous basis during the evolution of sciences, is usually ignored or underestimated. This omission would appear to be particularly serious, seeing that music would enable us to represent in a better and more characteristic way the main controversies at the basis of this history. For example, the question of the so-called irrational numbers, like $\sqrt{2}$, may have a very simple, direct musical representation.

This book will thus bring into full light some pages dealing with musical subjects, that are scattered throughout the most famous scientific texts. The complementary point of view is relatively more widespread, that is to say, the one that presents the history of music as traversed also by the study of physical sound, for example frequencies and harmonics. This happened because, for better or worse, science and technology have succeeded in influencing the world we live in, unfortunately, more than music, and thus they have also influenced music. At this point, it has become necessary to recall that also music was capable, on the contrary, of playing a role among the sciences and among scientists.

There is another not insignificant defect in the histories of the modern-day sciences. Apart from, in the best of cases, a few brief mentions in the opening chapters, the evolution of sciences seems to be taken place exclusively in Europe, or to have reached its definitive climax in Europe. However, despite Euclid, Galileo Galilei, Descartes, Kepler, Newton, Darwin and Einstein, it actually had other important scenarios: China, India, the countries of the Arab world. The idea that the sciences were practically an exclusively Western invention is due to a Eurocentric prejudice. The reasons for this commonplace, which does not stand up to careful historical examination, are manifold. They will emerge, if necessary, in due time. But one of these, in view of its general character, deserves to be discussed immediately. Scientific results, which are more often called discoveries

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than inventions, are in a certain sense made independent of the social, cultural, economic, political, national, linguistic and religious context. Deprived of all these characteristics, which are those that can be observed in historical reality, that is to say, in the environment where the inventors lived, the sciences are described as elements of an ideal, unreal world, which may also be called the justification context. This book is alien, not only to this philosophy of history, but also to all others. Partly because it serves to arrive at the idea of a (rhetorical) scientific progress and neutral sciences, for which the authors are not, after all, responsible.

Here, on the contrary, the sciences are shown to be rooted in the various cultures, and to contain their values in some form, which is to be verified time by time. Consequently, the contributions that come from countries outside Europe not only appear to be significant, and not at all negligible, but their value lies above all in the fact that they are characteristic, and different from Western contributions. The language represents the deepest aspect of each culture, because it is through language that each presents and cultivates its own system of values. Thus, the first element, and one of the most important that we must underline, is which language the scientific texts examined here were written in. This means that our multicultural history of sciences necessarily also becomes a review of the various dominant languages used in the different historical contexts. Just as the scientific community generally expresses itself nowadays in English, in other periods, for several centuries it had expressed itself in Greek or in Latin, and elsewhere in Arabic, in Sanskrit, or in Chinese. Often the language used by a scholar to write his text was not his own mother tongue, but that of the dominant culture of the area. For example, various Persian scholars wrote in Arabic. The Swiss mathematician and physicist of the eighteenth century, Leonhard Euler, who actually spoke German, has left us texts written in Latin.

The attention dedicated here to cultural differences, in relationship to the various scientific traditions, will also lead us to deal with the question of how the characteristics of the languages influenced the relative inventions. Thus we shall find arguments in a linear form, like the deductions from axioms, in a alphabetic language like Greek, but also another kind of visual demonstration, expressed in Chinese characters. An anthropologist and scholar of the *hopi* language like Benjamin Lee Whorf (1897–1941) wrote "… linguistics is fundamental for the theory of thought and, ultimately, for all human sciences".[1] Here the famous Sapir-Whorf hypothesis is even extended to the mathematical sciences.

Moved by the best intentions, other historians have taken great pains to recall the great inventions of Arabs, Indians and Chinese. They have often presented them, however, as contributions to a single universal science. Faith in this thus led them to overlook cultural differences and consequently to deal with insoluble, absurd questions of priority and transmission from one country to another.[2] On the contrary,

[1] Whorf 1970, p. 64.

[2] A good example of how one can limit one's studies to problems of transmission, completely ignoring cultural differences and music, is offered by the great, in many ways fundamental classic,

national pride animated the historians of countries unjustifiably ignored, leading them to offer improbable, not to say incorrect, dates for the texts that they study.

Apart from the cases with sufficient documentation, the history of science has, on the contrary, all to gain from the idea of independent inventions made in different contexts. In general, every culture generates its own sciences. Among these, it then becomes particularly interesting to make comparisons. However, it is advisable to avoid constructing hierarchies, which inevitably depend on the values of the historian making the judgements, but are extraneous to the people studied. A famous anthropologist like Claude Lévi-Strauss complained, "... it seems that diversity of cultures has rarely been seen by men as what it really is: a natural phenomenon, resulting from direct or indirect relationships between societies; rather, it has been seen as a sort of monstrosity, or scandal."[3]

This does not mean renouncing the characteristics of sciences compared with other human activities. Simply, they are not to be distinguished by making them independent of the people who invent their rules, laws or procedures, thus transporting them into a mythical transcendent world (imagined, naturally, to be European), or into the present epoch, with its specialisations of an academic kind.

While this book does not tell the story of the evolution of the sciences as if it took place in an ideal world alien to history, it does not proceed, either, as if there were never confrontations, unfortunately usually tragic, between the various cultures and peoples. Even the idyllic islands of Polynesia saw the arrival, sooner or later, of (war)ships that had set sail from Europe.

It is no desire of mine to deny that here in the West, the development of the sciences received a particularly fervid impulse, starting from the seventeenth century. Nor do I wish to ignore their capacity to expand all over the world, establishing themselves, for better or for worse, in the lifestyles of many populations.

But this does not constitute a criterion of superiority for Western sciences. Rather, the historical events that have led up to this situation indicate as the ground for a confrontation that of power and warfare. It is only on this basis that a hierarchic scale can be imposed on different values, each of which is fruitful and effective within its own culture, and each of which it is largely impossible to measure with respect to the others. Briefly, when we are tempted to transform the characteristics of Western sciences into an effective superiority, we need to realise that we are implicitly accepting the criterion of war as the ultimate basis for the comparison. As a result, this book reserves an equal consideration for extra-European sciences as for those that flourished in Europe, for the same *moral* reasons that lead us to repudiate the strength of arms and military success as a valid criterion to compare different cultures that come into conflict.

Otto Neugebauer 1970. This German scholar typically considers only astronomy as the leading science of the ancient world, and does not even remember that Ptolemy had also written a book about music; see Sect. 2.6.

[3]Lévi-Strauss 2002, p. 10.

Actually, Western sciences penetrated into China, thanks mainly to the Jesuits, precisely because they proved to be useful for the noble art of arms and war. This was clearly spelled out in the Western books of science which had been translated at that time into Chinese. And this, unfortunately, was to find practical confirmation, both when the Ming empire was defeated by the Qing (also known as the Manchu) empire, half-way through the seventeenth century, and two centuries later, when the latter imperial dynasty was subdued by Western imperialistic and colonialistic powers during the infamous Opium Wars.[4] With the precise aim of exposing the deepest roots of Eurocentric prejudice, on these occasions, the various reasons connected with arms and warfare, which had influenced the evolution of the sciences, were not ignored or covered up (as usually happened).

In Part I, dedicated to the ancient world, the Chap. 2 tells the story, in pages dealing with music, of the Pythagorean schools, Euclid, Plato, Aristoxenus and Ptolemy, and how that orthodoxy was created in the Western world, which was to prohibit the use of irrational numbers. The consequences of this choice persisted for 2,000 years, and came to be the most important characteristics of Western sciences: these included the typical dualisms of a geometry separated from numbers, and a mathematics that transcended the world where we live. The dominant language in that period was Greek. Nor can we overlook Lucretius, on the grounds that he was outside the predominant line, like Aristoxenus.

In Chap. 3, the Chinese mathematical theories of music based on reed-pipes reveal a scientific culture dominated, instead, by the idea of an energetic fluid, called *qi*. Omnipresent and pervasive, it gave rise to a *continuum*, where it could carry out its processes, where it could freely move geometric figures, and where it could execute every calculation, including the extraction of roots. Accordingly, the leading property of right-angled triangles was proved in a different way from that of Euclid. Also the dualism between heaven and earth, with the transcendence which was so important in the West, was lacking here. During the sixteenth and seventeenth centuries, these two distant scientific cultures were to enter into contact in a direct comparison. The relative texts were composed in Chinese characters.

In the Chap. 4, India comes on the scene, with its sacred texts written in Sanskrit. Here, the need for a particular precision, motivated by the rituals for the construction of altars, led to geometric reasoning. The fundamental property of right-angled triangles was exploited, and it was explained how to calculate the area of a trapezoid altar. Music, too, acquired great importance thanks to the rituals based on singing. But, by a curious unsolved paradox, which marks the culture that invented our modern numbers, their theory of music does not seem to demand exactness through mathematics, but rather trusts its ears.

In Chap. 5, the Arabs appear, with these famous numbers brought from India, and their books translated from, and inspired by, a Greek culture that had too long been ignored in the mediaeval West. By now, scholars, even those from Persia, left books usually written in Arabic. Their predominant musical theory was inspired by

[4] Tonietti 2006a, pp. 175–179 e 197.

that of the Greeks, above all by Pythagorean-Ptolemaic orthodoxy. Some of their terms, such as "algebra" and "algorithm" were to change their meaning in time, and to enter into the current modern usage of the scientific community.

In Chap. 6, we return to Europe, recently revitalised by Oriental cultures, whose influence is increasingly cited, even more than that of Greece. Its lingua franca, with universal claims, had become Latin. Here, the musical rhythms were now represented on the lines and spaces of the stave. Variations on Pythagorean-Ptolemaic orthodoxy were appearing, and Vincenzio Galilei at last remembered even the ancient rival school of Aristoxenus. Euclid still remained the general reference model for mathematical sciences, he now began to be flanked by new calculating procedures for algebraic equations, and the new Indo-Arabic numbers.

Appendix A contains a translation of the musical pages contained in the famous Chinese manual of mathematics, *Suanfa tongzong [Compendium of calculating rules]* written by Cheng Dawei in the sixteenth century, and discussed in Chap. 3. This is followed, in Appendix B, by a translation of a short text about music by Umar al-Khayyam, which is discussed in Chap. 5. Lastly, Appendix C contains a translation of the manuscript entitled *Musica*, handed down to us among the papers of Francesco Maurolico and presented in Chap. 6. In appendix D, the Chinese characters scattered in the text are given.

In Part II, which is dedicated to the scientific revolution, Chap. 8 narrates the evolution of the seventeenth century through the writings on music of Stevin, Zhu Zaiyu, Galileo Galilei and Kepler. The German even included in the title of his most important work his idea of harmony in the cosmos. The equable temperament for instruments was now also represented by means of irrational numbers.

The Chap. 9 is taken up by Mersenne, Pascal, Descartes, Beeckman, Wallis, Constantin and Christiaan Huygens, and their discussions about music, God, the world, and natural phenomena. Together with Latin, which still dominated in universities, national languages were increasingly used to communicate outside traditional circles. We now find texts also in Flemish, Chinese, Italian, French, German and English. Above all (as a consequence?), a new typically European mathematical symbolism was adopted, as writing music on staves had been.

In Chap. 10, we discover that even Leibniz and Newton, not to mention Hooke, had continued for a while to deal with music, and had ended up by preferring the equable temperament, at least in practice. With them, mathematical symbolism gained that (divine?) transcendence which was necessary to deal better with infinities and infinitesimal calculations with numbers.

In Chap. 11, music enjoys its final season of excellence among the great scientists of the eighteenth century. As happened at court and in diplomatic circles, French became the language most widely used among scientists, though some of them still insisted on Latin. Euler based his neo-Pythagorean theory on prime numbers. His opponent, d'Alembert, at his ease among the musicians of Paris, preferred, on the contrary, to follow his own ears. Both of them, however, were to come up against the musician Jean-Philippe Rameau, while the Illuminists of the *Encyclopédie* also took part in the discussion.

In the following period, harmony was overshadowed by the din of the combustion engine. Consequently, in Chap. 12, the ancient harmony became a not-so-central part of acoustics, together with harmonics and Fourier's mathematics. Bernhard Riemann, Helmholtz and Planck vied in explaining to us the sensitivity that our ears were guided by. Finally, the correspondence that passed between the musician Schoenberg and the famous physicist Albert Einstein shows us the (great?) nature of the period between the end of the nineteenth and the twentieth centuries. Their language had become German. With the pianoforte, all music now followed the equable temperament.

In Chap. 13 of Part III, only the caustic language of the Venusians would succeed in expressing the impossible dream of finding harmony in the age of warfare and violence. For this reason, we also need to remember the forgotten, destroyed cultures of Africa, pre-Columbian America and Oceania.

In the fourteenth and last chapter, with all the knowledge acquired by the mathematical sciences today, we speculate whether the asteroid Apophis and nuclear bombs will allow us to continue to enjoy music (and life).

Part I
In the Ancient World

Chapter 2
Above All with the Greek Alphabet

2.1 The Most Ancient of All the Quantitative Physical Laws

I would like to begin with an argument which may be stated most clearly and most forcefully as follows:

> Music was one of the primeval mathematical models for natural sciences in the West.

The other model described the movement of the stars in the sky, and a close relationship was postulated between the two: the music of the spheres.

This argument is suggested to us by one of the most ancient events of which trace still remains. It is so ancient that it has become legendary, and has been lost behind the scenes of sands in the desert. A relationship exists between the length of a taut string, which produces sounds when it is plucked and made to vibrate, and the way in which those sounds are perceived by the ear. The relationship was established in a precise mathematical form, that of proportionality, which was destined to dominate the ancient world in general. Given the same tension, thickness and material, the longer the string, the deeper or lower the sound perceived will be; the more it is shortened, the less deep the sound perceived: the length of the string and the depth of the sound are directly proportional. If the former increases, the latter increases as well; if the former decreases, the latter does as well. Or else, the sound could be described as more or less acute, or high. In this case, the length of the string generating it would be described as inversely proportional to the pitch. The shorter the string, the higher the sound produced. None of the special symbols employed in modern manuals were used to express this law, but just common language. If the string is lengthened, the height of the sound is proportionally lowered.

Two thousand years were to pass until the appearance of the formulas to which we are accustomed today. It was only after René Descartes (1596–1650) and subsequently Marin Mersenne (1588–1648), that formulas were composed of the kind

$$\nu \propto \frac{1}{l}$$

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where the height was to be interpreted as the number of vibrations of the string in time, that is to say, the frequency ν, and the length was to be measured as l.

The first volume will accompany us only as far as the threshold of this representation, that is to say, up to the affirmation of a mathematical symbolism increasingly detached from the languages spoken and written by natural philosophers and musicians, and this will be the starting-point for the second volume. Furthermore, it is important to remember that fractions such as $\frac{1}{l}$ or $\frac{3}{2}$ were not used in ancient times, but ratios were indicated by means of expressions like '3 to 2', which I will also write as 3:2. The ratio was thus generally fixed by two whole numbers. Whereas a fraction is the number obtained by dividing them, when this is possible.

The same relationship between the length of the string and the height of the sound would appear to have remained stable up to the present day, about 2,500 years later. Is this the only natural mathematical law still considered valid? While others were modified several times with the passing of the years? "... possibly the oldest of all quantitative physical laws", wrote Carl Boyer in his manual on the history of mathematics.[1] That "possibly" can probably be left out.

In Europe, a tradition was created, according to which it was the renowned Pythagoras who was struck by the relationship between the depth of sounds and the dimensions of vibrating bodies, when he went past a smithy where hammers of different sizes were being used. However, the anecdote does not appear to be very reliable, mainly because the above ratio regarded strings.

In any case, the sounds produced by instruments, that is to say, the musical notes perceived by the ear, could now be classified and regulated. How? Strings of varying lengths produced notes of different pitches, with which music could be made. But Pythagoras and his followers sustained that not all notes were appropriate. In order to obtain good music, it was necessary to choose the notes, following a certain criterion. Which criterion? The lengths of the strings must stand in the respective ratios 2:1, 3:2, 4:3. That is to say, a first note was created by a string of a certain length, and then a second note was generated by another string twice as long, thus obtaining a deeper sound of half the height. The two notes gave rise to an interval called *diapason*. Nowadays we would say that if the first note were a *do*, the second one would be another *do*, but deeper, and the interval is called an "octave", and so it is the *do* one octave lower. The same ratio of 2:1 is also valid if we take a string of half the length: a new note twice as high is obtained, that is to say, the *do* one octave higher. But musical notes were to be indicated in this kind of syllabic manner only from Guido D'Arezzo on (early 1000s to about 1050).[2]

The other ratios produced other notes and other intervals. The ratio 3:2 generated the interval of *diapente* (the fifth *do* – *sol*) and 4:3 the *diatessaron* (the fourth *do* – *fa*). Thus the ratios established that what was important for music was not the single isolated sound, but the relationship between the notes. In this way, harmony was born, from the Greek word for 'uniting, connecting, relationship'.

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[1]Boyer 1990, p. 65.

[2]See Sect. 6.2.

At this point, the history became even more interesting, and also relatively well documented, because in the whole of the subsequent evolution of the sciences, controversies were to develop continually regarding two main problems. What notes was the octave to be divided into? Which of the relative intervals were to be considered as consonant, that is to say ‘pleasurable’, and consequently allowed in pieces of music, and which were dissonant? And why? The constant presence of conflicting answers to these questions also allows us to classify sciences immediately against the background of the different cultures: each of them dealt with the problems in its own way, offering different solutions.

Anyway, seeing the surprising success of our original mathematical law model, it was coupled here and there with other regularities that had been identified, and was posited as an explanation for other phenomena. The most famous of these was undoubtedly the movement of the planets and the stars; this gave rise to the so-called music of the heavenly spheres, and connected with this, also the therapeutic use of music in medicine. This original seal, this foundational aporia remained visible for a long time. All, or almost all, of the characters that we are accustomed to considering in the evolution of the mathematical sciences wrote about these problems. Sometimes they made original contributions, other times they repeated, with some personal variations, what they had learnt from tradition. It might be named Pythagorean tradition, so called after the reference to its legendary founder, to whom the original discovery was attributed, or the Platonic or neo-Platonic tradition. This was even to be contrasted with a rival tradition dating back to Aristoxenus. In any case, many scholars felt an obligation to pay homage to tradition in their commentaries, summaries, and sundry quotations, or in their actual theories.

In this second chapter, we shall review the Pythagoreans, and other characters who harked back to their tradition, such as Euclid and Plato, but also significant variations like that of Claudius Ptolemaeus (Ptolemy), or the different conception of Aristoxenus. In Chaps. 6, 8–11, we shall see that the interest in the division of the octave into a certain number of notes, and the interest in explaining consonances passed unscathed, or almost so, through the epochal substitution (revolution?) of the Ptolemaic astronomic system with the Copernican one during the seventeenth century. It might be variously described as musical theory, or acoustics, or as the music of mathematics, or the mathematics of music. All the same, it continued without any interruption in the Europe of Galileo Galilei, Kepler, Descartes, Leibniz, and Newton. It was not completely abandoned, even when, during the eighteenth century, figures like d’Alembert and Euler felt the need to perfect the new symbolic language chosen for the new sciences, and to address them in a general systematic manner.

2.2 The Pythagoreans

> Pythagoras, ... constructed his own $\sigma o\phi\iota'\alpha$ [wisdom] $\pi o\lambda\upsilon\mu\alpha\theta\iota'\alpha$ [learning] and $\kappa\alpha\kappa o\tau\epsilon\chi\nu\iota'\alpha$ [art of deception].
>
> Heraclitus.

The mathematical model chosen by the Pythagoreans, with the above-mentioned ratios, selected the notes by means of whole numbers, arranged in a "geometrical" sequence. This means that we pass from one term to the following one (that is to say, from one note to the following one) by *multiplying* by a certain number, which is called the "common ratio" of the sequence. Thus, in the geometrical sequence 1, 2, 4, 8, 16, ... we multiply by the common ratio 2. In "arithmetic" sequences, instead, we proceed by *adding*, as in 1, 2, 3, 4, 5, ... where the common ratio is 1, or in 1, 4, 7, 10, 13, ... where the common ratio is 3. Thus the Pythagoreans had also introduced the "geometrical" or "proportional" mean, with reference to the ratio $1:2=2:4$. That is to say, the intermediate term between 1 and 4 in this sequence is obtained by multiplying $1\times 4=4$ and extracting the square root $\sqrt{4}=2$. The arithmetic mean, on the contrary, is obtained by adding the two numbers and dividing by 2. In other words, in the above arithmetic sequence, $\frac{1+7}{2}=4$.

Lastly, this same kind of music loved by the Pythagoreans also suggested "harmonic" sequences and means. Taking strings whose lengths are arranged in the arithmetic sequence 1, 2, 3, 4, ... notes of a decreasing height are obtained in the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. Consequently, the third mean practised by the Pythagoreans, called the harmonic mean, is obtained by calculating the inverse of the arithmetic mean of the reciprocals.

$$\frac{1}{\frac{1}{2}(2+4)}=\frac{1}{3} \text{ or } 2\frac{\frac{1}{2}\times\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}}=\frac{1}{3}$$

In faraway times, and places steeped in bright Mediterranean sunshine, rather than the pale variety of the Europe of the North Atlantic, the Pythagoreans had thus generally established the arithmetic mean $a=\frac{b+c}{2}$, the geometric mean $a=\sqrt{cb}$ and the harmonic mean $\frac{1}{a}=\frac{1}{2}(\frac{1}{b}+\frac{1}{c})$, that is to say, $a=2\frac{b\times c}{b+c}$. Taking strings whose length is 1, 2, 3 we obtain (if the tension, thickness and material are the same) notes of a decreasing height $1, \frac{1}{2}, \frac{1}{3}$, that is to say, the notes that gave unison, the (low) octave, the fifth (which could be transferred to the same octave by dividing the string of length 3 into two parts, thus obtaining $\frac{2}{3}$). The arithmetic sequence (whose common ratio is $\frac{1}{2}$) $1, \frac{3}{2}, 2$ generates the harmonic sequence $1, \frac{2}{3}, \frac{1}{2}$. On these bases, the mystic sects that harked back to that character of Magna Graecia (the present-day southern Italy) called Pythagoras, divided the single string of a theoretical musical instrument called the monochord. They believed that the only consonances (symphonies) were unison, the octave, the fifth and the fourth, because they were generated by the ratios 1:1, 2:1, 3:2, 4:3. For them, the fact that music made use of the first four whole numbers, and furthermore that added together, these made $10=1+2+3+4$, the *tetraktys*, acquired a profound significance. It seemed to be the best proof that everything in the world was regulated by whole numbers and their derivatives.

Games with whole numbers and means were very popular. The preferences for notes became 6, 8, 9, 12. These include the octave 12:6, the fifth 9:6, the fourth 8:6

and the tone 9:8.[3] Furthermore, 9 is the arithmetic mean between 6 and 12, while 8 is the harmonic mean.

$$6 : 9 = 8 : 12$$

In general

$$b : \frac{b+c}{2} = 2\frac{b \times c}{b+c} : c$$

In other words, the ratio between b, the arithmetic mean and c is completed by the harmonic mean.

The points of the *tetraktys* were distributed in a triangle, while 4, 9, and 16 points assumed a square shape. Geometry was invaded by numbers, which were also given symbolic values: odd numbers acquired male values, and even ones female; $5 = 3 + 2$ represented marriage. And so on.

If it had depended on historical coincidences or on the rules of secrecy practised by initiated members of the Italic sect, then no text written directly by Pythagoras (Samos c. 560–Metaponto c. 480 B.C.) could have been made available to anybody. It is said that only two groups of adepts could gain knowledge of the mysteries: the *akousmatikoi*, who were sworn to silence, and to remembering the words of the master, and the *mathematikoi*, who could ask questions and express their own opinions only after a long period of apprenticeship.

But in time, others (the most famous of whom was Plato) were to leave written traces, on which the narration of our history is based.

Thanks to the ratios chosen for the octave, the fifth and the fourth, the Pythagorean sects rapidly succeeded in calculating the interval of one tone $fa - sol$: the difference between the fifth $do - sol$ and the fourth $do - fa$. In the geometric sequence at the basis of the notes, adding two intervals means compounding the relative ratios in the multiplication, whereas subtracting two intervals means compounding the appropriate ratios in the division. Consequently, the Pythagorean ratio for the tone became

$$(3 : 2) : (4 : 3) = 9 : 8.$$ [4]

At this point, all the treatises on music dedicated their attention to the question whether it was possible to divide the tone into two equal parts (semitones). The Pythagorean tradition denied it, but the followers of Aristoxenus readily admitted it. Why? Dividing the Pythagorean tone into two equal parts would have meant

[3] See below.

[4] Even if he is guilty of anachronism, in order to arrive more rapidly at the result, the reader inured by schooling to fractions will easily be able to calculate $\frac{3}{2} : \frac{4}{3} = \frac{9}{8}$. However, the use of fractions in music had to await the age of John Wallis (1617–1703), Part II, Sect. 9.2. After all, the Greeks used the letters of their alphabet $\alpha, \beta, \gamma \ldots$ to indicate numbers ...

admitting the existence of the geometric mean, a ratio between 9 and 8, that is to say, $9 : \alpha = \alpha : 8$, where $9 : \alpha$ and $\alpha : 8$ are the proportions of the desired semitone. What would the value of α be, then? Clearly $\alpha = \sqrt{9.8}$, and therefore $\alpha = 3.2.\sqrt{2}$! Thus the most celebrated controversy of ancient Greek mathematics, the representation of incommensurable magnitudes by means of numbers, which nowadays are called irrational, acquired a fine musical tone.

The problem is particularly well known, and is discussed in current history books, though it is narrated differently. What is the value of the ratio between the diagonal of a square and its side? In the relative diagram, the diagonal must undoubtedly have a length.

But if we measure it using the side as the natural meter, what do we obtain? In this case, in the end the ratio between the side and the diagonal was called "incommensurable", for the following reason. If we reproduce the side AB on the diagonal, we obtain the point P, from which a new isosceles triangle PQC is constructed (isosceles because the angle $P\hat{C}Q$ has to be equal to $P\hat{Q}C$, just as it is equal to $C\hat{A}B$). By repeating the operation of reproducing QP on the diagonal QC, we determine a new point R, with which the third isosceles right-angled triangle CRS is constructed. And so on, with endless constructions. In other words, this means that it is impossible to establish a part of the side, however small it may be, which can be contained a precise number of times in the diagonal, however large this may be. There is always a little bit left over. The procedure never comes to an end; nowadays we would say that it is infinite.

And yet the problem would appear to be easy to solve, if we use numbers. Because if we assign the conventional length 1 to AB, then by the so-called (in Europe) theorem of Pythagoras (him again!), the diagonal measures $\sqrt{1+1} = \sqrt{2}$. It would be sufficient, then, to calculate the square root. But, as before, the calculation never comes to an end, producing a series of different figures after the decimal point: $1,414213\ldots$. Convinced that they could dominate the world by means of whole numbers, just as they regulated music by means of ratios, the Pythagoreans had hoped to do the same also with the diagonal of the square and $\sqrt{2}$. But no whole numbers exist that correspond to the ratio between the diagonal and the side of a square, or which can express $\sqrt{2}$, in the same way as we use 10:3. Also the division of 10 by 3 never comes to an end (though it is periodic); however, it can be indicated by two whole numbers, each of which can be measured by 1. Accordingly, the Pythagoreans sustained that $\sqrt{2}$ was to be set aside, and could not be considered or used like other numbers. Therefore the tone could not be divided into two equal parts. They even produced a logical-arithmetic proof of this diversity.

On the contrary, let us suppose for the sake of argument that $\sqrt{2}$ can be expressed as a ratio between two whole numbers, p and q. Let us start by eliminating, if necessary, the common factors; for example, if they were both even numbers, they could be divided by 2. As

$$\frac{p}{q} = \sqrt{2}, - - then - -, p^2 = 2q^2$$

Consequently, p^2 must be an even number, and also p must be even. It follows that q must be an odd number, because we have already excluded common factors. But if p is even, then we can rewrite it as $p = 2r$. Introducing this substitution into the hypothetical starting equation, we now obtain $4r^2 = 2q^2$, from which $q^2 = 2r^2$. In the end, the conclusion that can likewise be derived from the initial hypothesis is that q should be also even. But how can a number be even and odd at the same time? Is it not true that numbers can be classified in two completely separate classes? It would therefore seem to be inevitable to conclude that the starting hypothesis is not tenable, and that $\sqrt{2}$ cannot be expressed as a ratio between two whole numbers. Here we come up against the dualism which is a general characteristic, as we shall see, of European sciences.

Maybe it was again due to secrecy, or to the loss of reliable direct sources, but even this question of incommensurability remains shrouded in darkness, as regards its protagonists. Various somewhat inconsistent legends developed, fraught with doubts, and narrated only centuries later, by commentators who were interested either in defending or in denigrating them. Hippasus of Metaponto (who lived on the Ionian coast of Calabria around 450 B.C.) is said to have played a role in identifying the most serious flaw in Pythagoras' construction, and is believed to have been condemned to death for his betrayal, perishing in a shipwreck.[5] A coincidence? The wrath of Poseidon? The revenge of the Pythagorean sect? This was a religious-mathematical murder that deserves to be recorded in the history of sciences, just as Abel is remembered in the *Bible*.

The fundamental property of right-angled triangles, known to everybody and used in the preceding argument, was attributed to the founder of the sect, and from that time on, everywhere, was to be called the theorem of Pythagoras. But this appears to be merely a convention, linked with a tradition whose origins are unknown. The same tradition could sustain, at the same time, that the members of the sect were to follow a vegetarian diet, but also that their master sacrificed a bull to the gods, to celebrate his theorem. And yet he can, at most, have exploited this property of right-angles triangles, like other cultures, e.g. the Mesopotamian one, because he did not leave any proof of it. The earliest proofs are to be found in Euclid.

We are relating the origins of European sciences among the ups and downs and ambiguities of an early conception, sustained by people who lived in the cultural and political context of Magna Graecia. How did they succeed in surviving (apart from Hippasus, the apostate!) and in imposing themselves, and influencing characters who were far better substantiated than them, like Euclid and Plato? Did they do so only on the basis of the strength of their arguments, or did they gain an advantage over their rivals by other means? Because, of course, the Pythagorean theory was not the only one possible, and it had its adversaries.

[5] *Pitagorici* 1958 and 1962. Boyer 1990, pp. 85–87. Cf. Centrone 1996, p. 84. The Pythagoreans are to be considered as adepts of a religious sect governed by prohibitions and rules, somewhat different from the mathematical community of today, which has other customs.

That a Pythagorean like Archytas lived at Tarentum (fifth century B.C.), becoming tyrant of the city, may perhaps have favoured to some extent the acceptance and the spread of Pythagoreanism? We are inclined to think so. The sect's insistence on numbers, means and music is finally found explicitly in his writings. This Greek offered a first general proof that the tone 9:8 could not be divided into two equal parts, by demonstrating that no geometric mean could exist for the ratio $n + 1 : n$. He gave rise to an organisation of culture which was to dominate Europe for the following 2,000 years. The subjects to study were divided into a "*quadrivium*" including arithmetic, geometry, music and astronomy, and a "*trivium*" for grammar, rhetoric and dialectics.

Archytas commanded the army at Tarentum for years, and he is said to have never been defeated. He also designed machines. He is a good example of the contradiction at the basis of European sciences. On the one hand, the harmony of music, and on the other, the art of warfare.[6] How could he expect to sustain them both at the same time, particularly with reference to the education of young people? It is true that in the Greek myths, Harmony is the daughter of Venus and Mars, that is to say, of beauty and war: we shall return to the subject of myths, not to be underestimated, in Plato.

On the other side of the peninsula, on the Tyrrhenian coast, lived Zeno of Elea (Elea 495–430 B.C.): he was not a Pythagorean, but rather drew his inspiration from Parmenides, (Elea c. 520–450 B.C.), the renowned philosopher of a single eternal, unmoved "being". Zeno's paradoxes are famous. How can an arrow reach the target? It must first cover half the distance, then half of the remaining space, and then, again, half of half of half, and so on. The arrow will have to pass through so many points (today we would define them as infinite) that it will never arrive at the target, Zeno concluded. The school of Parmenides taught that movement was an illusion of the senses, and that only thought had any real existence, since it is immune to change. "... the unseeing eye and the echoing hearing and the tongue, but examine and decide the highly debated question only with your thought ...".[7] Zeno's ideal darts were directed not only against the Heraclitus (Ephesus 540–480 B.C.) of "everything passes, everything is in a state of flux", but also against the Pythagoreans, his erstwhile friends, and now the enemies of his master.

Could our world, continually moving and changing, be dominated and regulated by tracing it back to elements which were, on the contrary, stable and sure, because they were believed to be eternal and unchanging? The Pythagoreans were convinced that they could do it by means of numbers; the Eleatics tried to prove by means of paradoxes that this was not possible in the Pythagorean style. Let us translate the paradox of the arrow into the numbers so dearly loved by the Pythagoreans. Let us thus assign the measure of 1 to the space that the arrow must cover. It has covered half, $\frac{1}{2}$, then half of half, $\frac{1}{4}$, then half of half of half $\frac{1}{8}$, and so on, $\frac{1}{16}$, $\frac{1}{32}$ ...

[6] *Pitagorici* 1958 and 1962. The adjective "harmonic" used for the relative mean, previously called "sub-contrary", is attributed to him.

[7] Thomson 1973, p. 299.

The single terms were acceptable to the Pythagoreans as ratios between whole numbers, but they shied away from giving a meaning to the sum of all those numbers which could not even be written completely; today we would define them as infinite. After all, what other result could have been obtained from a similar operation of adding more and more quantities, if not an increasingly big number? Two thousand years were to pass, with many changes, until a way out of the paradox was found in a style that partly saved, but also partly modified the Pythagorean programme. Today mathematicians say that the sum of infinite terms (a sequence) like $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ ... gives as a result (converges to) 1. Thus the arrow moves, and reaches the target, even if we reduce the movement to numbers, but these numbers can no longer be the Pythagoreans' whole numbers; they must include also 'irrationals'.

Anyway, the members of the sect had encountered another serious obstacle to their programme. If whole numbers forced them to imagine an ideal world where space and time were reduced to sequences of numbers or isolated points, then the real world would seem to escape from their hands, because they would not be able to conceive of a procedure to put them together.

There were also some, like Diogenes (of Sinope, the Cynic, 413–327 B.C.), who scoffed at the problem, and proved the existence of movement, simply by walking. Heraclitus started, rather, from the direct observation of a world in continuous transformation; and adopting an opposite approach also to that of the Eleatics, he ignored all the claims of the Pythagoreans, who were often the object of his attacks. "They do not see that [Apollo, the god of the cithara] is in accord with himself even when he is discordant: there is a harmony of contrasting tensions, as in the bow and the lyre." The Pythagoreans combined everything together with their numerical means, whereas among all the things, Heraclitus exalted tension and strife. "Polemos [conflict, warfare] is of all things father and king; it reveals that some are gods, and others men; it makes some slaves, and sets others free." The λόγος logos [discourse, reason] of Heraclitus developed in a completely different way from that of the Pythagoreans. "What can be seen, heard, learnt: that is what I appreciate most."[8]

In the contrasts between the different philosophers, we see the emergence, right from the beginning, of some of the problems for mathematical sciences which are to remain the most important and recurring ones in the course of their evolution. What relationship existed between the everyday world and the creation of numbers with arithmetic, and of points or lines with geometry? By measuring a magnitude in geometry, we always obtain a number? But do numbers represent these magnitudes appropriately?

The whole numbers of the Pythagoreans, or the points of their illustrated models, are represented as separate from each other. We can fit in other intermediate numbers between them, $\frac{3}{2}$ between 1 and 2, for example, but even if it diminishes, a gap still remains. Thus numerical quantities are said to be "discontinuous" or "discrete". If, on the contrary, we take a line, we can divide it once, twice, thrice, ... as many

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[8]Thomson 1973, pp. 278, 281.

times as you like, obtaining shorter pieces of lines, which, however, can still be further divided. The idea that the operation could be repeated indefinitely was called divisibility beyond every limit. This indicated that the magnitudes of geometry were "continuous", as opposed to the arithmetic ones, which were "discrete". And yet there were some who thought that they could find even here something indivisible, that is to say, an atom: the point. Thus in the quadripartite classification of Archytas, music began to take its place alongside arithmetic, seeing that its "discrete" notes appeared to represent its origin and its confirmation in applications. In the meantime, astronomy/astrology displayed its "continuous" movements of the stars by the side of geometry.

So was the everyday world considered to be composed of discrete or continuous elements? Clearly, Zeno's paradoxes indicated that the supporters of discrete ultimate elements had not found any satisfactory way of reconstructing a continuous movement with them. Could they get away with it simply by accusing those who had not been initiated into their secret activities of allowing their senses to deceive them? Why should numbers, or the only indivisible being, lie at the basis of everything?

Those who, on the contrary, trusted their sight or hearing, and used them for the direct observation of the continuous fabric (the so-called *continuum*) of the world might think that both the Pythagorean numerical models and the paradoxes of the Eleatics were inadequate for this purpose. The process of reasoning needed to be reversed. As the arrow reaches the target, the sum of the innumerable numbers must be equal to 1. But this would have required the construction of a mathematics valued as part of the everyday world, not independent from it. On the contrary, the most representative Greek characters variously inspired by Pythagoreanism generally chose otherwise. Their best model appears to be Plato.

We have already demonstrated above that the discussion about the *continuum*, whether numerical or geometrical, had planted its roots deep down into the field of music, in the division (or otherwise) of the Pythagorean tone into two equal parts. The numerical model of the *continuum* contains a lot of other numbers, besides whole numbers and their (rational) ratios. It does not discriminate those like $\sqrt{2}$, which are not taken into consideration by the Pythagoreans, seeing that they do not possess any ratio (between whole numbers), and are thus devoid of *their* $\lambda o\gamma o\varsigma$. Others preferred to seek answers in the practical activity of the everyday world, and thus directly on musical instruments as played by musicians, rather than in the abstract realm of numbers (and soon afterwards, that of Plato's ideas). They had no doubt that it was possible to put their finger on the string exactly at the point which corresponded to the division into two equal semitones. This string thus became the musical model of the *continuum*. We shall deal below with Aristoxenus, who was their leading exponent.

Here began a history of conflict which was to continue to evolve constantly, without ever arriving at a definite solution. It is also one of the main characteristics of European sciences compared with other cultures, which, as we shall see, represented the question in very different ways.

I have found only one book on the history of mathematics[9] which proposes an exercise of dividing the octave into two equal parts and discussing what the Pythagoreans would have thought of the idea.

2.3 Plato

. . . if poets do not observe them in their invention,
this must not be allowed.

Plato

The Plato(on) of the firing squad.

Carlo Mazzacurati

Socrates (469–399 B.C.) showed only a marginal interest in the problems of mathematical sciences, with perhaps one interesting exception which we shall see. However, he was not fond of Pythagoreanism. His disciple Plato (Athens 427–Athens 347 B.C.), on the contrary, became its leading exponent. During his travels, the famous philosopher met Archytas, and was deeply influenced by him. Plato was even saved by him when he risked his life at the hands of Dionysius, the tyrant of Siracusa. Thus we again meet up with numbers, means and music in this philosopher, as already presented by the Pythagoreans.

The most reliable text, that believers in the music of the heavenly spheres could quote, now became Plato's *Timaeus*, with the subsequent (much later) commentaries of Proclus (Byzantium 410–Athens 485), Macrobius (North Africa, fifth century) and others. According to the Greek philosopher, when the demiurge arranged the universe in a cosmos, he chose rational thought, rejecting irrational impressions. Consequently, the model was not visible, or tangible; it did not possess a sensible body, but was on the contrary eternal, always identical to itself. Linked together by ratios, the cosmos assumed a spherical shape and circular movements. The heavens thus possessed a visible body and a soul that was "invisible but a participant in reason and harmony".

Given the dualism between these two terms, the heavens were divided in accordance with the rules of arithmetic ratios, into intervals (like the monochord), bending them into perfect circles. The heavens thus became "a mobile image of eternity . . ., an image that proceeds in accordance with the law of numbers, which we have called time". "And the harmony which presents movements similar to the orbits of our soul, . . ., is not useful, . . ., for some irrational pleasure, but has been

[9] Cooke 1997. Although Centrone 1996 is a good essay on the Pythagoreans, he too, unfortunately, underestimates music: he does not make any distinction between their concept of music and that of Aristoxenus. This limitation derives partly from the scanty consideration that he gives to the Aristotelian *continuum* as an essential element, by contrast, to understand the Pythagoreans. Without this, he is left with many doubts, pp. 69, 196 and 115–117. Cf. von Fritz 1940. *Pitagorici* 1958, 1962, and 1964.

given to us by the Muses as our ally, to lead the orbits of our soul, which have become discordant, back to order and harmony with themselves."

Lastly (on earth) sounds, which could be acute or deep, irregular and without harmony or regular and harmonic, procured "pleasure for fools and serenity for intelligent men, thanks to the reproduction of divine harmony in mortal movements."[10] Thus for him, the harmony of the cosmos was modelled on the same ratios as musical harmony and the influence of the moving planets on the soul was justified by the similar effects due to sounds.

Together with the ratios for the fifth, 3:2, the fourth, 4:3, and the tone, 9:8, already seen, Plato also indicated that of 256:243 for the "diesis". This is calculated by subtracting the ditone $do - mi$, 81:64, from the fourth, $do - fa$, that is to say, (4:3):(81:64) = 256:243. The Pythagorean "sharp" does *not* divide the tone into two equal parts, but it leaves a larger portion, called "apotome".[11] He even allowed himself a description of the sound. "Let us suppose that the sound spreads like a shock through the ears as far as the soul, thanks to the action of the air, the brain and the blood ... if the movement is swift, the sound is acute; if it is slower, the sound is deeper ...".[12]

The classification of the elements according to regular polyhedra is famous in the *Timaeus*. A late commentator like Proclus attributed to the Pythagoreans the ability to construct these five solids, known from then on also as Platonic solids. They are: the tetrahedron made up of four equilateral triangles, the hexahedron, or cube, with six squares, the octahedron with eight equilateral triangles, the dodecahedron with 12 regular pentagons, and the icosahedron with 20 equilateral triangles.

A regular dodecahedron found by archaeologists goes back to the time of the Etruscans, in the first half of the first millennium B.C.[13] In reality, leaving aside the Pythagorean sects and the Platonic schools, which presumed to confine mathematical sciences within their ideal worlds, we find hand-made products, artefacts, monuments, temples, statues, paintings and vases, which undoubtedly testify to far more ancient abilities to construct in the real world what those philosophers then tried to classify and regulate.

On a plane, it is possible to construct regular polygons with any number of sides. But in space, the only regular convex solids with faces of regular polygons are these five. Why? The explanations that have been given are, from this moment on, a part of the history of European sciences. They are an excellent example of how the proofs of mathematical results changed in time and in space, coming to depend on cultural elements like criteria of rigour, importance and pertinence. In other words, with the evolution of history, different answers were given to the questions: when is a proof convincing and when is it rigorous? How important is the theorem? Why does this property provide a fitting answer to the problem?

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[10] Plato 1994, pp. 25–27, 31–33, 61, 129–131.

[11] Plato 1994, p. 37.

[12] Plato 1994, p. 103.

[13] Heath 1963, p. 107.

Plato's arguments were based on a breakdown of the figures into triangles and their recombination. He also posited solids which corresponded to the four elements: fire with the tetrahedron, air with the octahedron, earth with the cube, and water with the icosahedron. He justified these combinations by reference to their relative stability: the cube and earth are more stable than the others. The fifth solid, the dodecahedron, represents the whole universe. Over the centuries, Plato's processes of reasoning lost credibility and the mathematical proofs modified their standards of rigour. Analogy became increasingly questionable and weak.

Regular polyhedra were studied by Euclid, Luca Pacioli and Kepler, among others. In one period, these solids were considered important because, with their perfection, they expresses the harmony of the cosmos. In another, they spoke of a transcendent god who was thought to have created the world, and to have added the signature of his "divine ratio".[14] At the time, this was considered to be necessary for the construction of the pentagon and the dodecahedron: "ineffable", because irrational, and also called "of the mean and the two extremes", or the "golden section".[15] For some, the field of reasoning was to be limited to Euclidean geometry, because the rest would not be germane to the desired solution. Subsequently, however, Euclid's incomplete argument was concluded by the arrival of algebra and group theory. I personally am attached to the relatively simple version offered last century by Hermann Weyl (1885–1955).[16]

In the *Meno*, Plato described Socrates teaching a boy-slave. He led him to recognize, by himself, that twice the area of the square constructed on a given line is obtained by constructing a new square on the diagonal of the first one.

We can interpret the reasoning of Socrates-Plato as an argument equivalent to the theorem of Pythagoras in the case of isosceles right-angled triangles. The first square is made up of two such triangles; the square on the hypotenuse contains four.[17]

The importance of Plato for our history derives from the role that was assigned to mathematical sciences and to music in his philosophy and in Athenian society. He enlarged on what he had learnt from the Pythagorean Archytas, to the point that his voice continues to be heard through the millennia up to today, marking out the evolution of the sciences. The motto, traditionally attributed to him, over the door of his school, the Academy, is famous: let nobody enter who does not know geometry. The fresco by Raffaello Sanzio "Causarum cognitio [knowledge of causes]", in the Vatican in Rome, is also famous; in this painting, together with Plato with his *Timaeus*, indicating the sky, and Aristotle with his *Ethics*, we can find allegories of geometry, astronomy and music.

In his *Politeia [Republic]*, Plato wrote that he wanted to educate the soul with music, just as gymnastics is useful for the body. He was discussing how to prepare the group of people responsible for safeguarding the state by means of warfare,

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[14] Pacioli 1509. See Sect. 6.4.

[15] In the pentagon, the diagonals intersect each other in this ratio.

[16] Weyl 1962.

[17] Heath 1963, p. 178. Fowler 1987, pp. 3–7.

both on the domestic front and abroad. Above all, he criticised poets, who, with their fables about the realm of the dead "do not help future warriors"; the latter risk becoming "emotionally sensitive and feeble". Laments for the dead are things for "silly women and cowardly men".[18]

Plato preferred other means to educate soldiers. Music could be useful, provided that languid, limp harmonies like the Lydian mode were eliminated, and the Dorian and Phrygian modes were used, instead. "… this will appropriately imitate the words and tones of those who demonstrate courage in war or in any act of violence … of those who attend to a pacific, non-violent, but spontaneous action, or intends to persuade or to make a request …". For this reason, the State organisation would not need instruments with several strings, capable of many harmonies [or, even less so, of passing from one to another, that is to say, modulating], and would limit itself to the *lyre*, excluding above all the lascivious breathiness of the *aulos*. Plato made similar comments about the rhythm. "Because the rhythm and the harmony penetrate deeply into the soul, and touch it quite strongly, giving it a harmonious beauty." Excluding all pleasure and every amorous folly, "the ultimate aim of music is love of beauty", the philosopher concluded. For the warriors of this state described by Plato, variety in foods for the body was as little recommended as variety in music. "… the one who best combines gymnastics and music, and applies them in the most correct measure to the soul, is the most perfect and harmonious musician, much more than the one who tunes strings together."[19]

In his famous metaphor of the cave, the Greek philosopher explained that with our senses, we can only grasp the shadows of things. We should break the chains, in order to succeed in understanding the true essence and reality, which for him lay in the realm of the ideas. "… We must compare the world that can be perceived by sight with the dwelling-place of the prison [the cave where we are imagined to be chained to the wall] … the ascent and the contemplation of the world above are equivalent to the elevation of the soul to the intelligible world …".

Thus Plato now presented the discipline that elevated from the "world of generation to the world of being …", and which was suitable to educate young people, who had occupied his attention since the beginning of the book. "Not being useless for soldiers", then. However, this could not mean gymnastics, which deals "with what is born and dies", that is to say, the ephemeral body. Nor was it music, which "procured, by means of harmony, a certain harmoniousness, but not science, and with rhythm eurhythmy". It was, instead, the "science of number and of calculation. Is it not true that every art and science must make use of it? … And also, maybe, …, the art of warfare?" After mocking Homer's Agamemnon because he did not know how to perform calculations, Socrates-Plato concluded. "And therefore, …, should we add to the disciplines that are necessary for a soldier that of being able to calculate and count? Yes, more than anything else, …, if he is

[18]Plato 1999, pp. 117, 119, 125, 145, 149.

[19]Plato 1999, pp. 179, 181, 209, 187, 191, 195, 211.

to understand something about military organizations, or rather, even if he is to be simply a man."[20]

Calculation and arithmetic are "fit to guide to the truth" because they are capable of stimulating the intellect in cases where it is necessary to discriminate between opposites. According to Plato, here "sensation does not offer valid conclusions". Thus, "we have distinguished between the intelligible and the visible". I will return at the end of this chapter to the hallmark of dualism thus impressed by this Greek culture.

"A military man must needs learn them in order to range his troops; and a philosopher because, leaving the world of generation, he must reach the world of being . . .". Thus he went so far as to impose mathematics by law, in order to be able to "contemplate the nature of numbers". Not for trading, "but for reasons of war, and to help the soul itself . . . to arrive at the truth of being", ". . . always rejecting those who reason by presenting it [the soul] with numbers that refer to visible or tangible bodies." Even if they discussed of visible figures, geometricians would think of the ideal models of which they are copies, "they speak of the square in itself and of the diagonal in itself, but not of the one that they trace . . .".[21]

Even geometry has an "application in war". But the philosopher criticised practical geometricians: "They speak of 'squaring', of 'constructing on a given line' . . .". Instead, "Geometry is knowledge of what perennially exists." Even astronomy is presented as useful to generals.[22]

Having rendered homage to the Pythagoreans for uniting astronomy and harmony, Plato criticised those who dealt with music using their ears. ". . . talking about certain acoustic frequencies [vibrations?] and pricking up their ears as if to catch their neighbour's voice, some claim that they perceive another note in the middle, and define that as the smallest interval that can be used for measuring . . . both the ones and the others give preference to the ears over the mind . . . they maltreat and torture the strings, stretching them over the tuning pegs . . .".[23]

Still more discourses, that Plato put into the mouth of Socrates, regard subjects that belong to the history of Western sciences. These will be found in numerous books of every kind and of all ages, as sustained by a wide variety of people: philosophers, scientists, educators, historians, professors, professionals and dilettantes. They end up by forming a kind of orthodoxy, which subsequently easily becomes a commonplace, a degraded scientific divulgation, a general mass of nonsense which is particularly suitable to create convenient caricatures, a celebratory advertisement for the disciplines.

Thus we find expressed here the distinction between sciences and opinions, beliefs. ". . . opinion has as its object generation, intellection has being." Sciences eliminate hypotheses and bring us closer to principles. To understand ideas, these

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[20]Plato 1999, pp. 457, 467, 469, 471.

[21]Plato 1999, p. 447.

[22]Plato 1999, pp. 471, 475, 477, 479, 481, 483.

[23]Plato 1999, pp. 491, 493.

should be isolated from all the rest, and “if by chance he glimpses an image of it, he glimpses with his opinion, not with science . . .”. Young people need to be educated to this, because those responsible for the State cannot be allowed to be extraneous to reason, like irrational lines”.[24]

The discourse undoubtedly has a certain logic, but it is not without clear contradictions. Education, in the State of the warrior-philosophers, would be imposed by law; and yet it was also noted that “no discipline imposed by force can remain lasting in the soul.” [Luckily for us!] Plato often used to repeat when he spoke of young people: “may they be firm in their studies and in war” . . . “. . . assuming the military command and all the public offices . . .” Therefore he was thinking of a State projected for warfare: the defeat suffered by Athens in 404 B.C. in the Peloponnesian War against Sparta weighed like a millstone on the text. It even assumed tones which may, at least for some of us, have hopefully become intolerable: “. . . we said that young children had to be taken to war, as well, on horseback, so that they could observe it, and if there was no danger [how good-hearted of him!], they were to be taken closer, so that they could taste the blood, like little dogs.”[25]

Our none-too-peaceable philosopher seemed to be less worried about armed violence than keeping young people away from pleasure: “habits that produce pleasure, which flatter our soul and attract it to themselves, but which do not persuade people who in all cases are sober”. Young men are to be educated to temperance, and to “remain subject to their rulers, and themselves govern the pleasures of drinking, of eating and of love.”[26] How unsuitable for them, then, Homer became (together with many other poets) who represented Zeus as a victim of amorous passion.

The myth of love, as narrated in the *Symposium [The banquet]*, appears to be interesting all the same, because it was used to explain medicine, music, astronomy and divination. The first of these was defined as “the science that studies the organism’s amorous movements in its process of filling and emptying”. The good doctor restores reciprocal love when it is no longer present: “. . . creating friendship between elements that are antagonistic in the body and . . . infusing reciprocal love into them . . . a warm coolness, a sweet sourness, a moist dryness . . .” For music, he criticised the Heraclitus quoted above,[27] who would have desired to harmonise what is in itself discordant. “It is not possible for harmony to arise when deep and acute notes are still discordant.” “Music is nothing more and nothing less than a science of love in the guise of harmony and rhythm.” . . . “And such love is the beautiful kind, the heavenly kind; Love coming from the heavenly muse, Urania. There is also the son of Polyhymnia, vulgar love . . .” . . . “men may find a certain pleasure in it, but may it not produce wanton incontinence.” In the seasons, cool heat, and moist

[24] Plato 1999, pp. 497, 499, 501.

[25] Plato 1999, pp. 505, 507, 513, 506, 507.

[26] Plato 1999, pp. 511, 155.

[27] See above Sect. 2.2.

dryness may find love for each other, and harmony. Otherwise, love combined with violence provokes disorder and damage, like frost, hail and diseases. The science which studies these phenomena "of the movements of stars and of the seasons", is called astronomy by Plato. Even in the art of prophecy, which concerns relationships between the gods and men, it is love that is dominant: "the task of prophecy is to bear in mind the two types of love".[28]

Diotima, a woman, then told Socrates how "that powerful demon" called love had originated: first of all, it was one of those demonic beings capable of allowing God to communicate with mortal man. Consequently, thanks to them, the universe became "a complex, connected unit. By means of the agency of these superior beings, all the art that foretells the future takes place . . . the prophetic art in its totality and magic." . . . "the one who has a sure knowledge of this is a man in contact with higher powers, a demonic figure." At the party for the birth of Aphrodite, there were also Poros, the son of Metis, and Penia. The latter decided to have a son with Poros, and in this way Love was born. He thus originated from want and his mother, poverty, but he was also generated by the artfulness and the expedients represented by his father. And then he inherited something from his grandmother, Metis, invention, free intuition. In order to reach his aims, in the end, Love must become a sage, a philosopher, an enchanter, a sophist.[29]

With minor modifications to the myth, we can now add that the necessities of life, linked with the capacities of invention, have produced the sciences. However, in the West, and as a result of the interpretation of Plato, these mainly are pushed towards the heavens populated by the ideas of the beautiful, of good and of immortality, causing man to forget that war and death are advancing, on the contrary, on earth.

The extent to which the Pythagorean and Platonic tradition was modified on its passage through the centuries, and was transmitted from generation to generation is narrated in the following history.

2.4 Euclid

> . . . the theorem of Pythagoras teaches us to discover a *qualitas occulta* of the right-angled triangle; but Euclid's lame, indeed, insidious proof leaves us without any explanation; and the simple figure [of squares constructed on the sides of an isosceles right-angled triangle] allows us to see it at a single glance much better than his proof does.
>
> Arthur Schopenhauer

A date that cannot be specified more precisely than 300 B.C., and a no-better-defined Alexandria witnessed the emergence of Euclid, one of the most famous mathematicians of all time. We hardly know anything about him, except that he

[28] Plato 1953, pp. 103–107.

[29] Plato 1953, pp. 128–130 and passim.

wrote in Greek, the language of the dominant culture of his period. But what will his mother tongue have been? Maybe some dialect of Egypt?

Euclid's *Elements* were to be handed down from age to age, and translated from one language into another, passing from country to country. For Europe, this was regularly to be the reference text on mathematics in every commentary and every dispute for at least 2,000 years. More or less explicit traces of it are to be found in school books, not only in the West, but all over the world. All books dealing with the history of sciences speak about him. While Plato represents the advertising package for Greek mathematics, Euclid supplies us with the substance. And here we find music again.

This scholar from Alexandria wrote a brief treatise entitled $KATATO$ $MH\ KANONO\Sigma$, traditionally translated into Latin as *Sectio Canonis*, which means *Division of the monochord*. The Pythagorean theory of music is illustrated in an orderly manner: theorem A, theorem B, theorem Γ, ... It was explained in the introduction that sound derives from movement and from strokes. "The more frequent movements produce more acute sounds and the more infrequent ones, deeper sounds ... sounds that are too acute are corrected by reducing the movement, loosening the strings, whereas those that are too deep are corrected by an increase in the movement, tightening the strings. Consequently, sounds may be said to be composed of particles, seeing that they are corrected by addition and subtraction. But all the things that are composed of particles stand reciprocally in a certain numerical ratio, and thus we say that sounds, too, necessarily stand in such reciprocal ratios."[30]

The beginning immediately recalled the Pythagorean ideas of Archytas. The third theorem stated: "In an epimoric interval, there is neither one, nor several proportional means." By epimoric relationship, he meant one in which the first term is expressed as the second term added to a divisor of it. A particular case is $n + 1 : n$. From this theorem, after reducing to the form of other theorems the ratios of the Pythagorean tradition translated into segments, Euclid finally derived the 16th theorem which states: "The tone cannot be divided into two equal parts, or into several equal parts."[31] The monochord was divided by Euclid into tones, fourths, fifths and octaves. And, of course, theorem number 14 stated that six tones are greater than the octave, because the ninth theorem had demonstrated that six sesquioctave intervals [9:8] are greater than the double interval [2:1].

Thus Euclid made a decisive contribution, not only to the creation of an orthodoxy for geometry, but also for the theory of music, which was to remain for centuries that of the Pythagoreans. In him, the distinction between consonances and dissonances continued to be justified by ratios between numbers. But here, instead of the *tetractis*, he invoked as a criterion that of the ratios in a multiple, or epimoric form, i.e. $n : 1$ or else $n + 1 : n$, like 2:1; 3:2; 4:3. Such a limpid, linear

[30] We use the 1557 edition of Euclid, with the Greek text and the translation into Latin. An Italian translation is that of Bellissima 2003. Euclid 1557, p. 8 and 14; Bellissima 2003, p. 29. Zanoncelli 1990. Euclid 2007, pp. 677–776, 2360–2379 and 2525–2541.

[31] Euclid 1557, p. 10 and 16; Bellissima 2003, p. 37.

explanation met with a first clear contradiction, which was later to be attacked by Claudius Ptolemaeus. The interval of the octave added to a consonance generates (for the ear) another consonance; thus the octave added to the fourth generates the consonance of the 11th. But its ratio becomes 8:3, which is not among the epimoric forms permitted. The 12th, on the contrary, possesses the ratio 3:1.

Euclid's mathematical model clashed with the reality of music. The theory did not account for all the phenomena that it claimed to explain. Was it an exception? Or was it necessary to substitute the theory? In this way, controversies arose, which were to produce other theories, setting the evolution of science in motion.

Some historians have taken an interest in the pages of Euclid quoted above, partly because they contain an elementary error of logic. One of the first person to realise this was Paul Tannery in 1904. According to Euclid, consonances are determined by those particular kinds of ratios. However in his 11th theorem ("The intervals of the fourth and the fifth are epimoric"), our skilful mathematician wrote that if the double fourth (a seventh) was dissonant, then it must be a non-multiple. As if the inverse implication were true: not consonant implies not multiple and not epimoric. But this is not possible, because it would imply that all epimoric ratios and their ratios are consonant, and so, for example, even the tone 9:8 would become consonant.

For Tannery, this error is sufficient to prove that the treatise on music was not by Euclid. But others are not so drastic; even Euclid may have fallen asleep.[32] After all, errors are commonly found in the work of other famous scientists. Pointing them out and discussing them would appear to be one of the most important tasks of historians.[33] In reality, they are often *lapsus* not noticed during the reasoning, which reveal aspects of their personality that would otherwise remain hidden. They are a precious help to better understand events that are significant for the evolution of the sciences, and not just useless details which become acts of *lèse-majesté* in the pages of hostile historians.

Tannery discovered the error at the beginning of last century, when European mathematical sciences were undergoing a profound transformation. Among other things, modern mathematical logic was developing, and some scholars were even re-considering Euclid in the light of the crisis of the foundations. The most famous of these was David Hilbert (1862–1943), who was polishing him up to make him meet the rigorous standards of the new twentieth-century scientific Europe.[34] The *Elements* were thus interpreted by means of an axiomatic deductive scheme, made up of definitions, postulates and theorems. However, this was an anachronistic reading of the ancient books, amid a dispute about the foundations of mathematics, the *Grundlagenstreit*, which took into consideration other positions, different from the formalistic one of the Hilbertian school of Göttingen.[35]

[32]Bellissima 2003. Euclid 2007, pp. 691–701.

[33]Tonietti 2000b.

[34]Hilbert 1899.

[35]Tonietti 1982a, 1983a, 1985a, 1990.

For 2,000 years, practically nobody read the works of Euclid from the point of view of a logician. Their importance lay in other fields. However, logic did not enjoy the favour of the Platonic schools, but appeared to be the prerogative of their Aristotelian rivals, with their well-known syllogisms. Now, what the *lapsus*-error betrays is not an apocryphal text falsely attributed to Euclid, but on the contrary, insufficient attention paid by him to the logical structure of the reasoning, and his adhesion (at all costs?) to the Pythagorean-Platonic theories of music.

Naturally, all this can be seen not only in the *Division of the monochord*, but above all in the *Elements*, the books of which (the arithmetic ones) are also used to argue the theorems of music.[36] From the *Elements* we shall extract only a couple of cases, which are most suitable for a comparison between cultures, which is what interests us here.

Euclid was the first to demonstrate here what was subsequently to be regularly called the theorem of Pythagoras, but would be better indicated by his name. In the current editions of the *Elements*, Euclid offered the following proof of the proposition: "In right-angled triangles, the square on the side opposite the right angle is equal to the squares on the sides enclosing the right angle."

The angle FBC is equal to the angle ABD because they are the sum of equal angles.

The triangle ABD is equal to the triangle FBC because they have equal two sides and the enclosed angle.

The rectangle with the vertices BL is equal to twice the triangle ABD.

The square with the vertices BG is equal to twice the triangle FBC.

Therefore the rectangle and the square are equal, because the two triangles are equal.

The same reasoning may be repeated to demonstrate that the rectangle with the vertices CL is equal to the square with the vertices CH.

As the square with the vertices DC is the sum of the rectangles BL and CL, it is equal to the sum of the squares BG and CH.

Quod erat demonstrandum.[37]

The proof bears the number 47 in the order of the propositions, and is followed by the inverse one (if 47 is true of a triangle, then it must be right-angled), which concludes the first book of the *Elements*. It is obtained by following the chain of propositions, like numbers 4, 35, 37, and 41. The demonstration is based on the other demonstrations; these demonstrations are based, in turn, on the definitions (of angle, triangle, square, ...), on the postulates (draw a straight line from one point to another, the right angles are all equal to one another, ...), on common notions (equal angles added to equal angles give equal angles, ...) and on the possibility of constructing the relative figures. Everything is broken down into shorter arguments,

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[36] Bellissima 2003, p. 31. Euclid 2007.

[37] Euclid 1956, pp. 349–350; Euclid 1970, pp. 146–150. Euclid 2007. Figure on every textbook.

reassembled and well organised in a linear manner; everything seems convincing; everything is well-known to every student.

The earliest commentators, like Proclus, Plutarch or Diogenes Laertius, attributed the theorem to Pythagoras, but none of them are eye-witnesses, indeed, they come many centuries after him, seeing that the period when Pythagoras lived was the fifth century B.C. Pythagoras did not leave anything written, but only a series of disciples and followers. Failing documentary evidence, we can believe or not believe the attribution of the accomplishment to Pythagoras.

Anyway, whoever the author was, Euclid's proof does not appear to be the most direct one, even in the field of the Pythagorean sects.

In the right-angled triangle ABC, by tracing the perpendicular AD to BC, two new triangles, ABD and ADC, are created, which are said to be similar to ABC, because their sides are reduced in the same ratio. In other words, for them, $BC : AB = AB : BD$. Consequently, by the rule of proportions, $BC \times BD = AB \times AB$. Having demonstrated the equality of square BG and rectangle BL, the argument continues in the same way as Euclid 47.

But Euclid did not follow this route, because he wanted to make his proof independent of the theory of proportions. This appears in the *Elements* only in books five and six. If he had used it (the necessary proposition would have been number eight of book six), he would have broken the linear chain of deductions, forming a circle that he would perhaps have considered vicious. Furthermore, he would have raised the particularly delicate question of incommensurable ratios, necessary to obtain a valid demonstration for every right-angled triangle. Euclid would succeed in avoiding the obstacles, but he would be forced to pay a price: following a route which seems as intelligent as it is artificial.

In his commentary, Thomas Heath, who has left us the current English translation of the *Elements*, considered Euclid's demonstration "... extraordinarily ingenious, ... a veritable *tour de force* which compels admiration, ...".[38] This British scholar compared it with various possibilities proposed in other periods and in other places by other people. At times he erred on the side of anachronism, because he also used algebraic formulas which only came into use in Europe after Descartes; but he seems to be worried above all about preventing some ancient Indian text (coming from the British Empire?) from taking the primacy away from Greece. In the end, he solved the question as follows: "... the old Indian geometry was purely empirical and practical, far removed from abstractions such as irrationals. The Indians had indeed, by attempts in particular cases, persuaded themselves of the truth of the Pythagorean theorem, and had enunciated it in all its generality; but they had not established it by scientific proof."[39]

Thanks to an article by Hieronymus Zeuthen (1839–1920), and to the books of Moritz Cantor (1829–1920) or David E. Smith,[40] Heath had access also to what they

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[38]Euclid 1956, p. 354.

[39]Euclid 1956, p. 364.

[40]Zeuthen 1896; Cantor 1922; Smith 1923.

thought contained an ancient Chinese text: the *Zhoubi [The gnomon of the Zhou]*. However, the British historian seems to see, in the Chinese demonstration of the fundamental property of right-angled triangles, only a way to arrive at the discovery of the validity of the theorem in the rational case of a triangle with sides measuring 3, 4, 5. "The procedure would be equally easy for any *rational* right-angled triangle, and would be a natural method of trying to *prove* the property when it had once been *empirically* observed that triangles like 3, 4, 5 did in fact contain a right angle."[41] Trusting D. E. Smith, he concludes that the Chinese treatises contained "... a statement that the diagonal of the rectangle (3, 4) is 5 and ... a rule for finding the hypotenuse of a 'right triangle' from the sides, ...".[42] But they ignored the proof of that.

It is easier to understand the common defence of Euclid undertaken by Heath, and his underestimation of the Chinese text, even with respect to the Arabs and the Indians. He is less excusable when he writes: "1482. In this year appeared the first printed edition of Euclid, which was also the first printed mathematical book of any importance."[43] However, Heath was led to his interpretations and judgements by his own Eurocentric prejudices. If we should want to take part in an absurd competition regarding priorities, it would be extremely easy to prove him wrong. We have evidence that *The gnomon of the Zhou* was first printed as long ago as 1084. A 1213 edition of the book is extant today in a library at Shanghai. Heath would only be left with the possibility of sustaining that *The gnomon of the Zhou* is not a book of mathematics, or that it speaks of a mathematics that is not important. Perhaps it is not important for Europe; but what about the world?

In the third chapter of this work, we shall show, on the contrary, that this ancient Chinese book in fact demonstrates the fundamental property of right-angled triangles. In the fourth chapter, we shall discuss other Indian demonstration techniques. It is true, we do not find in the Indian or Chinese texts the theorems that school has accustomed us to, but simply other procedures to convince the reader and help him find the result. Cultures that are different from the Greek one followed different arguments, which, however, are to be considered equally valid.

On the basis of what criterion may we expect to impose a hierarchy of ours from Europe? Unfortunately, we shall see that historical events offer only one. Then it must be the one sustained by Plato: war. But will our moral principles be prepared to accept it?

For more than 2,000 years, in Europe, Euclid will be the model that was generally shared to reason about mathematics. Rivers of ink have been consumed for him. I will not yield here to the temptation of making them more turbid, or better, more limpid, or of deviating them. However, in order to prepare ourselves for the comparison with different models of proof, we need to examine the procedure followed more closely.

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[41]Euclid 1956, pp. 355–356.

[42]Euclid 1956, p. 362.

[43]Euclid 1956, p. 97.

Our famous Hellenistic mathematician wished to convince his readers that: “In right-angled triangles the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.” To achieve this goal, he had defined the right angle at the beginning of book one. In the list of definitions, the tenth one proclaims: when a straight line conducted to a straight line forms adjacent angles that are equal to each other, each of the equal angles is a right angle, and the straight line conducted on to the other straight line is said to be perpendicular to the one on to which it is conducted. Not satisfied with this, Euclid included among the postulates also the one numbered four: all right angles are equal to one another.[44] He also defined the angle, the triangle and the square. He postulated that a straight line could be drawn from one point to another. Among the common notions, he included the properties of equality. He explained how to construct a square on a given segment of a straight line. All this was either defined, or postulated or demonstrated in the *Elements*. Demonstrating, then, means tracing back to some other property, already defined, or postulated, or demonstrated. And so on.

Euclid scrupulously sought certainty and precision. It seems that he did not want to trust either evidence or his intuition. Who would find it obvious that right angles are equal? Intuition would seem to lead us immediately to see how to draw straight lines, triangles, squares. But what would it be based on? What if it led us to make a mistake? Our Greek mathematician would like to avoid using his eyes, or working with his hands, or believing his ears. For him, the truth of a geometrical proposition should be made independent of the everyday world, practical activities or the senses. The organs of the body would provide us with ephemeral illusions, not properties that are certain and eternal. As in the myth of the cave described by Plato, Euclid would like to detach himself from the distorted shadows of the earth, which are visible on the wall, to arrive at the ideal objects that project them. It was only on these that he based the truths of his geometry, which were thus believed to have descended from the heavens of the eternal ideas. He would like to demonstrate every proposition by describing the procedure to trace it back to them. He would like to, but does he succeed?

Thus Euclid followed this dualism and hierarchy, whereby the earth is subject to the heavens. His model of proof must therefore avoid making reference to material things that are a part of the everyday world, where people use their hands, eyes and ears to live their lives. The truth of a proposition descends from the heavens on high: it is deduced. And yet, luckily for us, even in an abstract scheme like this, limitations filtered through, and echoes could be heard of the ancient origins among men living on the earth.

A line is length without width.[45] Anybody would think of a piece of string which becomes thinner and thinner, or of a stroke made with a pen whose tip gets increasingly thinner. How can we imagine the *Elements* without the numerous

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[44]Euclid 1956, pp. 349, 153–154.

[45]Euclid 1956, p. 153.

figures? But isn't it true that we see the figures? Or only with the eyes of our mind? In any case, it does not seem to have been sufficient for Euclid to think of an object of geometry, or to describe its properties. In order to make it exist in his ideal world as well, it was necessary for him to construct it by means of a given procedure at every step. That of Euclid is an ideal, abstract geometry, but not completely separated from the world. In it, properties are deduced from on high, but they also need to be constructed at certain points. It is made up of immaterial symbols, but they can be represented on the plane that contains them.

Having stated in book seven that a prime number "... is one which is measured by the unit alone",[46] in book nine, Euclid demonstrates proposition 20: "Prime numbers are more than any assigned multitude of prime numbers." Let the prime numbers assigned be represented by the segments A, B, C. Construct a new number measured by A, B, C [the product of the three prime numbers]. Call the corresponding segment obtained DE, which is commensurable with A, B, C. Add to DE the unit DF, obtaining EF. There are two possibilities:
either EF is prime, and then A, B, C, EF are greater than A, B, C,
or EF will be measured by the prime number G. But in this case, it must be different from the prime numbers A, B, C. Otherwise, G would measure both DE and EF, and consequently also their difference. However, the difference is the unit which cannot be measured any further. As this would be "absurd", G must be a new prime number. A, B, C, G form a quantity greater than A, B, C. *Quod erat demonstrandum*.[47]

Note that the numbers are represented by segments, and by ratios between segments. As a result, even this numerical proof is accompanied by a figure.

The previous proof is a procedure that makes it possible to obtain a new prime number. It really constructs the quantity of primes announced in the proposition. Having obtained a new prime number and added it to the previous ones, the procedure can be repeated as many times as is desired. The proof thus constructs, step by step, continually new prime numbers.

Besides the usual anachronistic algebraic translation, Heath concludes in his commentary: "*the number of prime numbers is infinite*."[48] But in this way, he annuls Euclid's peculiar style, because he transfers the subject among the mathematical controversies of the nineteenth and twentieth centuries. It was not Euclid, but rather these mathematicians who discussed about 'infinite' quantities, which were used in every field of mathematics, above all in analysis.

Only at that time did people like Richard Dedekind (1831–1916), Georg Cantor (1845–1918) and David Hilbert start to use infinity, after defining it formally by means of the characteristics property which cancelled Euclid's fifth common notion: The whole is greater than the part.[49] Up to that moment, it had been considered a paradox that, for example, whole numbers and even numbers could be counted in

[46]Euclid 1956, II, p. 278.

[47]Euclid 1956, II, p. 412.

[48]Euclid 1956, II, p. 413.

[49]Euclid 1956, I, p. 155.

parallel; because in that way, they would appear to be equally numerous, whereas intuition tells us that even numbers are only a part, there are fewer of them. The paradox became the new definition, which, in the new language that had now entered even primary school textbooks, stated: a set is called infinite when it admits a biunique correspondence with one of its own parts. In other words, when the whole is ‘equal’ to a part, it is a case of infinity.

Euclid shows what operation to perform in order to construct step by step a increasingly large quantity, but he avoided calling it infinite. Dedekind, Cantor and Hilbert defined as ‘infinite’ quantities that they could not succeed in constructing. There’s a big difference.

Like the Plato painted by Raphael in the Vatican for the Athens school, Euclid pointed his finger upwards; and yet he still maintains some connections with the world, both in pictures and in his constructions. After Hilbert[50] and his undeniable success with the mathematical community of the twentieth century, it became all too common to interpret Euclid axiomatically. And yet we have seen, with a clear example, that this is an anachronistic distortion. Euclid also used other approaches, and not everybody would like to cancel his construction procedures.

Hieronymus Zeuthen saw Euclid better together with the ‘problems’ *à la* Eudoxus (Cnidos, died c. 355 B.C.)[51] than together with the ‘theorems’ *à la* Plato. “... Euclid: he is not satisfied with defining equilateral triangles, but before using them, he guarantees their existence by solving the problem of how to construct these triangles: ...”[52] Anyone who believed in the existence of the geometrical object before examining it (in the world of the ideas) would not need to construct it in order to convince himself of its reality (on earth). “But the Greeks used constructions much more widely than we are used to doing, and specifically also in cases where its practical use is wholly illusory. [...] In order to arrive at a certainty on this matter, and at the same time to understand what the theoretical significance of constructions was at that time, they need to be observed from their first appearance in Euclid onwards. The idea will thus be found to be approved that *constructions, with the relative proof of their correctness, served to establish with certainty the existence of what is to be constructed.* Constructions are prepared by Euclid by means of *postulates*.”[53] In ancient geometry, therefore, proofs of existence were supplied by geometrical constructions.

This scholar’s interpretations were more or less closely taken up by people like Federigo Enriques[54] (1871–1946) and Attilio Frajese.[55] In 1916, also Giovanni Vacca published his translation of Book 1 of the *Elements*, with the parallel Greek text. But the fact that Euclid’s proofs were based on constructions was completely

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[50] Hilbert 1899.

[51] Boyer 1990, pp. 78, 99, 105–110.

[52] Zeuthen 1902, pp. 72–73.

[53] Zeuthen 1896, pp. 222–223.

[54] Euclid 1925, pp. 135–141.

[55] Euclid 1970, p. 147.

ignored.[56] Also Alexander Seidenberg stated that Euclid did not practise "the famous axiomatic method". Even though "he was meticulous in the constructions to abstract from the old 'peg and cord' (or 'straight edge and compass') constructions."[57] This scholar dedicated a whole work to rebutting the (anachronistic) idea that Euclid had developed Book 1 of the *Elements* axiomatically.[58] Rather, the ancient Hellenistic geometrician constructed the solution of problems.

It has been confirmed by David Fowler that the historical and real Euclid could not be taken up into the Olympus of orthodox formal axiomatic systematizers without falsifying him: "... their geometry dealt with the features of geometrical thought-experiments, in which figures were drawn and manipulated, ...".[59] The same line of reasoning is also followed initially by Lucio Russo, who refers directly to Zeuthen. "Mathematicians did not create, ..., new entities by means of pur abstract definitions, but they considered their real geometrical constructibility indispensable, ...".[60] However, this Italian mathematician, also interested in classical studies, then creates an excessive contrast between the construction procedures and Euclid's definitions, because he interprets them in a strictly Platonic sense. Thus he makes an effort to show that the latter are not authentic, but added by others. This is possible, considering the long chain of copies and commentaries on the codices that have been handed down to us.[61] But why should the alternative only be between a Platonizing Euclid, for whom the ideas really exist, and one who considers them just conventional names?

Isn't it true that in the definitions and all the figures, we can already perceive the representation and the inspiration of the everyday world? Russo tries to give the *Elements* a consistency which they do not possess in this sense, in order to assimilate them to his own, modern, post-Hilbertian definition of science, limited to a "rigorously deductive structure."[62] Luckily for us, the sciences and the arts of demonstration are more varied, as we shall soon see more clearly.

Book 1 of the *Elements* converged towards the proof of the theorem of Pythagoras. We may consider that all 13 books merged together in calculating the angles of regular polyhedra inscribed in a sphere. The 18th proposition of Book 13 reads: "To set out the sides of the five figures and to compare them with one another." ... "I say next that *no other figure, besides the said five figures, can be constructed which*

[56]Euclid 1916.

[57]Seidenberg 1960, p. 498.

[58]Seidenberg 1975.

[59]Fowler 1987, p. 21.

[60]Russo 1996, p. 73.

[61]Russo 1996, pp. 235–244.

[62]Russo 1996, p. 32. Mario Vegetti finds that Euclid's approach is used by Galen and Claudius Ptolemaeus as an axiomatic Platonic model. And yet, even this professor of ancient philosophy, though levelling out the procedure too much, realises that Euclidean rationality has to come to grips with Aristotle: "In the first place, the ontological obligation to consider the forms as transcendent, or at least as external to the empirical, disappears." Vegetti 1983, p. 155.

is contained by equilateral and equiangular figures equal to one another."[63] In this way, the mathematician from Alexandria seemed to have succeeded in making considerable progress in demonstrating what Plato had only outlined with his famous solids. But here, the idea of universal harmony, glimpsed by the philosopher through the dialogues, was now reached by means of a tiring ascent from one theorem to another, from one step to another, less esoteric and more scholastic.

Mathematical sciences were to evolve in Europe with several, sometimes profound, changes. Yet Euclid's *Elements* succeeded in surviving and adapting to the different periods. They represent the backbone of Western history, which the different kinds of sciences inherited from one another. The English logician Auguste de Morgan (1806–1871) could still write in the nineteenth century: "There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy of the name, which has any material departures (we do not speak of *corrections*, or *extensions*, or *developments*) from the plan laid down by Euclid."[64] But geometry had changed, and was practised with the powerful means of infinitesimal analysis or projection methods. Euclid's geometry was of interest above all as a logical scheme of deductive reasoning, and was to be readjusted, also as such. Only towards the middle of the twentieth century did a group of French mathematicians, united under the pseudonym of Bourbaki, try to substitute the geometrical figures of Euclid's *Elements* with the formal algebraic structures inspired by Hilbert. The new *Eléments de Mathématique*, however, met with far less success than the model whose place they wanted to take. The work remained on the scene for a few decades, and nowadays is found covered with dust mainly on the shelves of Maths Department libraries.[65]

2.5 Aristoxenus

Aristoxenus (Tarentum 365/75–Athens? B.C.) is seldom remembered in science history books. When he is mentioned, writers admit that they were forced to include him because in antiquity, the theory of music was a part of the *quadrivium* mentioned above. But it is immediately added that "he turned his back upon the mathematical knowledge of his time, to adopt and propagate a radically 'unscientific' approach to the measurement of musical intervals."[66] This judgement stems from a widespread prejudice. It should be underlined, however, that this Greek from Tarentum left us some important books on harmony and musical rhythm. They continue to be particularly interesting, also for historians of the mathematical

[63] Euclid 1956, III, pp. 503–509.

[64] Euclid 1956, I, p. v.

[65] Tonietti 1982b, pp. 11–21.

[66] Winnington-Ingram 1970, p. 282.

sciences, precisely because they did not belong to the Pythagorean or Platonic school.

"Tension is the continual movement of the voice from a deeper position to a more acute one, relaxation is the movement from a more acute position to a deeper one. Acuteness is the result of tension, and deepness of relaxation." Thus Aristoxenus considered four phenomena (tension, acuteness, relaxation and deepness), and not just two, because he distinguished the process from the final result.[67] He criticised "those who reduce sounds to movements and affirm that sound in general is movement". For Aristoxenus, instead, the voice "moves [when it sings], that is to say, when it forms an interval, but it stops on the note". Thus Aristoxenus does not appear to be interested in the movement (invisible to the eye) of the string that generates the sound, or to the movement of sound through the air, but only in the movement (perceptible with the ear) with which the passage is made from one note to another.[68]

This last movement has its limits: "The voice cannot clearly convey, nor can the hearing perceive, an interval less than the smallest diesis ($\delta\iota\epsilon\sigma\iota\varsigma$, a passage, a quarter of a tone) . . ."[69] After distributing the notes along the steps of the scale, our Greek theoretician listed the "symphonies", or in other words the consonances, to distinguish them from the "diaphonies", the dissonances. The former are the intervals the fourth, the fifth, the octave, and their compounds with two or more octaves.[70] "The smallest consonant interval [the fourth] is determined, . . ., by the very nature of the voice." The largest consonances are not established by theory, but by "our practical usage – by this I mean the use of the human voice and of instruments – . . ."[71]

In his reasonings, Aristoxenus never made any reference to ratios between whole numbers or magnitudes, as the Pythagorean sects, Archytas and Euclid did. He also made a distinction between rational $\rho\eta\tau\alpha$ and irrational $\alpha\lambda o\gamma\alpha$ intervals, but he did not explain the difference in the *Elementa Harmonica* as handed down to us. From his *Ritmica*, it is only possible to infer that by "rational" intervals, he intended those that could be performed in music, assessing their range, whereas the others are "irrational". Consequently, below a quarter of a tone, the intervals are "irrational", while all the combinations of quarters of a tone are "rational" for him.[72]

The definition of the tone and its parts now became crucial. "The tone is the difference in magnitude between the first two consonant intervals [between the fifth and the fourth]. It can be divided into three submultiples, one half, one third and one quarter of a tone, because these can be performed musically, whereas it is not

[67] Aristoxenus 1954, p. 19.

[68] Aristoxenus 1954, pp. 20–21.

[69] Aristoxenus 1954, p. 22.

[70] Aristoxenus 1954, pp. 30–31.

[71] Aristoxenus 1954, p. 31.

[72] Aristoxenus 1954, pp. 24–25.

possible to perform any of the intervals smaller than these."[73] Euclid, in his 16th theorem, denied the possibility of dividing the tone into equal parts, on the basis of the non-existence of the proportional mean in whole numbers. Here, on the contrary, Aristoxenus calmly performed this division. Was the former "scientific" because he used proportions and ratios in his arguments, and the latter "non-scientific" because, on the contrary, he ignored them following his ear? Certainly not. Rather, these are differences of approach to the problems, which reflect cultural, philosophical and social features, in a word, values, that are very distant from each other.

In Book 2 of the *Elementa Harmonica*, Aristoxenus became explicit. "… the voice follows a natural law in its movement and does not form an interval by chance. And, we shall, unlike our predecessors, try to give proof of this which is in harmony with the phenomena. Because some talk nonsense, disdaining to make reference to sensation, because of its imprecision, and inventing purely abstract causes, they speak of numerical ratios and relative speeds, from which the acute and the deep derive, thus enunciating the most irrelevant theories, totally contrary to the phenomena; others, without any reasoning or proof, passing each of their affirmations off as oracles …" "Our treatise regards two faculties; the ear and the intellect. By means of the ear, we judge the magnitudes of intervals, by means of the intellect, we realise their value."[74]

With musical intervals, in his opinion, "it is not possible to use the expressions that are typically used for geometrical figures … For the geometrician does not use his faculties of sensation, he does not exploit his sight to make a correct, or incorrect evaluation of a straight line, a circle or some other figure, as this is the task of a carpenter, a turner or other craftsmen. For the $\mu o \upsilon \sigma \iota \kappa o \varsigma$ [musician], however, the precision of sensible perception is, on the contrary, fundamental, because it is not possible for a person whose sensible perception is defective to give an adequate explanation for phenomena that he has not succeeded in perceiving at all."[75]

Having chosen the ear as judge, Aristoxenus repeated even more clearly: "as the difference between the fifth and the fourth is one tone, and here it is divided into equal parts, and each of these is a semitone, and is, at the same time, the difference between the fourth and the ditone, it is clear that the fourth is composed of five semitones."[76]

In the Pythagorean sects, worshippers of whole numbers were trained as adepts; in the Academy, Plato desired to educate the soul of young warriors to eternal being by means of geometry. Now Aristoxenus appealed to musicians, who use their hands and ears to play their instruments. We are faced with a variety of musical scales, modes, melodies, which, however, in practice were difficult to play all on the same instrument, and thus it did not appear possible to pass from one to the other, i.e.

[73] Aristoxenus 1954, p. 32.
[74] Aristoxenus 1954, p. 47.
[75] Aristoxenus 1954, p. 48.
[76] Aristoxenus 1954, p. 79.

to modulate, either.[77] Plato did not even perceive the problem, because he limited melodies to those (Doric and Phrygian) he considered suitable for the order of his State. He did not tolerate free modulations. Aristoxenus, on the contrary, made them possible with his theory, and facilitated them.

If the fourth were divided into five equal semitones, the octave (the fourth plus the fifth) would be composed, in turn, of 12 equal semitones. On instruments tuned in this way (and not in the Pythagorean manner), semitones, tones, fourths, fifths, and octaves can be freely transposed (transported) along the various steps of the scales, maintaining their value, and thus permitting a full variety of melodies, modes and modulations. It is like what happens today with modern pianos tuned in the equable temperament. But this was to be adopted in Europe only in the eighteenth century, thanks to the efforts of musicians like Johann Sebastian Bach (1685–1750) and Jean Philippe Rameau (1683–1764). Yet even this clear advantage of his theory has recently been denied to Aristoxenus by hostile historians. "... although modulation was exploited to some extent by virtuosi of the late fifth century B.C. and after, there is no reason to think that it created a need for a radical reorganization of the system of intervals, or that such could have been imposed upon the lyre players and pipe players of the time." As he was opposed by the Pythagoreans of his time, our theoretician from Tarentum continues to be judged badly by the Pythagoreans of today.[78]

Some of his other characteristics tend to deteriorate his image in the eyes of certain science historians. Euclid considered sounds as "compounds of particles".[79] In the *Elementa Harmonica*, on the contrary, sounds appear to form a *continuum*, and accordingly Aristoxenus stated: "... we affirm without hesitation that no such thing as a minimum interval exists."[80] In theory, therefore, the tone could be divided up beyond every limit [*ad infinitum*]. But, guided by his ear, the musician stopped at a quarter of a tone for the requirements of melodies. For him, therefore, music is to be taken out of the group of discontinuous, discrete sciences, and included among the continuous ones, thus disarranging the *quadrivium*. Also in this, the philosophical roots of Aristoxenus are not those of Plato. His whole concept recalls rather the principles of Aristotle (Stagira 384–Calchis 322 B.C.), who was actually mentioned by name at the beginning of Book 2. This offers us a testimony that Aristotle had attended Plato's lessons, and that Aristoxenus himself had then become a direct pupil of Aristotle: "... as Aristotle himself told us, he gave a preliminary account of the contents and method of his topic to his listeners."[81]

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[77] Aristoxenus 1954, pp. 53–55.

[78] Winnington-Ingram 1970, p. 282. This writer shows the origin of her/his prejudices, because she/he immediately adds that "'temperament' would distort all the intervals of the scale (except the octave) and, significantly, the fifths and the fourths". For her/him, the 'correct' intervals are, on the contrary, those of Pythagoras. See Part II, Sects. 11.1 and 11.3.

[79] See above Sect. 2.4.

[80] Aristoxenus 1954, p. 67.

[81] Aristoxenus 1954, p. 45.

Thus we have also met the other famous philosopher who, together with Plato, and with alternating fortunes, was to have a significant influence on European culture, profoundly conditioning even its scientific evolution. After representing orthodoxy for centuries in every field of human knowledge, co-opted by Christian and medieval theologians such as Thomas Aquinas (Aquino 1225–Fossanova 1274), with the scientific revolution of the seventeenth century, Aristotle became, at least in the travesty of scholastic philosophy, the idol to be destroyed. Since then, his name has been a synonym in the modern scientific community for error, a process of reasoning based on the authority of books (*ipse dixit*), without any reference to the direct observation of the phenomenon studied, and suffocation of the truth and research by a metaphysics made up of finalistic and linguistic rules, a backward-looking, irrational environment that hinders the progress of knowledge. All these judgements, however, are, on the contrary, ill-founded anachronistic commonplaces. This famous teacher of Alexander the Great displays, together with the usual presumed demerits, also some interesting characteristics for the more attentive historian, though we shall not deal with them in detail. We will recall only his naturalistic writings, which made him worthy of being considered by Charles Darwin (1809–1882) as one of the precursors of evolutionary theory,[82] and his logic based on syllogisms. He would deserve a little attention here, above all because his ideas of the world, of mathematical sciences and of the sciences of life constantly made reference to a continuous substrate: nature does not take jumps, it abhors a void, and so on.

Aristotle criticised indivisibles, sustaining, on the contrary, an infinite divisibility, and tried to confute the paradoxes of Zeno the Eleatic, not just by using common sense. The paradoxes were expressed in the following terms: "Zeno posed four problems about movement, which are difficult to solve. The first concerns the non-existence of movement, because before a body in motion reaches the end of its course, it must reach the half-way point . . . The second, called 'the Achilles', says that the faster runner will never overtake the slower one, because the one who is behind first has to reach the point from which the one who is ahead had started, and thus the slower runner is always ahead . . . The third is . . . that the arrow in flight is immobile. This is the result of the hypothesis that time is composed of instants: without this premise, it is impossible to reach this conclusion." Then Aristotle confuted them. "This is the reason why Zeno's paradox is incorrect: he supposes that nothing can go beyond infinite things, or touch them one by one on a finite time. Distance and time, and all that is continuous, are called 'infinite' in two senses: either as regards division, or as regards [the distance between] the extremes. It is not possible for anything to come into contact in a finite time with objects that are infinite in extension. However, this is possible if they are infinite in subdivision. In this sense, indeed, time itself is infinite."[83] Aristotle also states that Pythagorean mathematicians [of his time] "do not need infinity, nor do they make use of it."

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[82]Tonietti 1991.

[83]Sambursky 1959, pp. 182–185.

The Pythagoreans fell into the trap of the paradoxes because they imagined space as made up of points, and time as made up of instants. Aristotle solved the paradoxes with the idea of the continuous which could be divided ad infinitum. With their discrete numbers, the Pythagoreans took phenomena to pieces, but then they couldn't put them back together again. Aristotle presents them to us as they appear to our immediate sensibility, maintaining continuity as their essential characteristic.

Nowadays, modern physics deals in its first few chapters with the movement of bodies in an ideal empty space (which becomes the artificial space of laboratories). The physics of Aristotle, on the contrary, dealt with a nature that is in continuous transformation and movement, observed directly and maintained where it is, that is to say, on earth. In the present-day scientific community, only a few heretical members of a minority have dared to sustain positions referable to Aristotle.[84] However, even though for opposite reasons, neither the ancient popularity of Aristotle, nor his current discredit could prevent us from recognizing as valid his contributions to the mathematical sciences: a supporter of continuous models as opposed to the discrete ones of the followers of Democritus and Pythagoras.

Aristotle found contradictions in the Pythagorean reduction of the world to whole numbers: "If everything is to be distributed among numbers, then it must follow that many things correspond to the same number, and that the same number must belong to one thing and to another ... Therefore, if the same number belonged to certain things, these would be the same as one another, because they would have the same numerical form; for example, the moon and the sun would be the same thing."[85] Here Aristotle manifested the conviction, not only that the essence of things could not be limited to numbers, but also that the world was more numerous than the whole numbers (because it is continuous), thus making it necessary to assign various things to the same number.

As he was connected with Aristotle, and because he did not make any use of numerical ratios, Aristoxenus became the regular target in treatises on music theory. He remained in the history of music, but he was removed from standard books on the history of sciences.[86] As regards these questions, orthodoxy was to be created around the Pythagorean conceptions, and was long maintained. In the next section, we shall see the most famous and lasting variant, so long-lasting that it accompanies us till the nineteenth century.

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[84] Boyer 1990, pp. 116–117. Thom 1980; Thom 2005; Tonietti 2002a.

[85] Aristotle 1982 [Metaphysics] N5, 1093a, 1.

[86] Some followers of Aristoxenus have been listed and studied in Zanoncelli 1990. Aristoxenus remains one of the main sources regarding the Pythagorean sects for many scholars, who, however, curiously seem to avoid accurately the musical writings that are contrary to the Pythagorean scale. von Fritz 1940. *Pitagorici* 1964.

2.6 Claudius Ptolemaeus

In looking at Claudius Ptolemaeus (Ptolemy) (Egypt, between the first and second centuries), we shall not start from his best-known book, but from another one, the *APMONIKA*, which would deserve to enjoy the same prestige in the history of science.

Here, from the very start, he opposed "$\alpha'\kappa o\eta'$, Auditus" [hearing] to the "$\lambda\acute{o}\gamma o\varsigma$, Ratio" [reason], criticising the former as only approximate. "... Sensuum proprium est, id quidem invenire posse quod est vero-propinquum; quod autem accuratum est, aliunde accipere: Rationis autem, aliunde accipere quod est vero-propinquum; & quod accuratum est adinvenire. [...] Jure sequitur, Perceptiones sensibiles, a rationalibus, definiendas esse & terminandas: Debere nimirum priores illas (...) istis (...) suppeditare sonituum Differentias; minus quidem accurate sumptas (...) ab istis autem (...) eo perducendas ut accuratae demum evadant & indubitatae." ["... it is undoubtedly typical of the senses to be able to find what is close to the truth; what is, instead, precise is obtained elsewhere: on the contrary, it is typical of reason to obtain elsewhere what is close to the truth, and to find what is precise. [...] It rightly follows that the perceptions of the senses are established and measured by the rational ones; it is no surprise that the former, rather than the latter, should supply the differences in sounds, but as they are undoubtedly taken less accurately (...), they are to be led back there by these [the rational ones] so that may become sure and undoubted."][87]

Ptolemy trusted "Ratio" because it is "... simple ... without any admixture, perfect, well ordered, ... it always remains equal to itself". Instead, "sensus" depends on "... materia ... mista, & fluxui obnoxia" ["mixed material ... subject to change"], and therefore unstable, which does remain equal, and needs that "Reformatione" [improvement] which is given by reason. Thus the ear, which is imperfect, is not sufficient by itself to judge differences in sounds. Just like the case of dividing a straight line accurately into many parts, a rational criterion is needed for sounds, too. The means used to do this was called the "$\kappa\alpha\nu\omega'\nu$ ‘$\alpha\rho\mu o\nu\iota\kappa\acute{o}\varsigma$, Kanon Harmonicus" [harmonic rule], which was to direct the senses towards the truth. Astrologers were to do the same, maintaining a balance between their more unrefined observations of the stars and reason. "In omnibus enim rebus, contemplantis & scientia utentis munus est, ostendere, Naturae opera secundum Rationem quandam causamque bene ordinatam esse condita, nihilque temere aut fortuito ab ipsa factum esse; & maxime quidem, in apparatibus hujusmodi longe pulcherrimis, quales sunt sensuum horum (Rationis maxime participum) Visus atque Auditus." ["For in all things, it is the duty of the one who contemplates and who

[87]Ptolemy 1682, pp. 1–3. We follow the edition of John Wallis, extracted from 11 Greek manuscripts compared together, with a parallel Latin translation: *Armonicorum libri tres [Three books on harmony]*. The famous Oxford professor so judged the Venetian edition of 1562 printed by Gogavino: "... versio ... obscura fuerit & perplexa ... a vero saepius aberraverit." ["... the version is obscure and confused ... it departs from the truth somewhat often."]

makes use of theory, to present the works of nature as things that have been created by reason, with a certain orderly cause, and nothing is done by nature blindly or by chance; this undoubtedly [happens] above all in the organs that are of a far nobler kind among these senses (which participate in reason at the highest level) sight and hearing."][88]

But the Pythagoreans speculated above all, and the followers of Aristoxenus were only interested in manual exercises and in following the senses. "... errasse vero utrique." ["... both the ones and the others [appear] ... to have erred."] Thus the Pythagoreans adapted "$\lambda\acute{o}\gamma o\upsilon\varsigma$, Proportiones" [proportions] which often did not correspond to the phenomena. Whereas the Aristoxenians put great insistence on what they perceived with their senses. "... obiter quasi Ratione abusi sunt." ["... as if they made use of reason [only] on special occasions."] And for Ptolemy, they succeeded in going both against the nature of reason, and against what was discovered by experience. "... quia Numeros (rationum imagines) non sonituum Differentiis applicant, sed eorum Intervallis. ... quia eos illis adjiciunt Divisionibus quae sensuum testimoniis minime conveniunt." ["... because they use their numbers (representations of ratios) not for the differences of sounds, but for their intervals. ... because they place them in those divisions which show very little agreement with the testimony of the senses."][89]

As regards the acuteness and deepness of sounds, Ptolemy described their origin in the quantity of resonant substance. "Adeo ut Sonitus Distantiis (...) contraria ratione respondeant." ["Such that the sounds correspond to the lengths in an inverse ratio."] Having made a distinction between continuous and discrete sounds, the former, represented by the lowing of cattle and the howling of wolves, were dismissed as non-harmonic: they would not be liable to being "... nec definitione nec proportione comprehendi possint: (contra quam scientiarum proprium est.)" ["... to being understood, either by definitions, or by ratios (contrary to what is typical of sciences)".] Among the latter, instead, which he called "$\Phi\theta\acute{o}\gamma\gamma o\iota$, Sonos" [tones], it was possible to fix the ratios of the relationships. Then, the combination of these latter ratios gave birth to the "$\varepsilon'\mu\mu\varepsilon\lambda\eta'\varsigma$, Concinni" [harmonious] and lastly the "$\Sigma\upsilon\mu\varphi\omega\nu\iota'\alpha\varsigma$, Consonantias" [consonances]: $\Delta\iota'\alpha$‘ $\tau\varepsilon'\sigma\sigma\alpha'\rho\omega\nu$, Dia-tessaron" [fourth], $\Delta\iota'\alpha$‘ $\pi\varepsilon'\nu\tau\varepsilon$, Dia-pente" [fifth] and $\Delta\iota'\alpha$‘ $\pi\alpha\sigma\tilde{\omega}\nu$, Dia-pason" [octave]. It was called Dia-pason [through all] and not $\delta\iota'$ $o\kappa\tau\omega'$ [through eight] because it contained the idea of all the melodies.[90]

The ear perceived as consonances the diatessaron [fourth], the diapente [fifth], the diapason [octave], the diapason united to the diatessaron, the diapason with the diapente and the double diapason. But the "$\lambda\acute{o}\gamma o\varsigma$, ratiocinatio" [reason] of the Pythagoreans excluded the interval of the octave with the fourth from the list of consonances because it did not correspond to the ratios considered as consonant by them: only the ratios termed "'$\varepsilon\pi\iota\mu o\rho\iota'\omega\nu$, superparticularium" [superparticular,

[88]Ptolemy 1682, pp. 3–8.

[89]Ptolemy 1682, p. 8.

[90]Ptolemy 1682, pp. 13, 16–18 and 213.

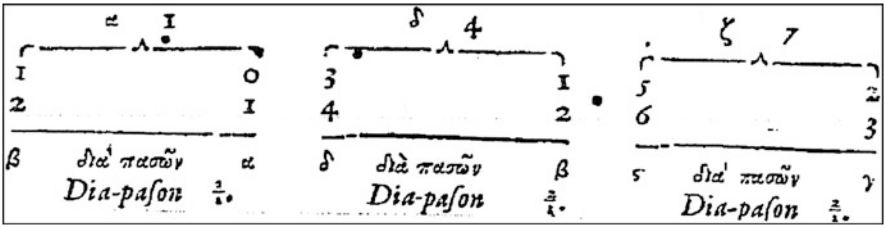

Fig. 2.1 The numbers used by Ptolemy for the ratios of musical intervals (Picture from Ptolemaei 1682, p. 26)

$n + 1$ to n] and “πολλαπλα′σι′ων, multiplicium” [multiple, n to 1]. The ratio judged dissonant was represented by the numbers 8 to 4 and 4 to 3, and consequently by the ratio 8 to 3, which is neither multiple nor superparticular. The octave with the fifth, on the other hand, gave 6 to 3 and 3 to 2, and thus 6 to 2, which is equivalent to 3 to 1, a multiple. The double octave was analogously 4 to 1. Therefore, the Pythagoreans’ hypothesis, that adding the octave to the fourth produced a dissonance, became a mistake for Ptolemy, because this was “definitely a clear case of consonance”. Indeed, in general, adding an octave did not change the characteristics of the interval.[91]

“… Prout etiam evidenti experientia compertum est. Non levem autem illis difficultatem creat.” [“… seeing that this is found even by plain experience. This creates a serious difficulty for them [the Pythagoreans] …”]. Ptolemy found it “absolutely ridiculous” to stop at the first four whole numbers, and ventured to count as far as six, thus arriving at the “senarius” which was to become famous only in the sixteenth century[92] (Fig. 2.1)

Playing with the new numbers, it was not difficult for Ptolemy to recover all the consonances that were pleasurable to his ear. It was thus necessary “… non ipsi [errores] λόγος-Rationis naturae attribuere, sed illis qui eam perperam adhibuerunt.” [“… not to attribute the errors to the nature of reason-ratio-discourse, but to those who erroneously made use of it.”] In the end, therefore, all those consonances were classified as indicated above, without supposing anything “in advance” about multiple or superparticular ratios.[93]

For the λόγος, Ptolemy searched for a “κανόνος, Canonem” [canon, rule], which he found in the “μονοχόρδου, monochordum” [monochord]. The other instruments of sound did not seem to be suitable to avoid the Pythagorean a priori criticisms. He expected “… ad summam accurationem perduci.” [“… to be conducted to a supreme precision.”] Consequently, he avoided listening to the sounds of the “αυ′′λων, tibia” [flute], or those obtained by attaching weights to strings. “Nam, in tibiis & fistulis, praeterquam quod sit admodum difficile omnem

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[91] Ptolemy 1682, pp. 19 and 23–24.

[92] Ptolemy 1682, pp. 25–26. See Sect. 6.6.

[93] Ptolemy 1682, pp. 29–33.

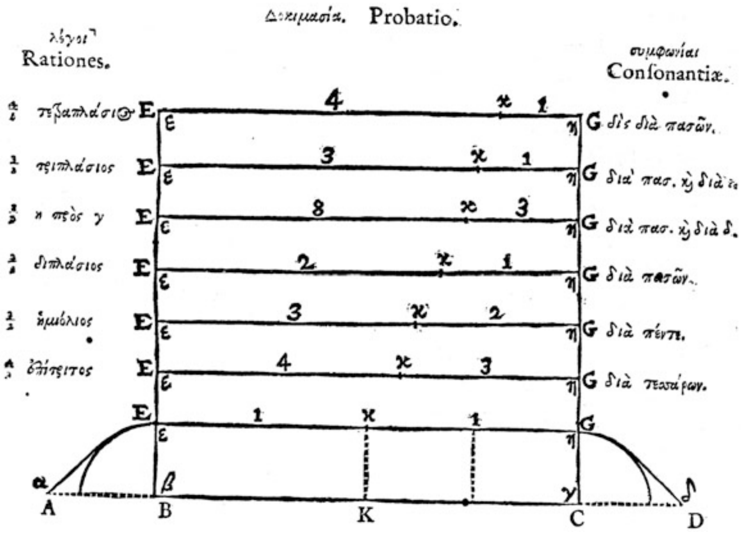

Fig. 2.2 Ptolemy's monochord (Picture from Ptolemaei 1682, p. 38)

irregularitatem inibi cavere: et am termini, ad quos sunt exigenda longitudines, latitudinem quandam admittunt indefinitam: atque (in universum) Instrumentorum inflatilium pleraque, inordinatum aliquid adjunctum habent; & praeter ipsas spiritus injectiones." ["For in flutes and reed-pipes, besides the great difficulty in avoiding every irregularity, the terms, whose lengths we must evaluate, admit a certain indefinite width; and (in general) the great majority of wind instruments have something disorderly, in addition to the input of breath."]

This famous astronomer-astrologer also condemned the experiment with weights, because it was equally imprecise, since it was impossible for "... ponderum rationes, sonitibus a se factis, perfecte accommodentur ..." ["... the ratios of weights with which sounds are produced to be perfectly proportional ..."]. Furthermore, the strings in this case would not remain constant, but would increase their length with the weight. This effect would need to be taken into consideration, besides the ratios between the weights. "Operosum utique omnino est, in his omnibus, materiarum omnem & figurarum diversitatem excludere." ["It is without doubt generally tiring to exclude, in all these things, every diversity of materials and shapes."] Therefore, precise ratios for consonances could only be obtained by considering the exact lengths of the strings. For this reason, he projected the monochord, by means of which he fixed the values of the various intervals under examination (Fig. 2.2).

Having excluded undesirable ratios, which he should have admitted, on the contrary, if he had operated also with weights and reed-pipes, in the end Ptolemy confirmed all the numbers of the Pythagoreans, adding 8:3 as well.[94]

[94]Ptolemy 1682, pp. 33–38; cfr. pp. 156–159.

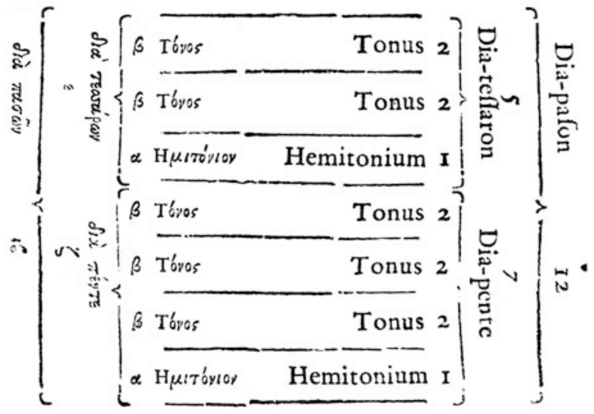

Fig. 2.3 The division of the octave by Aristoxenus into equal semitones, as related by Ptolemy (Picture from Ptolemaei 1682, p. 41)

Then he went on to criticise the Aristoxenians, much more however than the Pythagoreans. The latter should not have been blamed by the former for studying the ratios of consonances, seeing that these were generally acceptable, but only for their way of reasoning. Instead, the Aristoxenians would not accept them, nor would they invent any better ratios, when they expounded their theory of music. And yet, although these musical impressions touch the hearing, the ratios that express the relationships between sounds should be recognized. However, the Aristoxenians did not explain, or study, how sounds stand in a relationship with one another.

> Sed, (. . .) specierum $[\epsilon\iota'\delta\tilde{\omega}\nu]$ solummodo Distantias inter se comparant: Ut videantur saltem aliquid numero & proportione facere. Quod tamen plane contrarium est. Nam primo, non definiunt (. . .) specierum per se quamlibet; qualis sit: (Quomodo nos, interrogantibus, quid est Tonus; dicimus, Differentiam esse duorum Sonorum rationem sesquioctavam continentium). Sed remittunt statim ad aliud quid, quod ad huc indeterminatum est: ut, cum Tonum esse dicunt, Differentiam Dia-tessaron & Dia-pente: (cum tamen Sensus, si velit Tonum aptare, non ante indigeat aut ipso Dia-tessaron, aut alio quovis; sed potis sit, differentiarum istiusmodi quamlibet, per se constituere). [But they compare together only the distances in external aspects, so that they are at least seen to be doing something regarding numbers and ratios. However, this is not really a point in their favour. First of all, they do not define (. . .) the nature of anything that is, in itself, an external aspect. (As we do when we answer anybody who asks us what a tone is, that it is the difference between two sounds whose ratio is a sesquioctave.) But they invariably make reference to something else which is equally indeterminate for the question: as when they say that the tone is the difference between the diatessaron and the diapente (when, however, the sense that desires to prepare the tone does not need, first of all, the diatessaron, or anything else, but is capable of creates by itself any difference of that kind).]

If they were invited to specify what the above difference is, they would say, if anything, that it is two, and that of the diatessaron is five, and that of the diapason is 12 (Fig. 2.3).

As Aristoxenus had not defined the numerical terms between which the differences were to calculated, the latter remained uncertain for Ptolemy. The whole procedure to identify the tone by varying the tension of the strings was judged by him as "… inter absurdissima …" ["… among the most absurd things …"]. He challenged Aristoxenus' way of measuring the diatessaron as composed of two and a half tones, the diapente of three and a half tones and thus the diapason of six. How? Of course, Ptolemy used the ratios calculated by the numerical procedures of the Pythagoreans, starting from the tone, 9:8. The excess of the diatessaron with respect to the ditone thus became for him the minor semitone. "Quippe cum, in duas aequales rationes (*numeris effabiles*) non dividatur, aut sesquioctava ratio, aut superparticularium quaevis alia: rationes vero duae proxime-aequales, sesquioctavam facientes, sint sesquidecimasexta & sesquidecimaseptima: …". ["As they are not divided into two equal ratios (*that can be expressed with numbers*)[95] or into the sesquioctave ratio [9:8], or any other superparticular ratio; whereas two ratios close to parity which form the sesquioctave are the sesquisixteenth [17:16] and the sesquiseventeenth [18:17]: …"][96]

Our renowned Alexandrian mathematician calculated how far the Pythagorean minor semitone, or limma, was lower than a semitone which corresponded to half of a tone. But he did it with whole numbers, without using any roots, probably because he would otherwise have moved music from the discrete side of the *quadrivium* to the continuous side, next to geometry. He obtained such a tiny difference that not even the followers of Aristoxenus, in his opinion, would say that they could hear it with their ears. Therefore, if it could happen that the sense of hearing was likewise mistaken (ignoring the difference), then even greater mistakes would be made in the hotchpotch of many presuppositions to be found in their explanations. The Aristoxenians had demonstrated the tone, 9 to 8, more easily than the ditone, 81 to 64, since the latter was "incompositum, inconcinnum" [without art, not harmonious], while the former was "concinnum" [harmonious]. "Sunt autem sensibus sumptu promptiora quae sunt magis Symmetra." ["After all, those things that are better proportioned can more easily be apprehended by the senses."][97]

The intentions of the Aristoxenians were made even clearer by the way that they treated the diapason [the octave, considered by them to be exactly six tones], "… (praeterquam ab illa Aurium impotentia)" ["… (as well as by the inability of their ears) …"]. And Ptolemy demonstrated, on the contrary, with Pythagorean ratios, that the octave contained less than six tones: Aristoxenus had not used numerical ratios to define the diatonic, chromatic and enharmonic genres, but only

[95]The brackets were added with the italics by Wallis. This enables us to measure the distance between the world of Ptolemy, where it was taken for granted that numbers were only those with a *logos*, rational and expressible, and the sixteenth century, when an equal existence and use would be granted also to non-expressible numbers, the irrationals.

[96](18:17) combined with (17:16) gives (18:16), which is equivalent to (9:8). Ptolemy 1682, pp. 39–48.

[97]Ptolemy 1682, pp. 49–50.

"$\delta\iota\alpha'\sigma\tau\eta'\mu\alpha\sigma\iota$, intervalla" [intervals]. This was the final comment of Ptolemy: "Ipsisque, differentiarum causis, pro non-causis, nihiloque, nudisque extremis, perperam habitis, comparationes suas inanibus & vacuis [intervallis] accomodat. Ob hanc causam, nil pensi habet, ubique fere, bifariam dividere Concinnitates: cum tamen, rationes superparticulares (...) nihil tale patiantur." ["Having wrongly disposed the causes of the differences in favour of non-causes, and by nothing less arranged simple extremes, he adapts his ratios to empty, baseless [intervals]. For this reason, he does not hesitate to divide the harmonic intervals, in practically all cases, into two parts, when, on the contrary, superparticular ratios do not allow anything of the kind."]

Instead, the division of the tetrachord [the fourth] of the Pythagorean Archytas of Tarentum was quoted without severe criticism, quite the opposite. Though he, too, deserved to be corrected in certain things, "... in plerisque autem, eidem adhaeret, ita tamen ut manifeste recedat ab eis quae sensibus directe sunt comperta ...". ["... in the majority, on the contrary, he is close to the same [purpose], with the result that he keeps well away from those things that are discovered directly with the senses ..."].[98]

And yet among all the possible ways of dividing the Greek tetrachord, Ptolemy sought those that were in harmony with the numerical ratios, and with the $\varphi\alpha\iota\nu\acute{o}\mu\epsilon\nu o\nu$ [apparent, phenomenon]. In short, among the infinite ways of choosing three ratios between whole numbers, which together would give 4 to 3, Ptolemy fixed the superparticular ones to be composed with 5 to 4, 6 to 5, 7 to 6, 8 to 7, 9 to 8. He distributed among these the enharmonic, chromatic and diatonic genres, in turn subdivided into "$\mu\alpha\lambda\alpha\kappa\acute{o}\varsigma$, molle" [soft, effeminate, dissolute] and "$\sigma\upsilon'\nu\tau o\nu o\varsigma$, intensum" [tense], with other intermediate cases. In these markedly Pythagorean games, it remains to be understood what role Ptolemy reserved for hearing, and for the phenomena with which he had stated that he wanted to find an agreement.[99]

> Quod autem non modo Rationi congruant praemissae generum divisiones, sed & sensibus sint consentaneae, licebit rursus percipere ex Octachordo canone Diapason continente; sonis, ..., accurate examinatis, tum respectu aequabilitatis chordarum, tum aequalitatis sonorum. [Furthermore, it will again be understood from the octachord canon containing the diapason, that the above divisions of genres are not only in agreement with reason, but are also compatible with the senses, ..., after accurately comparing the sounds, with respect both to the uniformity of the strings and to the identity of the sounds.]

He believed that his procedure would stand the test of all the "... musices peritissimi ..." ["... most expert musicians ..."]. "... quin potius, in hanc circa aptationem syntaxi [$\sigma\upsilon\nu\tau\alpha'\xi\iota\varsigma$] naturam [$\varphi\upsilon'\sigma\iota\varsigma$] admiremur: Quippe cum, secundum hanc, tum ratio fingat quasi & efformet melodiae conservatrices differentias, tum Auditus quam maxime Rationi obsequator; Utpote, per ordinem qui inde est, eo adactus; atque agnoscens, ..., quod sit peculiariter gratum. Quique hujus improbandae partis authores fuerint; neque divisiones secundum rationem aggredi

[98] Ptolemy 1682, pp. 61–62.

[99] Ptolemy 1682, pp. 66–78.

per se potuerint; neque sensu patefactas adinvenire dignati fuerint." ["... rather, we will admire nature for her availability regarding this adaptation: seeing that in conformity with this, both the ratio practically models and adapts the differences to be maintained to the melody, and, as much as possible, the hearing obeys reason; since it is led to do so by the order that is thus created; and it recognizes, ..., what is in particular agreeable. And those who would have sustained that this part is to be rejected will neither be able to arrive at the divisions by themselves using reason, nor will they think it worthy to make them known by the senses."][100]

Our Ptolemy wrote that he had put the various genres to the test, finding all the diatonic ones suitable for the ears. But in his opinion, they would not be gladdened by the freer modes, such as the soft enharmonic or chromatic ones. "Praeterea, quantum ad totius tetrachordi in duas rationes sectionem, desumitur ea, in hoc genere, ab eis rationibus quae ad aequalitatem proxime accedunt, suntque sibi invicem proximae; nimirum sesquisexta [7 to 6] & sesquiseptima [8 to 7], quae quasi bifariam dividunt totum extremorum excessum. Ipsum igitur, propter ante dicta, tum auditui videtur acceptius, tum & nobis suggerit aliud adhuc genus: Festinantibus utique ab ea concinnitate quae secundum aequalitates jam constituta est, & dispicientibus, siqua haberi poterit, ipsius Dia-tessaron grata compositio, ipsum jam prima vice dividendo in tres rationes prope-aequales, cum aequalibus itidem differentiis." ["Furthermore, in this [diatonic] genre, as regards the division of the whole tetrachord into two ratios, it is derived from those ratios that are closer to parity, seeing that they are the closest together too. Without any doubt, these are the sesquisixth [7 to 6] and the sesquiseventh [8 to 7], which divide all the distance between the extremes roughly into two [equal] parts. Thus, on the basis of what has been said above, this genre seems so much the more pleasurable to the hearing, inasmuch as it suggests yet another genre to us: encouraged in particular by that harmoniousness which has already been created on the basis of equalities, and inclined [as we are] to examine what could be considered a pleasurable composition of the diatessaron itself, having already divided it into three almost equal ratios, together with differences that are likewise almost equal."]

Then Ptolemy reviewed various divisions of the fourth and the fifth into intervals that were constrained to be close to parity in their ratios. He thus came to divide the octave among the numbers 18, 20, 22, 24, 27, 30, 33, 36 (Fig. 2.4).

> Sumpta vero aequitonorum, secundum hos numeros sectione, comparebit modus quidem inexpectatior & quasi subrusticus, alias autem satis gratus & magis adhuc auribus accommodus, ut haberi despicatui minime mereatur, tum propter melodiae singulare quid, tum propter bene ordinatam sectionem; tum etiam quia, licet per se canatur, nullam infert sensibus offensionem. [Indeed, having assumed a division of equal tones in accordance with these numbers, a way will appear, which is quite unexpected and somewhat rustic, but otherwise quite pleasant and even more suitable for the ears, such as to deserve not to be at all despised, both because of its particular kind of melody, and its orderly division, and because, even if it is sung, it does not in itself procure any offence to the senses.]

[100]Ptolemy 1682, pp. 78–79.

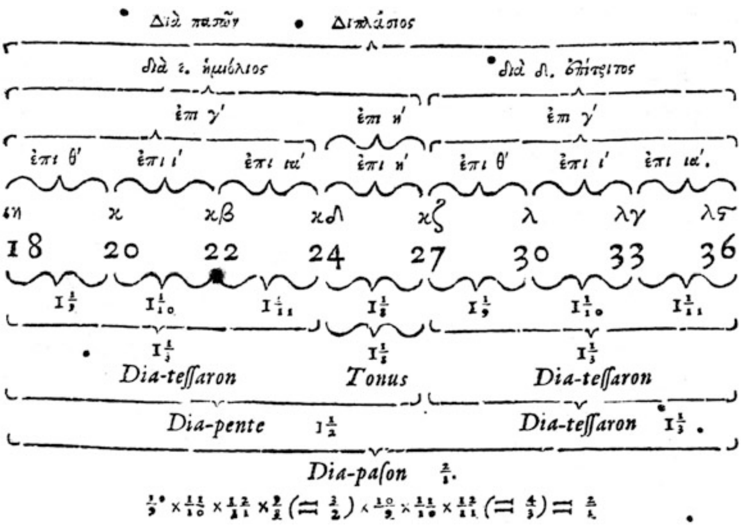

Fig. 2.4 Division of the octave using the numbers of Ptolemy (Picture from Ptolemaei 1682, p. 82)

At that time, it was called: “διαʹτονον o’μαλόν, Diatonum Aequabile” [Uniform diatonic]. It divided even the ratio 5 to 4 into 9 to 8 and 10 to 9, allowing some exchanges among the variety of possible appropriate ratios, without the ears suffering “… ulla notabilis offensio …” [“… any discomfort worthy of note …”].[101]

To that Ptolemy restricted his search for an agreement between numerical ratios and the ears. The event that he accepted the sense of hearing, as one of the criteria to choose between genres, would seem to detach him from the most orthodox Pythagorean tradition. But for him, the ear remained subordinate to the *logos*, and to the ratios of whole numbers; in spite that he repeated several times in his book here and there that, even first of all, he took into consideration the judgement offered by the hearing. The task of the canon should be, for all strings, and using only reason, “… omne aptare quod aptaverint musices peritissimi aurium ope.” [“… to adapt all that the most expert musicians have prepared using their ears.”] He stimulated them with the lyre and the cithara, or with an instrument called a Ελικωʹν [helicon], “… (a Mathematicis constructum ad exhibendas Consonantiarum rationes) …” [“… (constructed by mathematicians [μαθημαʹτικός] in order to demonstrate the ratios of consonances) …”].[102]

We should also notice that in the hands of Ptolemy, that theoretical musical instrument called the monochord, accompanied by the Apollonian helicon, undoubtedly inspired by the Muses, even became a test apparatus. It was not only capable

[101] Ptolemy 1682, pp. 79–85.

[102] Ptolemy 1682, pp. 89ff., 97, 156ff., passim, and 218.

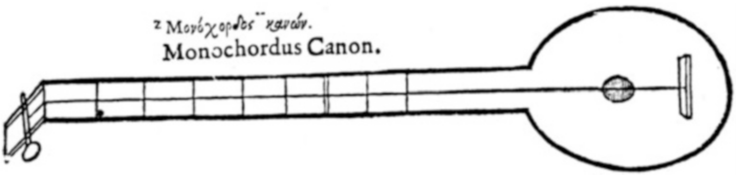

Fig. 2.5 Standard monochord of Ptolemy (Picture from Ptolemaei 1682, p. 159)

of producing sounds, but it was even authorised to verify the agreement between the ratios of numbers and the ears. Was this then an experimental apparatus? However, all the ambiguities in him, always solved in favour of rational numerical ratios, and his attitude towards the musical practice of instruments, are made clear in his subject "De incommodo Monochordi Canonis usu" ["On the deleterious use of the monochord canon"]. Here, the previous theoretical monochord became a real instrument in the hands of musicians (Fig. 2.5).

At the time, it was full of defects and inaccuracies, which were covered up or amplified by the event that it was played together with imprecise, and unreliable (for the Pythagorean canon) wind instruments. Experience also allowed our Alexandrian mathematician to criticise Didymus, the musician.[103]

Subsequently, the procedure followed even led him to find pleasure in divisions of the octave, using whole numbers, into tones that were, as far as possible, equal. And yet he lacked that certain something to take a further step along the same road. However, nobody should ever suspect that one of the most influential and famous natural philosophers, and mathematicians, of the ancient world was not able to use square roots for his calculations: the safest and most precise mathematical way, acceptable to the ears, to divide the octave into equal parts. This self-limitation seems to be particularly interesting, because, on the contrary, he calculated the ratios precisely, also by means of geometrical constructions.[104] Geometry was thus to allow him to give, with equal precision, even the proportional mean between 9 and 8: to divide the tone into two exactly equal parts, as the vituperated Aristoxenians claimed to do on their instruments. But, for Ptolemy, harmony was to remain a discrete science, which could use only discrete means, and the thing to avoid was "... sonituum motus continuus (alienissimam ab harmonia speciem continens, ut quae nullum stabilem & terminatum sonum exhibet) ..." ["... the continuous movement of sounds (which contains an aspect that is remote from harmony, like the one that does not manifest any sound that is stable or well specified) ...".][105] For the equable temperament, Europe and the Western world have to wait until the sixteenth century, but the world is round, and we shall first embark on a voyage to visit other cultures.

[103]Ptolemy 1682, pp. 156–166.

[104]For example, Ptolemy 1682, pp. 97–98ff.

[105]Ptolemy 1682, p. 158.

Harmony, in the words of Ptolemy, showed a "$\delta\upsilon'\nu\alpha\mu\iota\varsigma$, facultatem" [power] of its own, and was connected with other things in the world. As sciences above all of ratios, harmony and astronomy were seen as "… cousins generated by the sisters, sight and hearing, and nourished by arithmetic and geometry." This power was to be typical of movements, especially of those present in the "$o\upsilon\rho\alpha'\nu\iota\alpha$, divinis corporum coelestium" [divine elements of heavenly bodies] and in the "$\psi\upsilon\chi\eta'$, mortalibus humanarum … animarum" [mortal elements of human souls]. But in order to be able to participate in the perfection of mathematical ratios, these new movements should take place in the "$\epsilon\tilde{\iota}'\delta o\varsigma$, forma" [ideal form] and not in the "$\upsilon\lambda\eta$, materia" [matter], since the power of ratios is not observed "… in eis motibus quibus ipsa materia alteratur, …, cum neque qualitas quae secundum eam sit, neque quantitas, (propter ejus inconstantiam), determinari possit: …" ["… in those movements by which the matter itself is changed, …, when neither the quality by virtue of which that happens, nor the quantity can be determined (due to its instability): …"].[106]

On the basis of these general premises, our renowned ancient astronomer prepared a classification of the effects that harmony should have on souls, and of their relationships with the movements of the heavenly spheres. In his sensitivity to coincidences in numbers, he linked the various faculties of the soul to the different consonances. The diapente [fifth], for example, should correspond to the five senses, the diapason [seven notes] to the seven faculties of the intellective soul: "… Imaginationem, … Mentem, … Cogitationem, … Discursum, … Opinionem, … Rationem, … Scientiam …". Morals began to come into the question with the diatessaron, which should influence the covetous soul, while the diapente should affect the rational element. Harmonious sounds reveal virtues, non-harmonious ones vices, and so on. "Animarum Virtus est earum quaedam concinnitas & Vitium inconcinnitas." ["The virtue of souls consists of a certain harmoniousness, but vice is found in lack of harmony."] Consequently, the diatessaron stimulates "… Temperantia, in contemptu voluptatum, Continentia, in sustinendis indigentiis, & Verecundia, in vitandis turpibus." ["temperance, in despising pleasures, continence, in helping the needy, and modesty, in avoiding turpitudes."] The diapente should regard, on the contrary, "… Mansuetudo, … Intrepidus animus, … Fortitudo, … Tolerantia, …" ["… meekness, … bravery of soul, … fortitude, … tolerance, …"], whereas the diapason should be linked with a whole series of seven other virtues: "Acumen, … Ingenium, … Perspicacia, … Judicium, … Sapientia, … Prudentia, … Peritia" ["shrewdness, … intelligence, … perspicacity, … judgement, … wisdom, … prudence, … competence."] All this also should make it possible to obtain a good condition of the body.

If the theoretical domain included three parts, the natural, the mathematical and the divine, and the practical realm three more parts, the ethical, the economic and the political, then there must be three harmonic genres, the enharmonic, the chromatic and the diatonic. Ptolemy coupled the enharmonic with nature and ethics,

[106] Ptolemy 1682, pp. 232, 236 and 238.

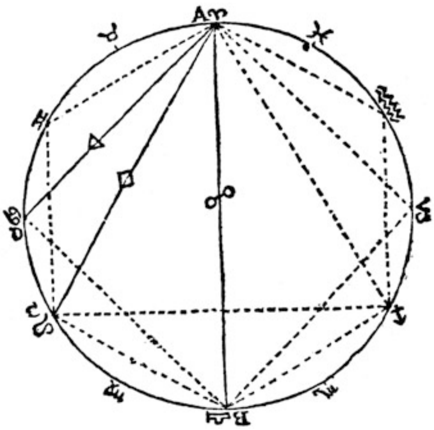

Fig. 2.6 Ptolemy's zodiac (Picture from Ptolemaei 1682, p. 254)

the chromatic with mathematics and economy and the diatonic with theology and politics, leaving space, however, for some overlaps. As a consequence of the conditions of life, in their continual alternation between peace and war, or indigence and abundance, our souls are influenced by the different modes, with passages from deep to acute. "Atque hoc ipsum credo Pythagoram considerasse, cum suaserit ut, primo mane exsuscitati, antequam actionem aliquam auspicarentur, musica uterentur & blando cantu." ["Furthermore I believe that Pythagoras was thinking precisely of this when he gave the advice to make use of music and a sweet song, after waking up early in the morning, before starting any kind of action."][107]

Partly for the influence that it had in Europe until the seventeenth century, the closing passage of book 3 of this *ΑΡΜΟΝΙΚΑ* should be remembered. Here the correlations with the "ζωδι′ων κυ′κλον, Zodiaci circuli" [circle of the zodiac] were based on numbers, and became more precise. "... Coelestium ... corporum hypotheses secundum rationes harmonicas confectas esse." ["... The principles of the heavenly bodies are composed in accordance with the harmonic ratios."] The order of sounds and their tension proceed in a straight line, but their power and their constitution are circular. As the revolutions of heavenly bodies are also circular, the ancient astronomer constructed correspondences between the 12 points of the zodiac and the musical notes, dividing the circle according to the proportions established by the musical harmony that had previously been explained[108] (Fig. 2.6).

Deep sounds are compared with the stars in the position where they rise and set, whereas in their highest position at midday, they are closer to acute sounds. And by so doing, Ptolemy distributed musical genres and modes among other astral features, such as the phases of the moon. He divided the circle into 360 parts in order to calculate conjunctions, oppositions and trines in accordance with their relative

[107]Ptolemy 1682, pp. 239–248.

[108]Ptolemy 1682, pp. 249–258. See Part II, Sect. 8.3.

harmonic ratios. He concluded that the sound of Jupiter with those of the Sun and the Moon formed diatessaron and diapason consonances, respectively, while the sound of Venus with the Moon formed one tone. "Evil" planets, like Saturn and Mars, together with those that have "beneficial" effects, like Jupiter and Venus, form a diatessaron consonance. And so on, with various other combinations among planets, consonances and dissonances, variously justified by numbers, by general principles and by dizzy analogies.[109]

In the *Astrological previsions addressed to Sirius*, also known as *Tetrabiblos, [The Four Books]*, Ptolemy classified the signs of the zodiac not only as male and female, but also based on their reciprocal affinities. And he derived these from their musical ratios, applying to their aspects, (that is to say, to the planets' angular arrangements with respect to one another), the sesquialtera musical ratio, 3:2, and the sesquithird ratio, 4:3. He obtained that the trine (120°) and the sextile (60°) were then $\sigma\upsilon'\mu\varphi\omega\nu o\iota$ [consonant], whereas the quadratures (90°) and oppositions (180°) were $\alpha'\sigma\upsilon'\mu\varphi\omega\nu o\iota$ [dissonant].[110]

We have dwelt in particular on the $APMONIKA$ of our renowned astronomer-astrologer-mathematician from Alexandria, because in general it is wrongly overlooked. On the contrary, other historians have studied, and undoubtedly continue to comment on his most widely used and best known book, the *Syntaxis mathematica [Mathematical Order]*. In Europe and the Near East, however, its title was to be completely changed from Greek to Arabic, *Almagest [The Greatest]*. The peoples on the southern shores of the Mediterranean were about to break into our history, and were to give this transliterated name to the Greek astronomical collection, *megiste* [greatest].[111] Thanks to its mathematical precision and its accuracy in observing more than one thousand stars, the book was to dominate astronomical and astrological discussion until the seventeenth century. Everybody read it and commented on it, but initially, nobody translated it directly from Greek, but rather from Arabic into Latin.[112]

Ptolemy had to carry out many calculations in order to represent the positions of the stars on the vault of the heavens. He obtained them by means of the chords of the circle, which he measured with great precision, proceeding by half a degree at a time in preliminary tables. However, these are not to be considered truly trigonometric, because *sines* and *cosines* only arrived thanks to the Indians, who made their calculations with semi-chords.[113] Of course, he also needed a good value

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[109]Ptolemy 1682, pp. 260–273. Cf. Barker 2000. He showed that "Ptolemy understood very well what conditions must be met if experimental tests are to be fully rigorous, ...". However, concerning "... how far the treatise is faithful to the principles it advertises, ... There are grounds for some scepticism here, ...". Therefore, in an independent way, my analysis does not side in Ptolemy's favour: because, with great probability, the Alexandrian did not test either the attunements of pipes, or Aristoxenus'.

[110]Ptolemy (Tolomeo) 1985, pp. 60–63; translation corrected by me.

[111]See Sect. 5.4.

[112]Boyer 1990, p. 294.

[113]See Sect. 4.3.

for the relationship between the circumference and the diameter (π), which made an improvement on that of Archimedes, 22 to 7, arriving at 377 to 120, equivalent to 3.1416 in decimal figures, which had not yet been introduced. His generally famous and widely discussed ideas include cycles, epicycles, eccentrics and equants, with which, without foregoing the musical harmony of circular movement, our astronomer-astrologer explained with a good degree of precision the complex movements of heavenly bodies, which were far from uniform and regular.

In his monumental *Geography*, he catalogued thousands and thousands of cities, rivers, and countries, situating them on the surface of the earth with their latitude and longitude. But he underestimated the size of the earth, and consequently overestimated the longitudinal size of his world.[114] No exact system had yet been found to calculate the longitude, as this was to appear only in the eighteenth century. Others after him were likewise to overestimate the size of the Mediterranean, and also the Northern part of the earth with respect to the South, through the projections chosen to represent the terrestrial sphere on the plane of geographic maps.[115] As there is more than one way of projecting a sphere on to a plane, every projection maintains certain characteristics of the figures on the sphere to the detriment of others. Thus the choice becomes subjective, and highlights not so much the geometrical ability as the practical interests and the culture of scholars. In general in that period, they revealed that they considered their own countries as the centre of the world. In the following chapters, we shall expose the limits of a similar Eurocentric vision, not only in geography.

> I do not consider it as the goal of historical writing to condense the complexity of historical processes into some kind of digest or synthesis. On the contrary, I see the main purpose of historical studies in the unfolding of the stupendous wealth of phenomena which are connected with any phase of human history and thus to counteract the natural tendency toward oversimplification and philosophical constructions which are the faithful companions of ignorance.
>
> Otto Neugebauer.

2.7 Archimedes and a Few Others

So far, we have ignored famous natural philosophers such as Democritus, Eudoxus, Archimedes, Apollonius, Diophantus, Heron, Theon, Hypatia, Pappus and others, because there is no mention of any theory of music in their extant texts which have luckily been handed down to us. Of course, this should not been turned into a value judgement about them, or about anyone else. For them, readers are simply referred to the many other history books that deal with them exhaustively. Let us recall only Democritus of Abdera (460–370 B.C.), who reasoned about fundamental

114 Boyer 1990, pp. 193–200.

115 Peters 1990.

elements, which cannot be broken down any further. These were thought to have formed all the things in the world, moving in a void: the famous atoms. Drawing his inspiration from the numerical atoms of the Pythagoreans, he sustained that even geometrical figures were composed of indivisible fundamental elements. How he would have coped with the *continuum*, incommensurable magnitudes and movement (remembering the paradoxes of the Eleatics), we are unable to say. His writings have been lost, or were treated too negligently by the rival schools of Plato and Aristotle, who did not take enough care to preserve them.[116]

Whether represented or not by means of music, the extent to which the problem of incommensurable ratios was felt to be important in Greek culture would be illustrated in the works written by one of Plato's disciples, Eudoxus of Cnidos (c. 408–c. 355 B.C.), if any were extant. In any case, he was credited with the invention of a method to compare together even ratios of incommensurable magnitudes. Furthermore, he successfully approximated curved figures, such as circles, by means of polygons with a large number of straight sides, obtaining results regarding their ratios of lengths, areas and volumes. In modern times, when the name of the author had been forgotten, as happens all too often, curiously, in the history of sciences, his procedure was to be given a name: Archimedes' exhaustion method. Increasing the number of sides, the polygon comes closer and closer to a circle, until it becomes one with it. To Eudoxus, lastly, we owe a model to represent the movement of stars, made up of concentric spheres in uniform movement, with the Earth immobile at the centre. This cosmology, substantiated by a perfect crystalline matter, was adopted by Aristotle and was to enjoy great success for thousands of years.[117]

Aristarchus of Samos (third century B.C.), on the contrary, said that the Earth was moving and the Sun immobile. But in antiquity, his model did not enjoy the same popularity. The only one who quoted it, for other reasons, was Archimedes, who, however, criticised it for its somewhat imprecise way of dealing with magnitudes.[118] This example will be sufficient to avoid recurring commonplaces about the ancient scientific world, and prepare us rather to understand those selective contexts which made one theory the orthodoxy promoted by the most famous philosophers, while rival theories were heresies worthy only of being forgotten.

As regards the renowned Archimedes of Siracusa (287–212 B.C.), we may recall his experiments on the equilibrium of liquids, his numerous mechanical inventions and his calculating ability, in a style that was not exactly that of Euclid, regarding curvilinear figures and bodies like spheres and cylinders. He was an expert in dealing with levers and balances, and, unlike others who were more theoretical, he was not averse to turning theory into practice. Exploiting his ability in calculations, this natural philosopher invented a procedure to obtain the length of the circumference, knowing the diameter, but the result was only approximate. He inscribed inside the circle a regular hexagon, whose perimeter was easy to calculate, as the figure

[116]Boyer 1990, pp. 94–96.

[117]Boyer 1990, pp. 105–110.

[118]Archimedes 1974, pp. 447–448.

was made up of six equilateral triangles. He obtained a perimeter whose length was 6, if the diameter of the circle was 2, and a ratio of 6 to 2 ($\pi = 3$), but the circumference, of course, was longer. Then he circumscribed another hexagon around it, but the perimeter was now too long. Then he transformed the hexagon into a regular dodecagon, constructing a triangle in the space that remained between the polygon and the circle.

The perimeter of the new polygon of 12 sides, both inscribed and circumscribed, gave a closer approximation to the circumference. Its side could easily be calculated from the hexagon, using the theorem of Pythagoras. Then the operation could be repeated, obtaining better and better values for the circumference. Archimedes made the calculation automatic, and thus a question of time and patience, using recurring formulas to obtain the new perimeter, by doubling the sides. If P_6 and p_6 indicate the perimeters of the circumscribed and inscribed hexagons, respectively, then P_{12} and p_{12} could be obtained by means of the formulas (in post-Cartesian symbolic notation)

$$P_{12} = \frac{2p_6P_6}{(p_6 + P_6)}, \ \ p_{12} = \sqrt{p_6P_{12}}.$$

That is to say, he turned them into the famous harmonic and geometric means of the Pythagorean musical tradition and so on. Another ratio made important by the theory of music, 3 to 2, returned in the result to which the mathematician from Siracusa would have liked to consign his remembrance and his fame. He had found that the same ratio held between the volume of a cylinder and that of a sphere inscribed in it, as also between the relative areas. Cicero was to relate that he had seen the figures engraved on his tomb. Three to two was also the ratio between the volume of the paraboloid of revolution and that of a cone with the same base and the same height. He found that 4 to 3 was the ratio between the area of the parabola and that of a triangle with the same base and the same height.[119]

Thus, together with Euclid's style of proof, the Pythagorean tradition continued to make its effects felt on Archimedes. And yet it would not be difficult to find in him also impulses and problems that might have separated him from it. How far would he have to go in multiplying the sides of polygons? When would we reach the final circle with certainty? Wasn't this reminiscent of a certain paradox of Zeno from Elea? Today it would be easy for us to answer: go on to infinity. However, this was the very notion that they tried to avoid in the Greek world of the period. In order to indicate it, they would have made use of the word $\alpha'\pi\epsilon\iota'\rho\omega\nu$, which means "boundless, without limits, unfulfilled, without means", while the verb $\alpha'\pi\epsilon\iota'\rho\omega$ means "to give up, to get tired, to succumb, to be forbidden". Our man from Siracusa seems to be reluctant to detach himself from Pythagorean whole numbers or from the geometrical theorems of Euclid. And yet he did so with his procedures to calculate the volume of a sphere, using the system which was later, in

[119]Napolitani 2001, pp. 21, 32–33, 36–37.

another epoch, defined by others as "exhaustion": $\frac{4}{3}\pi r^3$. But he cannot have been completely sure about it.

Instead of daring to take a bold step off limits into the infinite, he rather used the "*reductio ad absurdum*". He demonstrated that the magnitude known to him must be greater than a certain value, and at the same time less than the same value, and therefore it must be equal to it. It appeared to be another typical choice between alternatives seen as incompatible, like the choice between even or odd numbers in the Pythagorean argument about the impossibility of measuring $\sqrt{2}$. Had he invented, or copied (from Eudoxus) arguments that he chose not to theorise, in order not to come into conflict with the environment, and his points of reference at Alexandria? He had looked beyond whole numbers and Euclid, but it would seem that he preferred not to use his *logos* to talk about it. How would he succeed in knowing in advance the result which he was starting to prove? As he has not left us anything written, historians have advanced various conjectures regarding the 'mechanical' heuristics of Archimedes. To these, I now add the theory of music, seeing that those his ratios practically always arrived at 3:2, 4:3, 3:1. Or else, let us consider that as polygons come closer to a parabola, they arrange themselves in a geometrical succession, like the ratios of musical intervals. Lastly, cylinders circumscribed around a paraboloid stand in a relationship to one another like the numbers 1:2:3:4:...

However, he lived in a world that was different from the sectarian mysticism of the Pythagoreans, and from the pure geometrical theories of Euclid. The problems to be solved arrived from a world that was far from being raised to the heavens of Plato. Some of these have remained famous: the hydrostatic force equal to the weight of the liquid moved, levers to launch heavy ships, the equilibrium of paraboloids subject to gravity immersed in liquids as if they could float, devices with the shape of a spiral screw, inclined so as to convey water upwards and distribute it into channels to irrigate fields, estimates of astronomical distances and of differences between metals. And his calculations provided solutions not only for all these various practical problems.

In a text entitled *The Approach*, discovered by chance only at the beginning of the twentieth century, Archimedes recounted that on the contrary, his main mathematical inventions derived from preliminary investigations of a 'mechanical' kind. He busied himself with balances, fulcrums and levers, which acted on the paraboloids, triangles, segments, sections of spheres and cones under examination, now treated as composed of heavy matter. He materialised ratios in the law of the lever, $l_1 : l_2 = p_2 : p_1$. A small weight p_1 at a great distance l_1 from the fulcrum is in equilibrium with a large weight p_2 at a smaller distance l_2. Weights and distances are therefore proportional, like the notes and the lengths of strings in music, *mutatis mutandis*. His was thus a terrestrial world subject to gravity, and within certain limits, it was even in movement. He generated the spiral that bears his name today by moving a point along the circumference while at the same time varying its distance from the centre. In this way, he even discovered tangents.

In writing about the law of the lever, his reasoning seems to be less sensible to vicious circles, and consequently less linear than Euclid, but equally indifferent to logic.[120] "Now I am persuaded that this [mechanical] method is no less useful for the proof of propositions; because some of them, which for me were clear from the beginning on the basis of mechanics, were subsequently proved by geometry, since an inquiry conducted with this method excludes a proof." Mechanics, wrote Archimedes, provided him with the idea of a correct conclusion. "This is why, recognizing by myself that the conclusion is not proved, but with the idea that it is exact, we shall, at the right place, provide a geometrical proof."

Therefore, in spite of all his inclinations and his faith in the world of mechanics, our renowned man from Siracusa continued to confirm his affirmations in the geometrical language of Euclid. The latter was to maintain his role of matchless warranter of the truth, providing the general scheme of argumentation in orderly lines of propositions, until Galieo Galilei and Isaac Newton. Archimedes lived in a world divided in half between the earth of phenomena and the perfect ideas of geometry, between the necessary approximations of the former and the exactness of the latter. He appears to be uncertain of where to take his stance, because he would like to stand on both sides. His Alexandrian interlocutors, like Eratosthenes (c. 276–c. 194 B.C.), the director of the famous library, had remained in the safe wake of Euclid, and expected from him those theorems that he provided them with. And yet our man from Siracusa also wrote that, thinking of the figures of geometry as part of the world of heavy matter, they would find other, new propositions not yet discovered. "Actually, in favour of this [mechanical] method, once it has been expounded, I am sure that propositions that have not yet appeared to me will be found by others, both those who are now alive, and scholars of the future."[121]

For various centuries, Archimedes was to remain an unheeded prophet, while everybody else, for one reason or another, (Pythagoreans, followers of Plato or Aristotle, Christians, Neoplatonics or Muslims) continued to study and to comment, with great respect, above all on Euclid's *Elements*. Then, with the scientific revolution of the seventeenth century, or in some cases even earlier, many scholars started to read the works of Archimedes again, inventing, as he had foretold, procedures, techniques and new mathematical theorems variously connected with astronomy and mechanics. But the text of *The Approach*, containing the prophecy that was being fulfilled, was not extant at that time, and was never to be studied by those who were to draw advantage from it. By a strange quirk of fate, it came to light again only at the beginning of the twentieth century, when the mathematical community had abandoned the heuristic methods of Archimedes, condemned for their lack of rigour, and was constructing a completely different orthodoxy. Going way beyond the Platonising abstraction of Euclid, the formalistic axiomatics of David Hilbert was now following a new criterion of rigour, according to which mathematical arguments were to be expressed by means of pure signs on paper,

[120] Napolitani 2001, pp. 43–44.

[121] Archimedes 1960, II, pp. 478–479 and 484.

without any meaning posited either by mechanics or by figures.[122] How could certain historians of mathematical sciences not be influenced after a similar change in research and teaching?

Left in his context, therefore, Archimedes could never appear to be an abstract academic, only interested in his theorems, even though a Platonic philosopher like Plutarch (first and second century) tried to depict him as such. According to him, Plato had even rebuked Archytas and Eudoxus because they ruined "… the excellence of geometry, abandoning it with its abstract ideal notions, to pass to sensible objects … This is how degenerate mechanics was separated from geometry, and, long despised by philosophy, it became one of the military arts."[123] It is only by believing this Greek philosopher that historians of mathematics might succeed in making Archimedes more similar to those mathematicians of our time, inspired by David Hilbert, than he was different from his Alexandrian interlocutors.

On the contrary, he was so present in the reality of his time that he suffered all its tragic consequences. Involved in the Second Punic War, against Rome and on behalf of Carthage, he defended the besieged Siracusa using catapults, powerful winches and his legendary burning-glasses. After the city had fallen into the hand of the enemy, Archimedes is said to have been killed by a Roman soldier. Should the episode be emblematic of the culpable indifference of Roman culture towards the mathematical sciences, to which it never made any significant contribution? It is, on the contrary, a good example not only of the many other faults of war, but also of the declared interest in the sciences, seen as particularly useful in military activities. Marcellus, the victorious general, had taken pains to give orders that the life of the famous natural philosopher should be spared; but in the heat of the looting and the general bloodshed which was the custom of the valiant Roman soldiers, his orders were not obeyed. Subsequently, Cicero ordered his tomb to be traced and repaired with the emblems of the sphere and the cylinder mentioned above. Today, however, undoubtedly as a result of innumerable other similar joyful events, which those in power take pleasure in offering us, it has again been destroyed.[124]

The most curious work by Archimedes would appear to be the *Stomachion* [the word is said to derive from 'stomach', but it is likely to be the name of a puzzle]. In this operation, the renowned mathematician divided a square into 14 pieces, demonstrating that they were commensurable parts, 1:2, 1:4, 1:6, 1:12, 1:24, 1:48. In this way, following the path opened up by Book 10 of Euclid's *Elements*, he was perhaps trying to recover some of the commensurability lost with the relative diagonals.[125]

The pages of Archimedes were treated worse than those of Euclid. Can we not take the extremely limited diffusion of the translations of William of Moerbeke (thirteenth century) or the failure of a printed edition by Johannes Mueller from

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[122] Tonietti 1982a, 1988, 1990, 1992b; Napolitani 2001.

[123] Authier 1989, p. 107.

[124] Authier 1989.

[125] Archimedes 1960, pp. 467–473.

Königsberg, nicknamed Regiomontanus (1436–1476), as a judgement also on the little interest shown in the work? In actual fact, Euclid was printed for the first time in 1484, and Archimedes, on the contrary, had to wait until the edition published at Basle in 1544.[126] Of his lost, or missing works, we know a few names, and sometimes the results contained in them. One has come down to us because it was translated into Arabic and then into Latin. The comprehensible part of the *Stomachion* derives from an Arabic manuscript. This may reveal the judgements to which the inventions of Archimedes were subjected. They were moving away from the orthodoxy of the period, if this may be represented by the original ancient *quadrivium*. Why has nothing connected with music remained of such a similar volcanic, polyhedric figure? His indifference towards that part of the *quadrivium* to which the Pythagoreans were most attached is a measurement of his distance from them and from other scholars who in various ways took their inspiration from them. However, it is difficult to exclude that a similar work, if it was written, may have been lost. If a text by Archimedes about music were to re-emerge, like *The Method*, from a palimpsest used for the liturgy of Orthodox Christianity, may we expect significant variants to the division of the diapason?

It is true that chance may guide events to unexpected conclusions. As in the case of the town of Pompeii, which was preserved better than all the others because it was destroyed by Vesuvius, so those who desired to cancel the ancient pagan philosopher, covering his text with edifying prayers, in the end obtained the opposite effect of preserving it. We shall also see later that the spread of Greek scientific culture, not only of Archimedes, in other countries to the east, was the result of the ban to which it was subjected in its original cradle, seeing that the new religion had formed an alliance with imperial power. Chance and heterogenesis of ends, as philosophers rather too obscurely and pompously call the art of achieving results which are totally different from those desired, may sometimes even become the source of happiness and surprising discoveries, not only for historians and not only for the history of sciences.[127]

To Apollonius of Perga (Asia Minor, c. 262–c.190 B.C.), who worked at Alexandria in Egypt for one of the kings named Ptolemy (they were descendants of the first general of Alexander the Great), we owe the terms in current use for conical sections: ellipse [lack], hyperbole [throwing beyond] and parabola [comparing by placing beside]. He drew on a terminology already used also in rhetorical discourse with analogous meanings.

Again, some of his works are extant because although they were lost, they were read with interest by Arabic scholars. Among the many works lost and reconstructed thanks to quotations and subsequent commentaries, there was also *Section of a ratio*, which might have disappointed us if it had dealt exclusively (or mainly?) with ratios between straight lines, ignoring music. But in the second book of the famous *Conics*,

[126] Napolitani 2001, pp. 67–77.

[127] Boyer 1990, pp. 143–165; Napolitani 2001.

we at least find the harmonic division as a ratio between segments distributed along the axis of the ellipse.[128]

Here in the *Conics*, he also left the favourite sentence of many mathematicians, when some profane individual asks them what the use is of all these theorems. "They deserve to be accepted for the sake of the proofs themselves, in the same way as we accept many other things in mathematics for this, and no other reason." But doesn't perhaps the answer of this emigrant at Alexandria reveal the problem that is present in a historical context that would have expected much more from its natural philosophers? Did he only dedicate his spare time to his conics? What about the heterogenesis of ends, then? What did he think about Plato's *Republic*? In this way, would he free himself from all moral responsibility? In any case, supposing that they had not already been stimulated, some of these abstruse properties of conics found a rapid justification in the (military? nautical? commercial? territorial expansion?) art of projecting a sphere on to a plane, for the purpose of making geographical maps.[129]

Everything that other Alexandrian scholars maybe disliked, or tried to hide, Heron of Alexandria (first century), on the contrary, confidently displayed. He dealt with practical problems, giving formulas to solve them, and ignoring theorems to prove them. He constructed machines for warfare, musical instruments such as wind organs, and various devices, and he loved to measure every kind of magnitude, without worrying too much about the theoretical constraints set by his more illustrious colleagues. Being this man far from the usual commonplaces about Greek mathematics, some have even tried to deport him, labelling him as Babylonian or pre-Arabic. Some formulas still bear his name, such as a procedure to extract square roots, already known (of course?) to the ancient Babylonians.[130] We are debtors to him for the following definition of mathematics. "Mathematics is a theoretical science of things understood by the mind and by the senses, which fall into its traps. Someone has said shrewdly and rightly of mathematics what Homer says of Eris, the goddess of strife. .... Thus mathematics starts from a point and a line, but then its action extends to the heavens, to earth and to all the beings of the universe."[131]

Another umpteenth inhabitant of the same city was Diophantus of Alexandria (maybe third century). Projecting, as usual, their own idea of the mathematical sciences on to the ancient character, or, even worse, in order to belittle the Arabs, some would already consider him to be an "algebraist". Like others, only half of his works have come down to us, but unlike the majority, he dedicated himself to the theory of numbers using non-geometrical procedures: original results are to be found in his *Arithmetic*. While everybody else discussed the subjects under examination with discourses and words taken from everyday Greek, even if loaded

[128] Cf. Fano & Terracini 1957, pp. 356–360.

[129] Boyer 1990, pp. 166–184.

[130] Boyer 1990, pp. 201–204.

[131] Heron, Heiberg edition, IV, 162.

with particular technical meanings, Diophantus, on the contrary, used "syncopated" words (from the Greek for "to break, to shorten") to indicate the powers of numbers and the number to be sought (the unknown). We may see in these the beginning of a special symbology for calculations in mathematics, separating it from the common expressions of daily life. In this way, he wrote sequences of terms and numbers that were almost the equivalent of modern polynomials. However, his *Arithmetic* appears to be a list of numerical problems for which he was trying to find complete, or rational, solutions. Seeing the rigorously numerical spirit that animated him, this other Alexandrian might seem to be a genuine Pythagorean who was totally unaware of problems with incommensurable magnitudes, and consequently did not need to have recourse to geometry in order deal with them. For this reason, we might also wonder if there may have been, among his lost works, even a numerical theory of musical intervals.[132]

As regards another mathematician and philosopher who used the Greek language, Nicomachus of Gerasa (first century), we may more confidently say that he was an orthodox Pythagorean. In his *Introduction to arithmetic*, we find the complete tradition of this sect: from division into even and odd numbers to ratios between whole numbers used for music. The following generations of Pythagorean musical theoreticians took their inspiration from him.[133]

Something musical re-emerged in the last great Alexandrian mathematician, Pappus (fourth century). He commented on Book 10 of Euclid's *Elements* in a work which would have been lost, as usual, if it had not been of interest for the Arabs, who translated it and preserved it. Here we find the problem of incommensurable ratios, though it is discussed with the idea that magnitudes are rational, or otherwise not rational, only by convention, and not as a result of their intrinsic nature. Euclid had chosen a segment with respect to which he measured the rationality of other segments. And he had also deliberately broadened the notion of "rationality" to that of "potential rationality", when the squares of segments proved to be rational. In this way, the side and the diagonal of the square became "potentially rational", in the ratio 1:2, taking the side as the measurement. Then he had classified the other irrational segments in various categories, which he then treated with additions and subtractions. He gave the name "apotome" to the difference between two magnitudes which were only potentially commensurable. Commenting on this, Pappus associated the apotome with harmony, whereas the other irrational segments were correlated with arithmetic and geometry. Thus he harked back to the *quadrivium* of the Pythagoreans, and for the rest, references were not lacking to Plato. In this way of dealing with irrational magnitudes only by means of geometry, Pappus remained in the wake of Euclid, and to leave this course, it will be necessary to wait for some time, until the arrival of subsequent contributions made by Arabic scholars.[134]

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[132] Boyer 1990, pp. 211–215.

[133] Boyer 1990, pp. 210–211.

[134] Ben Miled 2002, pp. 351–352. See Chap. 5.

The *Collection* of this other natural philosopher from Alexandria is also rich in precious historical details about a world that was fading away, together with interesting new theorems; even though it remains a work of classical geometrical accuracy. In book 3, our musical means were represented in an original manner on the same semicircle. DO is the banal arithmetic mean between AB and BC, and DB a well-known geometrical mean, whereas the representation of the harmonic in DF appears to be an original idea of Pappus.

The geometer generalised the theorem of Pythagoras to include also all kinds of non-right-angled triangles, on sides where he constructed all kinds of parallelograms. He attributed a certain mathematical intuition to bees, seeing that they were capable of literally constructing hexagonal prisms, with which they realised an economy of material: given the same perimeter, the hexagon includes a larger area than polygons with a smaller number of sides. The largest area would be that of the circle. He studied curves created in relationship to distances from a growing number of sides. Taking his cue from this problem, Descartes will arrive at his *Geometry* in the seventeenth century. With Pappus, a "back to front" method of proof called "analysis" became explicit. In this method, we start from the property that is sought, and we derive other consequences from it. If these include the starting premise, then the property is considered as proved, but if properties considered impossible are obtained, then also the property sought is considered impossible.

In the cultural context of Alexandria, many other singular figures were born. Among them, we may mention Theon (fourth century), who wrote commentaries on some of the above-mentioned books, including Euclid, and we owe to him and to this activity of his the existence of the most ancient editions of the *Elements*. His daughter Hypatia (fourth and fifth century) continued her father's work, but in 415 she was lynched by a crowd of Christians, who did not tolerate that she had maintained such a great admiration for those aspects of classical Greek culture which they hated so much. Furthermore, it must be significant that she was one of the very few members of the female sex in our history.[135] This tragic episode brought to light the contrasts between ancient tradition and the new form of Christian religion, which was changing the historical context. Episodes of intolerance and censure towards disapproved cultural aspects were to assume a formal character in the edict of the Christian emperor Justinian, who officially closed the pagan schools of Athens in 529. Also Proclus (410–485), a scholar who studied Plato, has left us a commentary on Euclid, together with historical details about ancient mathematicians, which the new context was cancelling.[136]

In our history, Harmony is not only the daughter of Venus, but also of a father like Mars. War, soldiers or political powers that were born from wars have already appeared several times, and cannot be omitted without compromising an understanding of events.

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[135] Cf. Boyer 1990, though here at p. 209 the Italian translator turned her into a man.

[136] Boyer 1990, pp. 215–225.

The commander and tyrant, Architas, gave the Pythagorean mark which was to continue to the end, passing through Plato's *Republic*, as an essential element to educate young soldiers. After the death of Alexander the Great in 323 B.C., his general Ptolemy seized the kingdom of Egypt and transformed Alexandria into the centre of the cultural and scientific world, making it particularly powerful in many other ways. Also Aristotle died in 322 B.C.

All the greatest natural philosophers that we have discussed were there or thereabouts they would have passed along. Archimedes and Heron arrived at the explicit design of war machines. Apollonius worked directly for Ptolemy Philadelphus as his Treasurer General. Yet, based on the little that we know, not all of them were born there, quite the opposite. But at Alexandria they reached their maturity and worked, becoming captivated by the place. How can we define a capacity like this, which attracted famous figures from the four corners of the Mediterranean? If the term 'scientific policy' seems too anachronistic, what should we think of the resources placed at the disposal of scholars here, the meetings that they expected to benefit from, the circulation of writings contained in the famous library? As king of Egypt, Ptolemy set up for this purpose the *Mouseion* [Casket of the Muses] and collected hundreds of thousands of papyri. Directing the great library was a prestigious task that was carried out by famous scholars.[137]

These scholars, though not always closely linked with Pythagorean ideas, were at least under the influence of Platonic philosophies, and underlined the ideal qualities of their research. Then, as today, scholars claimed their independence, guided only by a love for the truth. This, of course, freed them from many other concerns, including, not to be overlooked, the assumption of responsibility for what they were doing, like all other common mortals. They often did not let their values become evident, and ignored, above all, moral values. We, on the contrary, shall follow the priceless advice of Albert Einstein (1879–1955), and contemplate not only what natural philosophers wrote on the subject, but above all the way that they acted and how they behaved during their lives.

2.8 The Latin Lucretius

Titus Lucretius Carus (c. 98–54 B.C.) is, like Aristoxenus, another figure famous for his absence from current histories of the sciences. His *De rerum natura [On the nature of things]* is generally excluded from them, with the exception that we shall see, because it does not correspond to the recurring models found in other writings on sciences. Lucretius composed verses in Latin instead of listing propositions in Greek. He described natural phenomena visible to everybody instead of proving geometrical theorems that could only be imagined. He did not refer back to the Pythagoreans, or to Plato, or to Euclid, but to Epicurus (fourth century B.C.).

[137]Napolitani 2001, p. 9.

In his poem, we do not find any figures, or numbers, or ratios, but "primordia" [primordials, fundamentals] and "inane" [void].

> Corpora sunt porro partim primordia rerum
> partim concilio quare constant principiorum.
> [Bodies are indeed partly primordia of things
> partly they are unions composed of fundamentals.]

These "primordia" are often translated by the "atoms" of Democritus and Epicurus.[138] "... nequeunt oculis rerum primordia cerni." ["... the primordia of things cannot be seen with the eyes."][139]

This natural philosopher and Latin poet allowed himself to be guided by his common sense and above all by his senses.

> Corpus enim per se communis dedicat esse
> sensus; cui nisi prima fides fundata valebit,
> haud erit occultis de rebus quo referentes
> confirmare animi quicquam ratione queamus.
> [For the event that the material body exists by itself is shown
> by common sense; a basic trust in this will act as a foundation, otherwise there will be no way to speak about hidden things
> in order to confirm something reasonable to the mind.][140]

> ... Quid nobis certius ipsis
> sensibus esse potest, qui vera ac falsa notemus?
> [... What can there be more sure for us
> than our very senses by which we distinguish true and false things?.][141]

These Latin verses reveal extraordinary intuitions, which only entered into the thinking of modern physics centuries later.

> Tempus item per se non est, sed rebus ab ipsis
> consequitur sensus, transactum quid sit in aevo,
> tum quae res instet, quid porro deinde sequatur.
> Nec per se quemquam tempus sentire fatendumst
> semotum ab rerum motu placidaque quiete.
> [Time in itself does not exist, but from things themselves
> derives its meaning, what has been accomplished in time,
> what thing still persists, and what will follow after.
> It must be admitted that nobody feels time by itself,
> separated from the movement of things and from peaceful repose.][142]

What could seem to return to an Aristotelian time, as a measurement of movement that is, was to come back again in the idea of Albert Einstein: that time depends on the matter distributed in the universe. And isn't his attempt to prove the existence of atoms (not just a mathematical make-believe ad hoc) with the Brownian

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[138] Lucretius I, 483–484; 1969, p. 32. The translations are mine, and Ron Packham's.

[139] Lucretius I, 268; 1969, p. 48.

[140] Lucretius I, 422–425; 1969, p. 28.

[141] Lucretius I, 699–700; 1969, p. 44.

[142] Lucretius I, 459–463; 1969, p. 30.

movement of pollen reminiscent of these verses of the Latin natural philosopher? Clearly, "primordia" cannot be seen so clearly. And yet, we see

> . . . corpora quae in solis radiis turbare videntur,
> quod tales turbae motus quoque materiai
> significant clandestinos caecosque subesse.
> Multa videbis enim plagis ibi percita caecis
> commutare viam retroque repulsa reverti
> [. . .]
> scilicet hic a principiis est omnibus error.
> [. . . specks that can be seen in the sunrays, moving confusedly,
> where this confusion indicates that there are
> also hidden, invisible movements of matter behind it.
> Here you will see many things, driven by invisible collisions,
> change their direction and turn back, repelled.
> [. . .]
> in other words, this movement derives from all the fundamentals. [the primordia][143]

Reading the following verses, what else could come to our mind, other than Galileo Galilei and falling bodies?

> . . . omnia quapropter debent per inane quietum
> aeque ponderibus non aequis concita ferri.
> [. . . all things, therefore, albeit unequal in weight, must be
> borne through the still void at equal speed.][144]

Lucretius spoke enthusiastically of a rich variety of phenomena displayed on the stage of an infinite world.

> Tantum elementa queunt permutato ordine solo.
> At rerum quae sunt primordia, plura adhibere
> possunt unde queant variae res quaeque creari.
> [This is what elements [common letters in words or verses] can do simply by changing their order.
> But those that are the primordia of things can unite many things
> so that all the other various things can be created.][145]

> . . . usque adeo, quem quisque locum possedit, in omnis
> tantundem partis infinitum omne relinquit.
> [. . . to the point that, whatever place anyone occupies,
> he still leaves all the infinite, equally large in every direction.][146]

The infinite void space, where the poet made his "primordia" move, was boundless.

> . . . omne quidem vero nil est quod finiat extra.
> [. . . in truth, indeed, nothing exists that limits everything from the outside.][147]

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[143]Lucretius II, 126–132; 1969, p. 78.

[144]Lucretius II, 238–239; 1969, p. 84.

[145]Lucretius I, 827–829; 1969, p. 52.

[146]Lucretius I, 966–967; 1969, pp. 60–62.

[147]Lucretius I, 987; 1969, p. 62.

Consequently,

> . . . nam medium nil esse potest, quando omnia constant
> infinita. . . .
> [. . . nothing can stand at the centre when everything
> is infinite. . . .].[148]

Forcing things just a little, isn't this a description of a spherical surface without any border and without any centre?

With material primordia in the infinite timeless void, Lucretius formed the world of phenomena without creation.

> . . . de nilo quoniam fieri nil posse videmus.
> [. . . for we see that nothing can be born from nothing.][149]

Our Latin natural philosopher shows too much trust in the senses, and too great an admiration for

> Aeneadum genetrix, hominum divumque voluptas,
> alma Venus, ...
> [Mother of the Romans, delight of men and gods,
> life-giving Venus, . . .].[150]

> . . . tactus enim, tactus, pro divum numina sancta,
> corporis est sensus, vel cum res extera sese
> insinuat, vel cum laedit quae in corpore natast
> aut iuvat egrediens genitalis per Veneris res,
> [. . . for touch, indeed, touch, by the sacred gods,
> is the sense of the body, both when something external
> penetrates, and when that which is born in the body wounds
> or delights, passing by the route of procreating Venus,].[151]

> Nec tamen hic oculos falli concedimus hilum.
> [. . .]
> hoc animi demum ratio discernere debet,
> nec possunt oculi naturam noscere rerum.
> Proinde animi vitium hoc oculis adfingere noli.
> [However we do not agree that the eyes be at all deceived.
> [. . .]
> after all, it is the reasoning of the soul that must discern,
> and eyes cannot know the nature of reality.
> Therefore, do not attribute to the eyes this fault of the mind.][152]

Among the examples given of illusions due to the mind, Lucretius included ships, suns, moons, stars, horses, columns, clouds that are sometimes still, and sometimes

[148] Lucretius I, 1070–1071; 1969, p. 68.

[149] Lucretius II, 287; 1969, p. 88.

[150] Lucretius I, 1–2; 1969, p. 3.

[151] Lucretius II, 434–437; 1969, p. 96.

[152] Lucretius IV, 379–386; 1969, p. 232.

in movement, which today might seem to be considerations about the principle of relativity and perspective.

> Nam nil aegrius est quam res secernere apertas
> ab dubiis, animus quas ab se protinus addit.
> [. . .]
> . . ., cum in rebus veri nil viderit ante,
> [. . .]
> Invenies primis ab sensibus esse creatam
> notitiem veri neque sensus posse refelli.
> [For there is nothing more onerous than distinguishing clear things
> from doubts, those things that the mind always adds by itself.
> [. . .]
> . . . when nothing true has previously been seen in things,
> [. . .]
> You will find that it is from the senses that
> the knowledge of truth is first created, and the senses cannot be disproved.][153]

Besides taste, smell and sight, the poet first spoke of sounds and hearing.

> Asperitas autem vocis fit ab asperitate
> principiorum et item levor levore creatur.
> Nec simili penetrant auris primordia forma,
> [Furthermore, the harshness of sound derives from the roughness
> of the primordia, and likewise soft sounds are created from smoothness;
> nor do the primordia enter into the ear with the same form,].[154]

> Praeterea partis in cunctas dividitur vox,
> ex aliis aliae quoniam gignuntur, ubi una
> dissiluit semel in multas exorta, . . .
> [. . .]
> At simulacra viis derectis omnia tendunt
> ut sunt missa semel . . .
> [Furthermore, sound is shared out everywhere,
> because other sounds are generated from one another, when
> a voice, once emitted, is divided into many, . . .
> [. . .]
> Images, on the contrary, all proceed in straight lines
> once they have been projected.][155]

Thus Lucretius could not dwell in the Pythagorean-Platonic tradition. However, music supplied him with ideas to narrate the variety of the world.

> . . . ne tu forte putes serrae stridentis acerbum
> horrorem constare elementis levibus aeque
> ac musaea mele, per chordas organici quae
> mobilibus digitis expergefacta figurant;
> [. . . lest you may believe that the rough vibration of the rasping saw
> is composed of smooth elements in the same way as

[153]Lucretius IV, 467–479; 1969, p. 236.

[154]Lucretius IV, 542–544; 1969, p. 240.

[155]Lucretius IV, 604–605 and 609; 1969, p. 244.

the musical melodies, which musicians create on their strings
modulating them with their agile fingers.][156]

However, the different forms of primordia could not be infinite for Lucretius. Otherwise his world would become too unstable.

. . . cycnea mele Phoebeaque daedala chordis
carmina consimili ratione oppressa silerent.
namque aliis aliud praestantius exoreretur.
[. . . the melodies of swans and the artistic songs of Apollo on the
strings would become silent, suffocated by perfectly similar rules.
For another song, more excellent than the others, would be created.][157]

Was our poet afraid that a variety without limits in music might lead to an excessive, paralysing uncertainty in the choice of melodies?

The celebrations for Mother-Earth were accompanied by the sound of drums, cymbals, horns,

. . . et Phrygio stimulat numero cava tibia mentis, . . .
[. . . and the hollow flute excites their minds with the Phrygian rhythm, . . .].[158]

We do not find any tendency in the book to reduce sounds to numbers by means of primordia. On the contrary, it was excluded that they could possess sensible properties, like smell or taste; they were also "... sonitu sterila ..." ["... devoid of sound ..."].[159] And [Pythagorean] "harmony" was rejected as an influence on the soul, necessary for "feeling", because for Lucretius, the spirit, mind and soul formed "unam naturam" ["a single nature"] with the parts of the body.[160] Thus, for him, music was not born from strings, but from flutes and shepherd's pipes.

Et zephyri, cava per calamorum, sibila primum
agrestis docuere cavas inflare cicutas.
[And the whistling of the wind through the empty reeds first
taught peasants to blow into hollow hemlock reed-pipes.][161]

The result was melodies to excite bodies in Bacchic dances. These Muses did not come down from Apollo's Helicon, but lived in the countryside; they did not bring the music of the spheres, but cultivated that of Mother Earth.

Tum caput atque umeros plexis redimire coronis
floribus et foliis lascivia laeta monebat,
atque extra numerum procedere membra moventis
duriter et duro terram pede pellere matrem;
unde oriebantur risus dulcesque cachinni,
omnia quod nova tum magis haec et mira vigebant.

[156] Lucretius II, 410–413; 1969, p. 94.
[157] Lucretius II, 505–507; 1969, p. 100.
[158] Lucretius II, 618–620; 1969, p. 106.
[159] Lucretius II, 845; 1969, p. 120.
[160] Lucretius III, 117–160; 1969, pp. 150–152.
[161] Lucretius V, 1382–1383; 1969, p. 366.

Et vigilantibus hinc aderant solacia somno,
ducere multimodis voces et flectere cantus
et supera calamos unco percurrere labro;
unde etiam vigiles nunc haec accepta tuentur
et numerum servare genus didicere, neque hilo
maiorem interea capiunt dulcedini' fructum
quam silvestre genus capiebat terrigenarum.
[Then joyful lasciviousness prompted them to adorn their heads
and shoulders with crowns of intertwined flowers and leaves,
and to advance shaking their members out of time
clumsily, and to stamp on mother earth with vigorous feet;
this gave rise to laughter and sweet peals of mirth,
these were all things that were new then and surprising.
And the sleepless found comfort for their rest
producing sounds in various ways and modulating tunes
and running their puckered lips over the fifes.
Thus also in our times watchmen stand guard over these traditions
and have learnt to observe the genre of melodies,
nor they take a fruit a whit sweeter
than that the country race of earth-dwellers used to pick up.][162]

The music that our philosopher-cum-poet enjoyed was clearly the opposite of the kind that Plato considered suitable for young soldiers.

Lucretius presented us with a world that was continually changing, and described it through the transformations that he observed in the rain-soaked earth, inhabited by herbs and plants, where animals, herds and human beings roamed. Here, life became food, and food, life.

. . . praeterea cunctas itidem res vertere sese.
[. . . in the same way, then, all things are transformed one into another.][163]

Iamne vides igitur magni primordia rerum
referre in quali sint ordine quaeque locata
et commixta quibus dent motus accipiantque?
[Can you not see, therefore, it is of great importance in what order
the primordia of things stand and how they are distributed
and mixed together, in order to produce and undergo changes?][164]

In this world, made up of material primordia continually jumbled up together in the void,

. . . scire licet gigni posse ex non sensibu' sensus.
[. . . it is possible to understand how senses can be born from non-senses.][165]

Our Latin natural philosopher avoided creation and a creator. His divinities appear to be poetical metaphors for sensible phenomena. Religious beliefs were presented by him as sources of suffering and unhappiness: the sacrifice of Iphigenia

[162]Lucretius V, 1399–1411; 1969, p. 368.

[163]Lucretius II, 874; 1969, p. 120.

[164]Lucretius II, 883–885; 1969, p. 122.

[165]Lucretius II, 930; 1969, p. 124.

for her father. A man could as well speak of Neptune, Ceres, Bacchus and the other gods,

> . . ., dum vera re tamen ipse
> religione animum turpi contingere parcat.
> [. . ., provided that in reality, he himself, on the contrary,
> bewares of contaminating his mind with foul religion.][166]

He terminated his celebration of his master, Epicurus, with these words:

> Quare religio pedibus subiecta vicissim
> obteritur, nos exaequat victoria coelo.
> [Consequently, religion was trampled down underfoot,
> while the victory raises us to the level of the heavens.][167]

"... desiperest ..." ["... it is folly ..."] to believe in the gods.[168] Those who appealed to them did not understand "... coeli rationes ordine certo ..." ["... the reasons in the fixed order of the heavens ..."]. Then ensued a mankind that was unhappy because: "Nec pietas ..." ["It was not devotion ..."] to shed the blood of animals on altars, "... sed mage pacata posse omnia mente tueri." ["... but there would be more piety, if everything could be considered with a serene mind."][169]

> Cum praesertim hic sit natura factus, ut ipsa
> sponte sua forte offensando semina rerum
> multimodis temere incassum frustraque coacta
> tandem coluerunt ea quae coniecta repente
> magnarum rerum fierent exordia semper,
> terrai maris et caeli generisque animantum.
> [Especially using that [serene mind] with nature, just as the seeds of
> things themselves bumping into one another spontaneously by chance,
> blindly driven in various ways fruitlessly and in vain,
> in the end grew those things which, thrown suddenly together,
> always became the beginnings of great things,
> the earth, the sea, the sky, and living creatures.][170]

> Nunc et seminibus si tanta est copia quantam
> enumerare aetas animantum non queat omnis, . . .
> [And now there is such a great abundance in the seeds that
> a whole life of living creatures would not suffice to count them, . . .].[171]

The same force would then have produced in other parts of the terrestrial globe various other generations of living creatures, plants, animals and different kinds of men.

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[166]Lucretius II, 659–660; 1969, p. 108.

[167]Lucretius I, 78–79; 1969, p. 6.

[168]Lucretius V, 146–165; 1969, pp. 292–294.

[169]Lucretius V, 1183–1203; 1969, pp. 354–356; cf. Lucretius VI, 54; 1969, p. 376.

[170]Lucretius II, 1058–1063; 1969, p. 132.

[171]Lucretius II, 1070–1071; 1969, p. 134.

As regards the soul and the spirit, our Latin poet does not appear to have suffered from the dualisms typical of Platonic philosophies, which were in the following centuries to become the orthodoxies of the Jewish and Christian religions in Europe.

> Haec eadem ratio naturam animi atque animai
> corpoream docet esse. . . ..
> [This same reason teaches us that the nature
> of the mind and of the soul is corporeal. . . ..].[172]

For him, those vital and spiritual elements are kinds of fluids contained in the body, as in a vase.

> Haec igitur natura tenetur corpore ab omni
> ipsaque corporis est custos et causa salutis;
> [. . .]
> discidium [ut] nequeat fieri sine peste maloque;
> ut videas, quoniam coniunctast causa salutis,
> coniunctam quoque naturam consistere eorum.
> ["This nature [the spirit] is thus contained in every body
> and is the guard of the body and the cause of its good health;
> [. . .]
> as no separation can take place without illness or ruin;
> you see, as the cause of its well-being is united,
> so also their nature remains linked.][173]

> Quippe etenim corpus, quod vas quasi constitit eius,
> cum cohibere nequit conquassatum ex aliqua re.
> [For the body, clearly, which almost is like the vase [of the mind],
> cannot detain it when it is damaged by something].[174]

Thus for Lucretius, as the soul, mind and spirit are born, so they die together with the body.

Trusting his senses, our Latin natural philosopher observed the phenomena of the atmosphere and the earth, trying serenely to find the reasons. He liked the clouds, and described their formation in the water cycle from the sea to rain. Having freed himself from Jupiter the Rain-bringer, and the Tyrrhenian [Etruscan] haruspices, he made thunder and lightning spring from friction between the clouds. ". . . ut omnia motu percalefacta vides ardescere, . . ." [". . . as you see that all things, heated up by movement, catch fire, . . ."]. He wrote that in this way even lead bullets could be liquefied, if they flew for a long distance.[175]

The colours of the rainbow were produced by the sunrays filtered through the vapours of the clouds.[176] He evoked in detail and in poetic tones the cyclones that formed over the sea, due to the winds that created a whirlwind, calling them by their

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[172]Lucretius III, 163; 1969, p. 152.

[173]Lucretius III, 323–324 and 347–349; 1969, p. 162.

[174]Lucretius III, 440–441; 1969, p. 168. Cf. III, 554–557; 1969, p. 174 and III, 579; 1969, p. 176.

[175]Lucretius VI, 177–179; 1969, p. 382.

[176]Lucretius VI, 524–526; 1969, p. 404.

Greek name of "prester".[177] He tried to explain how the magnet sticks to iron by emitting tiny invisible particles, which are, however, capable of shifting the air and creating voids, subsequently filled by the body attracted. For him, even earthquakes were produced by the swirling of air in the caves of the earth.[178]

The natural world of Lucretius was dominated by the phenomena that move in eddies, and those for which friction is important. He extended his models to the movements of heavenly bodies, without following the greatest Greek philosophers in their classic separation between terrestrial and astral phenomena.

> . . . quanto quaeque magis sint terram sidera propter,
> tanto posse minus cum caeli turbine ferri.
> [. . . the closer each star is to the earth
> the less it will be attracted by the whirling of the sky.][179]

The senses faithfully transmitted a complex variety of that world which our Latin poet was concerned to preserve for us.

> Nam veluti tota natura dissimiles sunt
> inter se genitae res quaeque, ita quamque necessest
> dissimili constare figura principiorum;
> [For, just as in nature all things generated are
> different from one another, so it is necessary that the fundamentals
> [primordia] be different in shape;][180]

Variously moved in the void by their own weight, colliding, mingling, separating and recombining in countless ways, particles of matter gave shape to a world in continuous change.

> Scilicet haec ideo terris ex omnia surgunt,
> multa modis multis multarum semina rerum
> quod permixta gerit tellus discretaque tradit.
> [That is to say, all these things thus arise from the earth,
> many seeds of many things in many ways
> the earth bears in itself, mixed together, and gives forth separate.][181]

The repeated *m's* of the attractive alliteration recall to our mind the *Alma mater Venus* as a general model of this universal generation without creation.

This natural philosopher *sui generis* indicated the particles of matter sometimes as "praecordia" [viscera], sometimes as "semina" [seeds]; here they became "materia" or "materies" [matter], there "principia" [primordia]: he never used the more contemporary term, current for us, of "atomos", as others did, for example Cicero. He observed particles of water in the clothes left on the sea shore, particles of a

[177] Lucretius VI, 423–450; 1969, p. 398.

[178] Lucretius 1969, pp. 404–408 and 426–434.

[179] Lucretius V, 623–624; 1969, p. 320.

[180] Lucretius II, 720–722; 1969, p. 112.

[181] Lucretius VI, 788–790; 1969, p. 420.

plague killed the inhabitants of Athens, particles of matter carried the smell of things to the nostrils.

Among these, the various ways of distributing the empty spaces further increased the variety.

Multa foramina cum variis sint reddita rebus,
dissimili inter se natura praedita debent
esse et habere suam naturam quaeque viasque.
[As the many spaces are assigned to various things,
they must possess a nature that is different one from the other,
and have each one its own nature and its own paths.][182]

This variety was regenerated by a continual changing, which does not encounter any reduction to a limited number of ultimate elements in the book.

. . . omniparens eadem rerum commune sepulcrum,
ergo terra tibi libatur et aucta recrescit.
. . . assidue quoniam fluere omnia constat.
[. . . the earth, mother of all things, and also common grave of things,
thus loses something, and grows again, richer for you.]
[. . . because it is recognized that all things are continually in a state of flux.][183]

Usque adeo omnibus ab rebus res quaeque fluenter
fertur et in cunctas dimittitur undique partis
nec mora nec requies interdatur ulla fluendi,
perpetuo quoniam sentimus, et omnia semper
cernere odorari licet et sentire sonare.
[To such an extent is everything carried forward by everything else
in a continual flow, and is dispatched everywhere in every direction
that there is no respite or rest in the flow,
because we perceive them incessantly, and we can always
see, smell and hear the sounds of everything.][184]

The substances that flow in *De rerum natura* are material; even if they were invisible, Lucretius offered indirect evidence that can be perceived by means of the senses.

. . ., ut aestus
pervolet intactus, nequeunt impellere usquam;
[. . ., when the exhalation
flies past without contact, they cannot push in any direction;][185]

Thus the magnet does not attract either gold or wood, because the force that it emanates passes through their interstices without touching them.

[182]Lucretius VI, 981–983; 1969, p. 430.

[183]Lucretius V, 259–260 and 280; 1969, pp. 298 and 300.

[184]Lucretius VI, 931–935; 1969, p. 428.

[185]Lucretius VI, 1059–1060; 1969, p. 434.

To convince his readers about the nature of things, Lucretius expounded his reasons in a captivating poetic guise: "... musaeo dulci contingere melle, ..." ["... to spread over them the sweet honey of the Muses ..."].[186] He recounted

... quibus ille modis congressus materiai
fundarit terram caelum mare sidera solem
lunaique globum; ...
[... in what ways that meeting of matter
founded the earth, the sky, the sea, the stars, the sun
and the globe of the moon; ...].[187]

The thunder that shakes the sky and earth together was presented as an argument for the unity of the world.

... quod facere haud ulla posset ratione, nisi esset
partibus aeriiis mundi caeloque revincta.
Nam communibus inter se radicibus haerent
ex ineunte aevo coniuncta atque uniter apta.
[... in no case could this happen for any reason if [the earth] were
not connected with the airy regions of the world and the sky.
For they have been attached together with common roots
ever since the beginning of the centuries, joined and linked in unity.][188]

Lucretius saw this same unity between the soul and the body; and at times he did not fail to elaborate analogies between certain phenomena and the human body. The water cycle is similar to the circulation of fluids in the body, the earthquake is like trembling caused by the cold.[189]

The things of the world follow an order, and are repeated, like the seasons, or the movement of the sun and the moon.[190] Our poet-cum-natural philosopher sought the "ratio" [reason] for this, which he sometimes called the "causa". He reserved the term "lex" [law] for the social rules needed to maintain a life in common among people. As regards the magnet, he wondered "... quo foedere fiat naturae ..." ["... by means of what pact of nature it happens ..."].[191] And, with poetic sensitivity, he admitted his doubt about the possibility of always finding the reasons for a phenomenon, because there might be many of them.

Sunt aliquot quoque res quarum unam dicere causam
non satis est, verum pluris, unde una tamen sit;
[There are also various things for which it is not sufficient to indicate
only one cause, but several, of which one, however, is the real one;]

Here he was referring to the floods caused by the Nile.[192]

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[186] Lucretius I, 147; 1969, p. 60.

[187] Lucretius V, 67–69; 1969, p. 288.

[188] Lucretius V, 552–555; 1969, p. 316.

[189] Lucretius VI, 498–503; 1969, p. 402. Lucretius VI, 591–595; 1969, p. 408.

[190] Lucretius 1969, pp. 324, 352, 370.

[191] Lucretius V, 906–907; 1969, p. 426.

[192] Lucretius VI, 703–704; 1969, p. 414.

However, for Lucretius, the particles of matter did not follow a deterministic order fixed by absolute laws.

> Nam certe neque consilio primordia rerum
> ordine se suo quaeque sagaci mente locarunt
> nec quos quaeque darent motus pepigere profecto
> sed quia multa modis multis primordia rerum
> ex infinito iam tempore percita plagis
> ponderibusque suis consuerunt concita ferri
> omnimodisque coire atque omnia pertemptare
> quaecumque inter se possent congressa creare,
> propterea fit uti magnum vulgata per aevum
> omne genus coetus et motus experiundo
> tandem conveniant ea quae convecta repente
> magnarum rerum fiunt exordia saepe,
> terrai maris et caeli generisque animantum.
> [For undoubtedly the primordia of things did not arrange themselves
> in order, each on the basis of its own decision, with shrewd judgement,
> nor did they negotiate, undoubtedly, what movements to cause,
> but as several primordia of things, in many ways
> set in motion already from time immemorial by collisions
> and by their own weight, have been used to be transported grouped
> together, and to join up in every way and to try all possibilities,
> whatever they could create by uniting together;
> thus it comes to pass that when they are diffused for a long time,
> by trying every kind of union and movement
> finally they merge, forming things suddenly brought together,
> which often become the beginnings of great things,
> the earth, the sea, the sky, animals and the human race.][193]

Even though, as he often repeated, everything was just a mixture of matter, which would inevitably fall through the void, sooner or later, as a result of the collisions that continue for a long period of time, every thing observed would find its occasion to be born. Lucretius conceived of the world as unstable, and therefore free: free both from a divine destiny, and from any absolute law which might determine movement once and for all.

> ... corpora cum deorsum rectum per inane feruntur
> ponderibus propriis, incerto tempore ferme
> incertisque locis spatio depellere paulum,
> tantum quod momen mutatum dicere possis.
> [... when the bodies are dragged down in a straight line through
> the void, by their own weight, at some unspecified moment,
> and in places not established in space, they deviate a little,
> enough for you to call it a change in movement.][194]

Otherwise collisions could not take place, and matter would not have the chance to generate and regenerate continually. Therefore,

[193]Lucretius V, 419–431; 1969, pp. 308–310.

[194]Lucretius II, 217–220; 1969, p. 84.

> . . . paulum inclinare necessest
> corpora; nec plus quam minimum, ne fingere motus
> obliquos videamur et id res vera refutet.
> [. . . it is necessary for bodies to incline a little;
> no more than the minimum that is sufficient, so that we will not seem
> to invent oblique movements which are refuted by reality.][195]

Furthermore, for our Latin philosopher, this would give rise to

> . . . exiguum clinamen principiorum
> nec regione loci certa nec tempore certo.
> [. . . a minimal inclination of the primordia
> neither in a sure place nor at a sure time.][196]

> Denique si semper motus conectitur omnis
> et vetere exoritur (semper) novus ordine certo
> nec declinando faciunt primordia motus
> principium quoddam quod fati foedera rumpat,
> ex infinito ne causam causa sequatur,
> libera per terras unde haec animantibus exstat,
> unde est haec, inquam, fatis avulsa voluntas
> per quam progredimur quo ducit quemque voluptas,
> declinamus item motus nec tempore certo
> nec regione loci certa, sed ubi ipsa tulit mens?
> [Lastly, if every movement is always connected
> and the new (always) arises with certainty from the old order,
> and the primordia do not, in their deviations, by movements make
> some kind of principle that breaks the bonds of destiny,
> so that cause does not follow cause everlastingly,
> where does this free will come from for living creatures in the world?
> And from where, I repeat, comes this will separated from destiny
> by which we go wherever our desire leads each of us,
> and also we modify our movements, not at certain moments,
> nor in certain places, but where our mind itself has brought us?][197]

Lucretius has restored to us the freedom of *voluptas* [pleasure] and with this, man returned to the best guide for his life.

> . . . ipsaque deducit dux vitae dia voluptas
> et res per Veneris blanditur saecla propagent,
> ne genus occidat humanum.
> [. . . the very guide of life, divine pleasure, has led
> and attracts by the ways of Venus, and generations are perpetuated
> so that the human race does not die out.][198]

By making them incapable of uniting *per Veneris res*, incapable of feeding, we were released from monsters that any strange hotchpotch perhaps might

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[195] Lucretius II, 243–245; 1969, p. 84.
[196] Lucretius II, 292–293; 1969, p. 88.
[197] Lucretius II, 251–260; 1969, p. 86.
[198] Lucretius II, 172–174; 1969, p. 80.

have been produced.[199] Whereas, instead,"... Venus in silvis iungebat corpora amantum;" ["... In the woods, Venus united the bodies of lovers;"].[200] In this way, a varied, multiform life had been perpetuated on earth, starting from the senseless collisions between primordia. The world had then been populated with every kind of phenomenon and every living creature.

> An, credo, in tenebris vita ac maerore iacebat,
> donec diluxit rerum genitalis origo?
> Natus enim debet quicumque est velle manere
> in vita, donec retinebit blanda voluptas.
> [And, I believe, did life not pine in darkness and sorrow
> until the sexual origin of things shone forth?
> For once born, everyone must desire to remain
> alive, as long as a pleasant desire attracts him.][201]

> Linquitur ut merito maternum nomen adepta
> terra sit, e terra quoniam sunt cuncta creata.
> [It remains that the name of 'maternal' be assigned deservedly
> to the earth, because all things were born from the earth.][202]

And yet, for Lucretius, this earth that was so happy and joyful already seemed to be starting to decline.

> Iamque adeo fracta est aetas effetaque tellus ...
> [Indeed, the age is already broken and the earth worn out ...].[203]

> ... hic natura suis refrenat viribus auctum.
> [... here nature curbs the growth with its own forces.][204]

> Sed quia finem aliquam pariendi debet habere,
> destitit, ut mulier spatio defessa vetusto.
> Mutat enim mundi naturam totius aetas
> ex alioque alius status excipere omnia debet,
> nec manet ulla sui similis res: omnia migrant,
> omnia commutat natura et vertere cogit.
> [...]
> Sic igitur mundi naturam totius aetas
> mutat et ex alio terram status excipit alter,
> quod, tulit ut nequeat, possit quod non tulit ante
> [But, as there must be some end to generating,
> [the earth] desisted, like a woman tired by old age.
> For age changes the nature of the whole world,
> another state must receive everything from yet another,
> and nothing remains similar to itself: everything changes,
> nature transforms all things, and forces them to vary.

[199] Lucretius 1969, p. 334.
[200] Lucretius V, 962; 1969, p. 340.
[201] Lucretius V, 175–178; 1969, p. 294.
[202] Lucretius V, 795–796; 1969, p. 330.
[203] Lucretius II, 1150; 1969, p. 138.
[204] Lucretius II, 1121; 1969, p. 136.

[. . .]
Thus, therefore, age changes the nature of the whole world,
and from one condition another one rules the earth, so that
what it bore should be negated, and it can bear what it had not before.][205]

Yet in the incessant dance of the primordia, in the succession of changing generations, our Latin natural philosopher introduced another dramatic protagonist, for which he also expressed his own moral judgement.

Denique tantopere inter se cum maxima mundi
pugnent membra, pio nequaquam concita bello,
nonne vides aliquam longi certaminis ollis
posse dari finem?
[In the end, when to such a labour the most mighty members of the
world fight among themselves, engaged in a thoroughly unjust war,
do you not see that some close may be put to their long struggle?][206]

Unfortunately, for him and for us, mankind was to enter into another epoch, after an initial period of peace.

At non multa virum sub signis milia ducta
una dies dabat exitio nec turbida ponti
aequora libebant [?] navis ad saxa virosque.
[But [in those times] many thousands of men led under the banners
were not slaughtered in one single day, nor did the surging
waters of the sea sacrifice men and ships on the rocks.][207]

Then the Iron Age arrived, when men laboured increasingly to invent new arms.

Sic alid ex alio peperit discordia tristis,
horribile humanis quod gentibus esset in armis,
inque dies belli terroribus addidit augmen.
[Then deadly discord generated one thing from another
which was to be terrifying for the nations of men in arms
and daily added an increase to the terrors of war.][208]

Tunc igitur pelles, nunc aurum et purpura curis
exercent hominum vitam belloque fatigant;
[. . .]
Ergo hominum genus incassum frustraque laborat
semper et (in) curis consumit inanibus aevum,
nimirum quia non cognovit quae sit habendi
finis et omnino quoad crescat vera voluptas.
Idque minutatim vitam provexit in altum
et belli magnos commovit funditus aestus.
[So then it was skins, now it is gold and purple that
tire the life of men with cares and torment them with war.
[. . .]

[205]Lucretius V, 826–836; 1969, p. 332.
[206]Lucretius V, 380–383; 1969, p. 306.
[207]Lucretius V, 999–1001; 1969, p. 342.
[208]Lucretius V, 1305–1307; 1969, p. 362.

Thus mankind labours uselessly and in vain,
and spends his age always worrying about nothing,
because, clearly, he has not learnt to recognize the purpose
of possessing, and above all how far true pleasure may grow.
And this has gradually dragged his life down to the depths
and has aroused great outbursts of war down inside him.][209]

This Latin poet, who was a witness of several wars, condemned the age of mankind, both his and ours.

. . . nequaquam nobis divinitus esse paratam
naturam rerum: tanta stat praedita culpa.
[. . . in no way has the nature of things been divinely
prepared for us: so great is the blame that it stands accused of.][210]

Then Lucretius imagined the end of the world. Was he perhaps somewhat relieved?

. . . una dies dabit exitio, multosque per annos
sustentata ruet moles et machina mundi.
Nec me animi fallit quam res nova miraque menti
accidat exitium caeli terraeque futurum,
[. . .]
succidere horrisono posse omnia victa fragore.
[. . . a single day will give over to perdition, and the mass
with the machinery of the world, sustained for many years, will fall.
Or it does not escape me how new and surprising for the mind
the future ruin of the sky and the earth will be,
[. . .]
all things can be overcome and destroyed with a terrible-sounding din.][211]

The only scholar who has recently taken Lucretius into consideration for the evolution of sciences was Michel Serres. This Frenchman underlined the model based on the flow of water and liquids, with the consequent whirlpools. He made of it "... in contrast with the enterprises of Mars ... a science of Venus, without violence or guilt, in which the thunderbolt is no longer the wrath of Zeus, ...".[212] He reinterpreted Lucretius, considering in particular the problems of stability brought to the attention of scholars by René Thom (1923–2002) during the '1970s of last century. But then he contaminated everything with an excessive dose of anachronism, setting it in a one-dimension history of sciences without any internal conflicts. Also for him, the *primordia* became the usual atoms; the deviations of the *clinamen* [inclination] were assimilated to Newton's fluxions and to the infinitesimals of Leibniz. In his opinion, Lucretius anticipated the combination of letters, numbers and notes, typical of this German philosopher. Our French scholar followed the ceaseless rhythm of the text of Lucretius. But unfortunately, he ignored

[209] Lucretius V, 1423–1435; 1969, pp. 368–370.

[210] Lucretius V, 198–199; 1969, p. 296.

[211] Lucretius V, 95–109; 1969, p. 290.

[212] Serres 1980, p. 127.

its complex variety, confusing music with the arithmetic of the Pythagoreans. Thus he made it reversible, as if were subject to a fixed pre-existent Newtonian concept of time.[213] On the contrary, it is the music of things and sounds that creates its own various rhythms and times.

Having connected Lucretius somehow with Archimedes, Serres relegated him to a precursor who invented modern physics. To me, on the contrary, Lucretius seems rather to be a poet and natural philosopher, distant, in his age, from the traditions of Pythagoras and Euclid, but similar in some ways to some non-orthodox figures of today.[214]

Among his arguments open to criticism, however, Serres scattered a few gems, which are worth the whole of the rest of the book: "Scientists foresee the exact time of an eclipse, but they cannot foresee whether it will be visible to them. Meteorology is a repressed part of history. [...] This interests only those people that scholars are not interested in: farmers and seamen. [...] For it is the weather of clouds where people should not have their heads, and which should not exist in their heads."[215] Even today, organisers can plan races between sailing-boats out on the sea, spending millions of euros or dollars, without succeeding in disputing them, due to lack of wind.

As our Frenchman underlined, the world of Lucretius was one which was continually renewed, out in the open, come rain, come shine. On the contrary, modern orthodox mathematical sciences have been separated from the world, closed inside laboratories, and fixed in the rigid formulas of laws.

In the end, according to the interpretation of Serres, the wholly justified pessimism of the Latin poet took this form: "Culture is the continuation of barbarism, using other instruments."[216] In our great epoch of wars and violence, this sentence should undergo just a tiny *clinamen* to be appropriate: orthodox sciences are the continuation of barbarism, using arms that are more powerful and more destructive.

> And there were Phobos [Fear], Deimos [Terror] and with them the restless Eris [Strife], the sister and companion of murderous Mars, who, though small at first, raises her crest high, and then points her head towards the heavens, while her feet are still on the earth.
>
> Homer, *Iliad* IV, 442–443.

> Nec me animi fallit Graiorum obscura reperta
> difficile inlustrare latinis versibus esse
> multa novis verbis praesertim cum sit agendum
> propter egestatem linguae et rerum novitatem
> sed tua me virtus tamen et sperata voluptas
> suavis amicitiae quemvis efferre laborem
> suadet et inducit noctes vigilare serenas.

[213] Serres 1980, pp. 159–163.

[214] Tonietti 2002a; Tonietti 1983b, pp. 279–280.

[215] Serres 1980, pp. 75–76.

[216] Serres 1980, p. 200.

[Nor am I unaware that it is difficult to illustrate the
obscure discoveries of the Greeks in Latin verse,
especially since we must use unfamiliar words, due to the
poverty of our language and the unfamiliarity of the theme;
however, your skill and the expected pleasure of your
sweet friendship persuade me to bear any toil
and induce me to spend serene nights awake.]

Titus Lucretius Carus, *De rerum natura*, I, 136–142.

2.9 Texts and Contexts

We would prefer to keep as far away as possible from well-known philosophies of history, whichever side they come from. But this is no reason to avoid asking ourselves general questions connected with these environments shared by many of our protagonists. After all, even the texts of mathematicians, in spite of the considerable efforts made by some of them, acquire sense only if they are set in their context, without which it would be impossible to understand them.

Unfortunately, while it is customary and required for every other cultural product to reconstruct its context, for the sciences, on the contrary, this appears to be curiously uncommon, and even discouraged, if not openly opposed, by the guardians of the disciplines. Why else do those mythical divinities become characters in the history once again, and, worse still, responsible for what they do?

A famous physicist like Erwin Schroedinger (1887–1961) asked the question "Do the sciences depend on the environment?" and presented arguments in favour of this thesis.[217] Paul Forman reconstructed the hostile environment that surrounded the German scientific community after the defeat in 1918, drawing from it some surprising effects for the invention of quantum mechanics.[218] In another period, from 1979 to 1983, a journal like *Testi & contesti [Texts & Contexts]* succeeded in coming to light and growing around similar ideas, until a hostile environment suppressed it. This does not appear to be only a necessity of mine, then. Recently, Nathan Sivin suggested "doing away with the border between foreground and context, and studying scientific change as an integral whole, what I call a cultural manifold."[219]

The historical context allows us to consider possible features that are shared by the characters we are studying, without masking their uniqueness and without reducing them to some arbitrary disembodied philosophical concept. The context also reveals the heretical minority positions, and if we can succeed in preserving

[217] Schroedinger 1963.

[218] Forman 2002.

[219] *Testi & contesti* 1979–1983. Sivin 2005, p. 58. The context was described as "essential for understanding" also by Vegetti 1983, pp. 11–12ff.

their detail, it would explain the reasons for their lower esteem compared with the formation of orthodoxy.

Let us then see what we should do, together with other histories of the Greek and Latin sciences, in order to achieve this purpose, bearing in mind, however, that if we had ignored music, we would have preserved a context that would be distorted in various ways. Furthermore, we should be careful not to confuse features that are attributed to the origins of European mathematical sciences with presumed universal characteristics, which rather serve to exclude other cultures that do not possess them.

Almost all our scholars, regardless of their various mother tongues, ended up by writing in Greek, because this was the dominant language accepted for culture, even in the Roman Empire. We may ask ourselves, therefore, if and how this language opened up spaces for other common characteristics of that cultural context, which ended up by being concentrated in Alexandria: with Euclid, Ptolemy and many others, medical doctors included. Athens remained the seat of the great schools of philosophy, for as long as they lasted: Plato's Academy, Aristotle's Lyceum and the Garden of Epicurus.

In the $\Sigma TOIXEIA$ [Letters, elements, principles, shadows of the gnomon], Euclid predicated innumerable properties for one figure or another. In this way, he expressed his thought succinctly, listing one geometrical characteristic after another. "… $\tau\epsilon\tau\varrho\alpha'\gamma\omega\nu o\nu$ '$\iota'\sigma o\nu$ '$\varepsilon\sigma\tau\iota'$ …" ["… the square is equal …"]. The series of theorems is built up, sustained by a continuous interplay of copulas '$\varepsilon\sigma\tau\iota'$ [is], $\epsilon\iota$'$\sigma\iota'\nu$ [are]. It is difficult to imagine the text without the possibility of conjugating the verb 'to be': "Let the right-angled triangle be … the square is … is straight … is equal … is twice … are on the same parallels …"[220]

In the $KATATOMH\ KANONO\Sigma$ [*Division of the monochord*], whereas, Euclid predicated regarding musical intervals and the relative ratios '$\varepsilon'\sigma\tau\omega$…. [let … be ….], '$\varepsilon\sigma\tau\iota'$ [is]. And he continued to range his theorems always with an underlying structure of copulas: "… $\delta\iota\alpha\pi\alpha\sigma\tilde{\omega}\nu$ … '$\varepsilon\sigma\tau\iota'$ …" ["… the diapason (the octave) … is …"].[221]

Also Latin expresses properties easily, using the verb 'to be', conjugated in the forms *est*, *sunt*, *esse*. We, who are their heirs in Europe, have grown so used to this event that we forget it, and ignore it. But this *is* (as I was saying), on the contrary, one of the characteristics of our culture, and our geographical area: it *is not* (again) universal. In the following chapter, we shall see another area where it *is* missing.

The term for the verb 'to be' also indicated the 'being' of existence. It was opposed to the $\gamma\iota'\gamma\nu o\mu\alpha\iota$ of becoming, being born, being generated or created. Thus '$o\omega$''ν [the one who is, the being] effectively represented the uncreated eternal being. This was predicated of a God outside the world of mortal creatures, and thus a transcendent God. Parmenides developed the affirmation "he is" to an even more stable "he is and he cannot not be." "The being is"; "the god is" and "he is one". Plato later wrote: "that discourse is true which says things that are". Aristotle put his

[220]Euclide 1916, pp. 110–114.

[221]Euclid 1557.

seal on it: "saying that being is not, or that non-being is, is false; saying that being is, and that non-being is not, is true: thus whoever says 'it is' or 'it is not' either tells the truth or speaks falsely." "... being is common to all things". Vitruvius will translate this idea into Latin and into practice in the motto: "fabros esse" [be originators].[222] Another aspect of the Greek language which is important here can be found in the letter α [a]; when it is placed in front of certain words, it indicates their absence: '$\alpha'\pi\epsilon\iota\varrho o\nu$ [without end], '$\alpha'\lambda o\gamma o\varsigma$ [without words, inexpressible, without reason, without a ratio].

I still continue to reject all forms of determinism that are traced back to language, together with every other kind of determinism or reductionism that simplifies the complex events of history. However, how can we not suspect that those continual discussions, of which we read in every field of that context, achieved a particularly convincing flow and sound in Greek? In their political assemblies, in their tribunals, in the schools of philosophy, among scholars of all subjects, there was a continual $\delta\iota\alpha\lambda o\gamma\eta'$ [dialogue, evaluating, arguing], a debate between opposing positions.[223] This is generally the form of the books written by Plato.

Also the Greek scientific environment was animated by countless disputes, in which everything was divided in half, confronted and in the end discriminated. The Graeco-Roman culture appears to us to be largely dualistic in its prevailing forms. Exceptions were rare and to find examples, there are a plethora to choose from.

Typical dualisms of this culture were those between the sky and the earth, and between rational and irrational. There was a desire to distinguish "... friend from enemy, to know the one, and not to know the other." For the alternative between good and evil, Plato, as usual, criticised Homer, who on the contrary, made "... a hotchpotch of them". The true was to be separated from the false, the eternal from the ephemeral and contingent. The former were the attributes of divinity. Parmenides contrasted the Way of truth to the Way of opinion, seen as misleading.[224] Euclid's theorems enjoyed the only alternative between true and false. The famous saying *tertium non datur* [there's no third way] became a part of logic: a theorem is either true or false. This was employed to derive an extremely useful final inversion in proofs by *reductio ad absurdum*. There is no need to insist on the dualism between soul and body, seeing that it has entered into the everyday language of Western culture, revived by various people in different epochs. Aristotle classified animals by dividing them in a dualistic manner. He distinguished the *logos* between meaning and truth; for the latter, only *apophantic*, or affirmative, discourses are valid, as distinct from the *epos* [poetic word]. Where was primacy to be assigned in the body? To the heart or to the brain? In either case, the soul would command the body. Greek society was reflected in a dualistic anthropology made up of the couples: "reason/desire; soul/body; male/female; master/slave; man/animal; ... Greek/barbarian." They also included the opposition sky/earth,

[222] Lloyd 1978, p. 38. Vegetti 1979, pp. 61–69, 76, 93, 102.

[223] Lloyd 1978. Lloyd & Vallance 2001.

[224] Lloyd 1978, p. 38.

parallel to male/female and upper/lower. Exceptions to this dualism, such as the monkey, created embarrassment.[225]

Among the Pythagorean sects, numbers could be only even or odd. Hence, as in the argument against $\sqrt{2}$, derived the alternative between *logos* and *alogos*. Empedocles (500–430 B.C.) wrote of the couple love/hate, attraction/repulsion, which generated the one or the multiple. Tangible qualities were distributed by Aristotle into couples of opposites such as hard or soft, rough or smooth, bitter or sweet. This gave rise to a medicine dominated by hot or cold, and wet or dry humours. For the Stoics, the basic opposition was between active and passive. The *pneuma* [spirit of life] was divided between *psyche* [soul anima] and *nous* [reason]. Human activities, above all the cognitive ones, were differentiated between theory and practice. We have already encountered discussions between reality and appearance, that is to say, between essence (being) and the $\varphi\alpha\iota\nu\acute{o}\mu\epsilon\nu o\nu$ [what is visible, what appears]. For the Pythagoreans, mathematics discriminated between the possible and the impossible. The opposition between continuous and discrete is not to be underestimated, as it divided the schools of Aristotle and the Stoics from those of Pythagoras and Democritus. Anaximander and Anaxagoras made the world begin with a separation between hot and cold, light and dark, dry and wet.[226]

Here we shall follow above all the dualism between truth and error. We have seen above that this derived from the discussion on ‘being’. How could the $'\alpha\lambda\eta'\vartheta\epsilon\iota\alpha$ [truth, reality] be attained, then? Everybody had something to say about this. Curiously, for this term two different etymologies are proposed: not latent, from $\lambda\alpha\nu\vartheta\alpha'\nu\omega$ [to remain hidden] or not forgotten, from $\lambda\eta'\vartheta\eta$ [oblivion]? A Pythagorean wrote: “No lies penetrate into numbers; for lies are adversaries and enemies of nature, just as the truth is innately typical of the species of numbers.” Democritus sentenced: “We know nothing according to truth; because truth is in the depths.” Talking about him, Galen (second century) quoted: “Opinion is the colour, opinion the sweet, opinion the bitter, truth the atoms and the void.” Everybody, wrote Aristotle, “... has posited contraries as principles, as if they were constrained by truth itself.” He described Empedocles in these terms: “... guided by the truth itself, he is forced to admit that natural realities are only the essence.” “Rather than as a historian, Aristotle behaves like an anatomist. The systems of thought undergo a double treatment of dissection.” The very principles for making distinctions are in turn classified. “The principle may be (a) one or (b) multiple; if it is one, it may be (a′) immobile or (a″) mobile. If there are many, they may be (b′) finite in number or (b″) infinite in number; in the second case, they may be (b‴) equal in kind or (b⁗) different in kind; and so on.” From doctors, Galen expected “... a loving folly for truth.”[227]

[225]Plato 1999, pp. 123, 133, 143, 471, 733, 781–782 ... Vegetti 1979, pp. 20–22, 61, 101, 113, 121, 125, 132. Vegetti 1983, pp. 53, 59ff., 85, 94, 122.

[226]Sambursky 1959, pp. 13, 23–24, 37, 113, 124ff., 163ff., 228, 235. Lloyd 1978, pp. 105, 171, 173, 239–240, 264, 307.

[227]Sambursky 1959, pp. 39, 160–162. Vegetti 1979, pp. 59–62, 85–86. Vegetti 1983, p. 118.

The weight of truth in Greek culture is to found also among their poets, such as Sophocles. His Tiresias is said to be "the only man in whom truth is innate". Jocasta, on the contrary, a woman, saw the world as dominated rather by $\tau\upsilon'\chi\eta$ [chance, fate, destiny].[228] In the myth, Tiresias received the ability to foresee the future as a series of violations and condemnations. For striking and disturbing two snakes tightly entwined in their love, he was transformed into a woman. Having thus learnt also the nature of female sexuality, he then became the arbiter of the dispute between Jupiter and Juno about which of the two experienced more delight in their married union. "Maior vestra profecto est quam quae contingit maribus . . . voluptas". ["Your delight [of females] is undoubtedly greater than that of males"].[229] For lifting the veil from the most intimate mystery in history, Tiresias was punished by the angry goddess, who struck him blind. But the king of the gods recompensed him with the gift of prophecy. The myth is a good representation of the *aporia* on which the problem of knowledge was being founded in the West. Are things understood by separating them or by uniting them? And what other secrets would we like to uncover? Tiresias separated, and the Greeks ended up by choosing in most cases the former of the two routes. Those who confuse them will be punished with ignorance, those who make distinctions will know their future, even if they may not be able to bear it. Pleasure was removed far from knowledge, as if it were an obstacle.

The alternatives needed to be separated. The presumed truth should stand only on one side. The only person who admitted a kind of pluralism of explanations seems to have been Epicurus. ". . . for these [the heavenly phenomena], many kinds of origins are proposed, and in accordance with the witness of the senses, different explanations can be given of their mode of existence. [. . .] unless, for the sake of the method of a single explanation, all the others are senselessly disregarded, without understanding what it is, or is not, possible for man to know, yearning thus to glimpse what can not be seen." On the contrary, Plato has been interpreted as follows: "Reason is an Apollinean impulse which introduces order, making distinctions and dividing things." The philosopher, whether it was Pythagoras or Empedocles, Parmenides or Plato, is ". . . a man who is at the same time able to wield a dissecting knife like a butcher, the *mageiros* – both the profane one used at the market, and the sacred one of the sacrifice and the hieroscopy."[230]

For Greek culture in general, knowing meant solving controversies by using the $\delta\iota\chi\alpha'\iota o\nu$ [that which settles]. For this reason, it is necessary that 'I $\delta\iota\chi\alpha'\zeta\omega$' [I divide into two, I separate]. In Latin, the word used was *discriminare* [which has

[228] Vegetti 1983, pp. 30–31.

[229] Ovidio 1988, pp. 164–167.

[230] Sambursky 1959, pp. 205–208. The faith in progress, towards our physical sciences of the twentieth century, continually led Sambursky to make anachronistic comparisons between the ancient Greeks and us, taken as touchstones. Here, as regards poor Epicurus, who is presented in a contradictory manner, as if he had been afraid of religion, his judgement was: ". . . he abolishes any possibility of arriving at a comprehensive scientific conclusion." and ". . . 'scientific failure' . . .". Lloyd 1978, p. 169; Vegetti 1979, pp. 92, 94.

passed unchanged into English, to discriminate]. Even if equally cruel, the knife was sometimes symbolic, like the *logos* that separated the Greeks from the barbarians, like a transcendent religious philosophy that separated the soul from the body, like a male-chauvinistic society afflicted by sex phobia, that separated the male from the female. Plato confessed in what kind of historical context he had elaborated his condemnation of mingling, and his cult of an Apollinean purity. “If the decision of the Athenians and the Spartans had not rejected the impending slavery, now almost all the Greek races would be mixed together, and the barbarians would be mixed with the Greeks and vice versa, just like the nations nowadays under the domination of the Persians, who are dispersed and mingled, and confusedly scattered.”[231]

In a discipline unfortunately ignored by us like medicine, divisions were actually made literally, with the bloody dissection of animals, and sometimes of living creatures, not only of human corpses, but also of living people. In this way, classifications were made even of dead animals. Only when he saw himself as a fisherman or a hunter did Aristotle consider them as alive. But otherwise his rationality killed them. Alexandrian doctors like Herophilus or Erasistratus (third century B.C.) and Galen (second century) investigated live patients as if they were dead.[232]

An equally cutting instrument, if not even more so, because the blood, hypocritically, was not visible, was the law. The Graeco-Latin culture was the culture of the law. The $\kappa\acute{o}\sigma\mu o\varsigma$ [order, decoration, cosmos] was separated from the $\chi\alpha' o\varsigma$ [yawning abyss, chaos] because it was ordered and guided by rules: $\nu\acute{o}\mu o\varsigma$ [tradition, law, rule, norm].[233] It is possible that ‘I $\lambda\epsilon'\gamma\omega$’ [I pronounce, declare, prescribe them]: $\nu\acute{o}\mu o\varsigma$ $\lambda\epsilon'\gamma\epsilon\iota$ [the law prescribes]. This was the derivation of the word *lex* in Latin, and of the word ‘legge’ in Italian.

Ever since the time of Thales (sixth century B.C.) and Solon, the laws of the cosmos were developed consistently with those of the city-state, like *The constitution of Athens*, which was edited by Aristotle.[234] When studies were carried out on the movements of the stars, with them both regularities and irregularities were discovered. But these, in turn, had to be explained by means of new regular movements, among which the spherical ones along circumferences enjoyed most prestige and success. In the *Laws*, Plato wrote that the movement of the sphere around the centre and the “... circular movement of intelligence ...” were similar “... in accordance with the same principle and the same order. [...] Every course and movement of the sky and of all the bodies in the sky is of a similar nature to the revolution and the calculations of intelligence.” “Time itself seems to be a kind of circle”, wrote Aristotle.[235] In the end, the famous epicycles and deferents

[231] Vegetti 1979, p. 133.

[232] Lloyd 1978, pp. 223–225; Lloyd & Vallance 2001, p. 552. Vegetti 1979, pp. 111, 113, 125. Vegetti 1979, pp. 14, 23, 27, 33, 37–40. Vegetti 1983, pp. 116ff.

[233] Lloyd 1978, pp. 20, 120.

[234] Lloyd 1978, pp. 16, 119–120.

[235] Sambursky 1959, pp. 85 and 296.

arrived. Through music (which is thus not to be ignored), reduced to norm in turn by the theories of Pythagoras, Euclid and Ptolemy, the law for the stars could be extended to the souls of human beings, and their relative behaviour, which could thus be ordered as well.

However, all this love of laws often had the consequence of imposing a certain disregard for discrepancies between the theories and the phenomena. The latter could not always be "saved" as such. Archimedes ignored friction in his machines.[236] But it could also be sustained that the imperfections derived from the corrupted earth, and that in the pure sky of the Platonic ideas, everything was perfectly regular. Ptolemy wrote: "... every study that deals with the quality of matter is hypothetical."[237] In this way, the laws could succeed in expressing truths that were eternal and universal, that is to say, that transcended contingent historical circumstances, which depended rather on living people.

Parmenides accompanied his truth with necessity, persuasion and $\delta\iota'\kappa\eta$ [justice]. One of Plato's followers explained the perfection of Greek astronomy compared with that of the barbarians, "because the Greeks possess the prescriptions thanks to the oracle of Delphi, and all the complex of divine worship set up by the laws." For Aristotle, the relationship was so close that it could be inverted. Then the law became "... reason free from desire" based on a divine order. The *logos* expressed the truth, the order and the law of the world. Plato presumed to control the inevitable carnal impulses for food and sex by means of "... fear, the law and true discourse". For him, children were bad because they followed their instincts and their nature. "The guardian should keep watch carefully, and pay particular attention to the education of the little ones, correcting their nature and always guiding it towards the good, in accordance with the laws." The judicial conception of science became explicit with Ptolemy. "Therefore, continuing the comparison [...] of the criterion with the tribunal, the sensible realities can be likened to those who are on trial; the contingent aspects of these realities are like the actions of the defendants; the sensor, like the trial documents; sensation, like the lawyers; [...] the intellect, like the judges; [...] reason, like the law [...]. Opinion can be compared to a sentence which is in a certain sense uncertain and dubious, against which it is possible to lodge an appeal; science, on the contrary, can be compared to a sentence that is absolutely certain and unanimous. And above all the purpose of truth is similar to that of society."[238]

In the project of controlling seeming phenomena by means of laws, therefore, the mathematical sciences played the leading role. But they did not succeed yet in achieving a general mastery over everything. They seemed to work best above all in the field of music and in certain areas of astronomy and astrology. Laws that were firmly anchored to eternal, universal truths could be shown above all in the

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[236]Lloyd 1978, pp. 219–220, 269, 278, 327.

[237]Lloyd 1978, p. 282.

[238]Sambursky 1959, pp. 197ff. Vegetti 1979, pp. 71–73, 90, 104–107, 110. Vegetti 1983, pp. 71ff., 170.

mathematical sciences. Orthodoxy grew up around these, though sceptical heretical characters remained, such as Xenophanes of Colophon (sixth century B.C.), with his "nobody knows or will ever know the truth about the gods, or about all the things of which I speak."[239]

Our Greek and Latin scholars arrived at the truth that they sought by means of those forms of reasoning called $\theta\epsilon o'\rho\eta\mu\alpha$ [vision, theory, theorem, demonstration]. But what kinds of demonstrations? The ones that followed the schemes of reasoning according to Aristotle, by means of logical syllogisms? Or those invented by Archimedes who took his inspiration from his machines and balances? No. It was above all Euclid's proofs that enjoyed success; the famous theorems of his *Elements* were to represent, in the following centuries (and even in a different culture like that of the Arabs[240]) the law for every process of reasoning that claimed, with the due authority, to arrive at the certainty of eternal, universal truths. The pathway followed in order to arrive at them was considered subjective and insignificant. In general, it appears to be absent from Greek texts. The essential thing was to expound the final result in the form of a theorem that could be deduced from other truths. The physician-cum-anatomist Galen prescribed: "... demonstration is to be learnt from Euclid and then, after learning that, come back to me; I will show you these two straight lines on the animal"; "... lastly, we will try to prove the theorems, not assuming anything other than what was established at the beginning." Euclid's way of defining and distinguishing with his propositions was taken as a general criterion by Galen, who opened up bodies (not always dead ones) with his sharpened knives. "Where will the proof come from, then? From no other source than dissection?" In their different fields, the stylus and the knife were the instruments used to make distinctions and to arrive at the truth.[241] For this reason, it was necessary to transcend the uncertain instabilities of life on this earth. Did they take their inspiration, then, to a certain extent from those divinities that were venerated by some religion?

At Delphi, on the pediment of the temple dedicated to Apollo, the E of $E\tilde{\iota}$ ["You are"] was visible. Parmenides thus identified 'being' with a god. In the tradition of the Hebrew *Bible*, this was "Ego sum qui sum" ["I am who I am"].[242] Not only myth, therefore, to subject the people to the laws, in Aristotle this god became "the prime mover". Philosophers often presented themselves as priests. Aristotle called metaphysics "the science of divine things". For him, in rising up towards Heaven, man is "... among the animals known to us, either the only one that participates in the divine, or the one that participates to the greatest extent." In the end, certain philosophers began to believe themselves divine: because the activity of thought was dedicated to the divine, and because thought itself has a divine nature.[243]

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[239] Lloyd 1978, pp. 264–265, 308 and 323.

[240] See Chap. 5.

[241] Lloyd 1978, pp. 190–191. Vegetti 1983, pp. 113ff., 151ff., 162, 167–168.

[242] *Bible*, "Exodus" III, 14.

[243] Vegetti 1979, pp. 66, 73, 91, 94, 95, 116, 138, 142–143.

Mathematics and religion appear to be closely linked by the followers of the Pythagorean sects. Also a famous physician like Galen (129–c. 199) presented himself as capable of revealing mysteries written in sacred books. His purpose was to build up a “rigorous theology”, and to lift up his “hymn to the gods”. Ptolemy justified his own astronomy: “… it can especially open the way to the theological field, seeing that it alone can correctly come close to a motionless, separate activity.” A disregard for the body and for the material world, together with a belief in immortality, were aspects to be found not only in Greek philosophy, but also in the relative religion. Plato wrote: “Every soul is immortal. For everything that is always in motion is immortal”. For him, incontrovertible proof was offered by the movement of the stars, and the relative music of the spheres. Some of these ideas were later to be found even among Christian writers. Origen (third century) expressed the wish: “I hope that you will learn from Greek philosophy things that will be of use for your general or preparatory studies for Christianity, and from geometry or astronomy things that may be of use for the interpretation of the holy scriptures.” Even Augustine of Hippo (354–430) respected the Platonism of the period, while condemning a Christian’s search for causes as vain and useless, since for him it was sufficient to have faith in the Creator.[244]

Doubtless, there were infinite discussions and diatribes about the truth of this or the other position. With equal certainty, scholars were divided about who, or what, should guarantee this truth. But the event that it seemed to descend from heaven could convince many, even if not all, of them. Why all this anxiety to attribute to others what they had invented? Why not enjoy all the merits themselves? Was it that they were afraid that otherwise they would have to assume also the defects, and thus be fully and solely responsible?

Eratosthenes (third century B.C.) worked for the Ptolemy family at Alexandria, taught their children and directed their library; he once wrote to a customer of his: “My invention [a machine for doubling the volume of a solid] may be useful also for those who desire to increase the size of catapults and martinets, because everything has to be increased proportionately, if we want the shot to be proportionately longer. This cannot be achieved without calculating the means.” However, it was his more famous correspondent, Archimedes, who invented the most renowned war machines, capable of keeping at bay the might of the Romans during the siege of Siracusa.[245] Even the leader of the attacking forces, Marcellus, had his own devices. These included one enormous machine called the “sambuca”.[246]

In any case, this represents one of the clearest episodes in which we can see that a context of war was capable of polarising everything, including the interests of people devoted to the sciences. Among the titles that we know of the books written by Democritus, there was also one on the technique of warfare. We have already dwelt on the way Plato supported to educate his young men through the mathematical

[244] Lloyd 1978, pp. 134, 303, 319. Vegetti 1983, p. 174. Samburky 1959, pp. 67ff.

[245] See above Sect. 2.7.

[246] Authier 1989, pp. 108, 116 and 123. Lloyd 1978, p. 194.

sciences in order to prepare them better for war.[247] We must note first of all that the Hellenistic age of the Ptolemies, when such important results were achieved, was not at all pacific. The first of the Ptolemies was a general of Alexander the Great.

In his *On the construction of the artillery*, Philo of Byzantium (third and second century B.C.) described the relative problems, contrasting the mistakes of the early archaic attempts with the successes achieved by the engineers of Alexandria. For their war machines, the latter calculated the proportions of the various parts, and verified the results experimentally. They "... received considerable help from sovereigns who were in search of glory, and amenable to the arts and crafts." In his *De Architectura*, Marcus Vitruvius Pollio (first century B.C.) projected war machines that he used in the imperial army of Octavian Augustus. The Heron of Alexandria (first century) that we have already encountered sponsored *On the construction of the artillery*.

Pappus of Alexandria (fourth century) was later to write in his *Mathematical Collection*: "The most necessary of the mechanical arts, from the point of view of everyday requirements, are as follows: (1) The art of pulley makers .... (2) The art of makers of war machines, who are also called mechanics. Missiles of stone, iron, or similar materials are projected for great distances by the catapults that they construct. (3) The art of makers of machines ...". In defining mechanics as "... the study of material objects that can be perceived by the senses ...", even Proclus (fifth century) included, under the first point, the construction of devices that were useful in war. "The priority assigned to the projecting and construction of war machines" may surprise only those ingenuous ones who continue to believe, by faith, in the sublime purity of disinterested scientific research performed by natural philosophers, motivated only by a love for truth. In his arguments, Aristotle would indulge in military comparisons. The $\tau\alpha'\xi\iota\varsigma$ [array, battle formation, order] of the world had to be guaranteed, like that of an army. "An army is in good conditions when it is in order, and when it has a general, and in particular when it has a general." Pliny the Elder (24–79) represented animals as a war or post-war spectacle, which was put on show during triumphs and in circuses. He besides wrote books on the military art, and on the wars in Germany.[248]

Plato described the $\sigma\omega'\mu\alpha$ [body] as the site of battles between humours. These give rise to illnesses, including those of the soul. Among these, we find '$\alpha\phi\rho o\delta\iota'\sigma\iota\alpha$ [sexual pleasure]. The same image was used by Hippocratic medicine.[249] Even the famous physician Galen, who cured gladiators, and followed the Roman soldiers in their campaigns against the Germans, declared: "What is more useful for a doctor in curing a war wound, extracting missiles, amputating bones .... than a detailed knowledge of all the parts of the arms and legs ...?". For him, scorpions, tarantulas and vipers were to be suppressed, because they were "... evil by nature, and not of their own free will. Logically, therefore, we hate evil men ... and we kill those

[247] See above Sect. 2.3.

[248] Lloyd 1978, pp. 47, 144–147, 240–248. Vegetti 1979, pp. 87, 95. Vegetti 1983, pp. 97ff.

[249] Plato 1994, pp. 142–143. Vegetti 1983, pp. 47–51.

who are irremediably evil for three good reasons: so that they will not commit evil, remaining alive; so that they will arouse the fear in their fellow-men that they will be punished for the evils that they commit, and thirdly, it is better for them to die, as they are so corrupt in their souls that they cannot be educated by the Muses, or improved by Socrates or by Pythagoras." In a treatise of Hippocratic medicine, *Airs, waters, places*, the writer intended to explain the weakness of Asian peoples. As they are "... subject to despots, they do not think of how to train themselves for warfare, but rather how to seem unsuitable for fighting. The dangers are clearly not the same. It is natural that in their case, they are forced to fight, suffer and die on behalf of their masters, ...". In order to curb the "wild beast" that urges man towards food and sex, Plato placed some "sentinels" in the heart, just as his Republic needed soldiers. Aristotle subjected all plants and animals to man. "... also the art of war will by nature be, in a certain sense, a technique of acquisition (and the art of hunting is a part of this), and it must be practised against those beasts and men that refuse to allow anyone to command them, even if they were born for this: because by nature such a war is right." "... dominating the barbarians is befitting for the Greeks."[250]

In general, science historians modestly avoid recalling the links between mathematical sciences and military arts, perhaps so that they will not have to admit the influence of war contexts on their evolution. And yet Giovanni Vacca did acknowledge it in the introduction to his edition and translation of "Book 1" of Euclid's *Elements*. He noted the "progress of mechanics" due to the military arts, and quoted Plato's *Republic* for "... the manifest usefulness of geometry in the art of war ...". He even identified in this the origin of the speculations dedicated by Tartaglia and Galileo Galilei to movement. This exception can easily be explained by the date of the edition: 1916. In that period, a part of the Italian population was labouring under the illusion that by entering into the world war and achieving an easy, rapid victory, the Italian Risorgimento would be rhetorically completed. In those years, therefore, this mathematician and historian, above all of Chinese matters,[251] considered war as a factor of patriotic, civil and social progress.[252] But as this did not happen then, leading to fascism and resuming in full force worse than before in 1939, neither will it happen today, now that the century of warfare and violence is continuing into the new millennium.

Far be it from us to fall into a totally pessimistic or consolatory philosophy of history, because we continue not to want to exclude from our history the heretics, chance (like Jocasta) and the heterogenesis of ends. However, senators, kings and emperors, for one reason or another, amplified the probability of obtaining the results that they desired by favouring these researches, compared with other forms of culture. We have been able to tell the story of the former together with the relative

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[250] Lloyd 1978, pp. 287 and 302. Plato 1994, pp. 108–111. Vegetti 1979, pp. 105, 112, 120, 134.

[251] See Chap. 3.

[252] Euclid 1916, pp. xiii–xiv. Sambursky 1959, pp. 283 and 292, on the contrary, complained that all those war machines had not produced a "serious, multifarious technological development".

circumstances. Who knows what happened to the latter, if they ever existed? The traces that have remained are undoubtedly scarce, and more difficult to find.

The name *sambuca* attributed to the war machine of the Roman general Marcellus is interesting, because in Greek, $\sigma\alpha\mu\beta\upsilon'\chi\eta$ had not only that meaning, but it also indicated a musical instrument with four strings, a kind of triangular harp. As in the myth of the birth from Venus and Mars, or in Plato, harmony and war were presented as variously linked.[253] Arms and harmony have a common origin in the Greek language. '$\alpha'\rho\mu\acute{o}\zeta\omega$ meant 'to join, to regulate, to govern, to be in agreement'. '$\alpha'\rho\mu\acute{o}\nu\iota'\alpha$ derived from this, with the meaning of 'connection, agreement, concord, musical harmony'. The same verb, with the same meaning, also gave rise to '$\alpha'\rho\mu\alpha$, which was not so much the amorous union of Aphrodite as a war chariot, and also to '$\alpha'\rho\mu\varepsilon\nu o\nu$, an instrument, and (in the plural) the equipment of a ship, its rig. For sailing-boats, still today, the Italian *armamento* [armament] means the way in which shrouds, halyards and sheets are connected to the mast and to the sails.

It is equally rare, however, for historians of the sciences to give due importance to musical harmony. We have shown, on the contrary, how much importance Greek scholars of the mathematical disciplines dedicated to it. But this is not just a pedantic, bureaucratic question of completeness. Only the music (of the spheres) explains the insistence on considering our souls as part of the heavens, making human beings similar to stars in the astrological and astronomic picture, and trying to represent the strokes of the pulse by numbers. How could rhythm be measured in that period apart from by numbers? Galen wrote: "... as musicians establish their rhythms in accordance with certain precise combinations of periods of time, contrasting the '$\alpha\rho\sigma\iota\varsigma$ [lifting, raising, arsis] to the $\vartheta\varepsilon'\sigma\iota\varsigma$ [downstroke, beat, thesis], so Herophilus [third century B.C.] supposed that the dilation of the artery corresponded to the arsis, and its contraction to the thesis."[254] Even in the seventeenth century, we shall find one of the main protagonists of the modern scientific revolution, who turns to music in order to measure the time of a physical phenomenon. Socrates was to compare Plato to a swan that sings and then flies away.[255]

By means of music, it is easier to understand how many, and what kinds of obstacles the Greek and Roman natural philosophers had created between

[253] Authier 1989, p. 116.

[254] Lloyd 1978, pp. 216, 219 and 228. Sambursky 1959, pp. 45–46ff., wrote that musical harmony was "... the first example of the application of mathematics to a basic physical phenomenon". Unfortunately, however, he added that the Pythagoreans had carried out "... authentic quantitative measurements, using wind instruments and instruments with strings of different lengths ...." This does not transpire from the completely different tradition that built up around them. Furthermore, if they had really done so, they would not have been able to maintain the ratios that were so dear to them; because reed-pipes and strings are tuned in accordance with different numbers, as will be seen in Sects. 3.2 and 6.7 below. It is clear that Sambursky does not seem to have had any direct experience with his ears, either.

[255] See Part II, Sect. 8.2. Vegetti 1979, p. 73.

mathematical sciences and the world of the senses. Free access to this world was forbidden by the orthodoxy that grew up around the Pythagorean-Platonic-Euclidean-Ptolemaic axis. They judged the harmony of Aristoxenus to be heretical, with its divisions of musical intervals into equal parts, which were attuned to the ears of the musicians. The prohibition of lascivious, effeminate music, which distracted young men from the virile military arts, was extended to the kind of theory of music that permitted it, thus offering a better justification in practice for micro-intervals. Consequently, the famous question of denying any numerical representation for incommensurable ratios, which was almost equivalent to the division of the tone into equal parts, also assumed the nature of a prohibition, and not just that of a distinction between ratios. Nowadays we would say that diversity was transformed into discrimination and inferiority. And it would be sufficient, then, to read Plato's *Republic* to discover the historical context responsible for discriminating between the two positions: the defeat of Athens (404 B.C.) in the Peloponnesian wars. Music is thus able to offer us new material, in order to re-discuss the *vexata questio* about the invention, or otherwise, of so-called experimental methods, and their relationship, or otherwise, with mathematics.

Aristoxenus was after to be taken into consideration again in Europe only by musicians centuries later, and before the division of the octave into equal parts was given its due importance by scholars of exact sciences. We can maintain for Greek culture the important place it deserves in the evolution of the sciences. We can likewise recognize that it took advantage of its characteristic inclination for discussions, facilitated by its language. But we must also add that we have identified in it powers and instruments of discrimination.

Not everything was left free to develop. In the Museum and the Library at Alexandria, the Ptolemies organised their explicit 'scientific policy', which did not range in all possible directions. Plato made Socrates say that not all sciences were equal, and that a hierarchy existed among them. Those at the basis of the art of building and trading were different from geometry and pure calculations. "Among them, the arts and sciences practised in the search for knowledge by true philosophers are far superior, in precision and truth in measurements and numbers." On that point, Eratosthenes, Galen, and above all Claudius Ptolemy referred to Plato. Ptolemy wrote that he had dedicated himself "... to mathematical theory as a whole, but with a particular preference for that part of it which deals with divine and heavenly things, because this branch alone investigates the things which always exist without changing." We are thus forced to conclude: how? "Well, in reality, science acts as a powerful device for the censure and exclusion of possibilities of discourse, and therefore the control of imaginable universes and images of the world: so much the more powerful, because it does not speak in the name of this or the other option of values, but in the name of the truth itself, ...". Some of these censures are well-known, like the movement of the Earth around the Sun for Aristarchus, or the infinite nature of the universe, and the intelligence of animals. Here I have added to these the division of the tone and the octave into equal parts, following the ear, as suggested by Aristoxenus.[256]

"... law also means obeying the will of one alone." Even in the *polis*, Aristotle distinguished men like kings, so perfect in virtues and capable in politics as to be like gods. "For them, given their nature, there is no law: they are the law, and it would be ridiculous to try to draw up a set of laws for them." There was a hierarchy between those who commanded and those who owed obedience, and had to submit. "Commanding and being commanded are not only necessary, but also beneficial; [...] this happens among living creatures in all nature; and there is a principle of command also in things that do not participate in life, like musical harmony." Thus, for Aristotle: "As the race of the Greeks occupies geographically a central position, so it participates in the character of both, because it has courage and intelligence, and thus it continually lives in freedom; it has the best political institutions and the possibility of dominating all others, provided that it maintains constitutional unity."[257]

Diogenes the Cynic preferred to eat like dogs, and laughed at Plato's definitions, "man is an unfledged biped", showing a cock that had been plucked.[258] As memories of him, we have, above all, anecdotes and caricatures.

Partly to understand better to what extent that historical context acted as a filter, we shall study in the following chapters how other great written cultures behaved in this regard. Let us start from the one that is most different and most distant: China.

> He turns the bow round and round in his hands!
> [...]
> Like a skilful singer who, having fastened
> The twisted catgut of his new lyre
> At both ends, without any difficulty
> Stretches the string by turning the peg;
> So he effortlessly strung the great bow.
> Then he decided to test the string: he opened
> His hand, and the string sang an acute note,
> Like a chirruping swallow's song.
>
> Homer, *Odyssey*, XXI, 480–493.

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[256] Sambursky 1959, pp. 55–56. Vegetti 1983, pp. 151ff., 156, 169ff., 175ff. Paul Tannery (1843–1904) did not contrast Aristoxenus sufficiently with the Pythagoreans and Platonics, putting them all together. But to the Frenchman should be recognized his great merit in attributing the correct role to music in the development of Greek mathematics. He went so far as to write: "... l'origine de la conception grecque de la mesure du rapport est essentiellement musicale, ..." ["... the origin of the Greek idea of measuring the ratio is essentially musical, ..."]; Tannery 1915 (1902), p. 73. Cf. Mathiesen 2004 who did not attribute *Sectio Canonis* to Euclid. Cf. Barker 2007 who believes that *Sectio canonis* is Euclid's.

[257] Vegetti 1979, pp. 108, 141, 119–121, 134.

[258] Vegetti 1979, pp. 43, 51. Vegetti 1983, p. 86.

Chapter 3
In Chinese Characters

Dao sheng yi yi sheng er er sheng san san sheng wan wu.
[The tao generates one, one generates two, two generates three, three generates ten thousand things.]

Wan wu fu Yin er bao Yang. Chong qi yi wei he.[1]
[The ten thousand things bring the Yin and embrace the Yang. Thanks to the qi, they then become harmony.]

Daodejing

3.1 Music in China, Yuejing, Confucius

Showing scrupulous respect for a scientific culture like that of China, so distant and so different from our own, means starting to consider its language as an aspect that cannot be ignored. Consequently, I will often quote the original Chinese words. But all the same, I will not be able to escape from a paradox: on the one hand, we do not want to measure these sciences here on the basis of European discourses, with their relative values. Thus we present them as incommensurable. But on the other hand, we narrate the events in another language, and by so doing, we are already modifying them. Even if we try to remain faithful in representing the differences, we betray them in our translations. In these, readers will come into contact, at the same time, with my interpretations of them. Every Chinese character, word and phrase is generally translated by me, even if other translations exist which are pregnant with other interpretations. Terms which are too different from ours are explained

[1]I am tempted to include also the Chinese characters, as they are pertinent to the argument that follows. However, for practical reasons connected with the difficulty of reproducing them in this edition, Chinese words and phrases will only be quoted in their official *pinyin* transliteration adopted in the modern-day People's Republic of China. However, the classical Chinese characters, which are consequently not simplified, can be found both in Needham et al. 1954 and Tonietti 2006a, as well as in the original copy of this text in Italian. In this edition they are shown in Appendix D.

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and paraphrased the first time, and then left in Chinese. Others, which at least have approximate Western equivalents, are repeated in translation. However, to start by pinpointing one of the most significant differences between China and the West, the paradox appears to be such, only as a result of the typically 'linear' European way of reasoning. On the contrary, it would disappear in the various 'circular' arguments in Chinese.

Also in China, it is possible to narrate the mathematical sciences by means of music. The most ancient text dealing with *yinyue* [music] is the *Yuejing*. [Classic for Music] In this context, the character *jing* means a book that is considered to be so important that it has assumed the status of a canon, becoming compulsory reading for scholars of the subject, a kind of *Bible*. The title could even be rendered as *The Bible of Music*, but in Europe, the term *Bible* would evoke too many religious connotations, which, as we shall see, are not present in Chinese culture.

Unfortunately, we can deal all too quickly with the *Classic for Music*, because it was no longer extant at the beginning of the Han Empire (221 B.C.–220 A.D.). It would have been the sixth *jing*, together with the other five famous works: *Shujing* [Classic for Books], *Shijing* [Classic for Odes], *Yijing* [Classic for Changes], *Liji* [Memories of Rites] and *Chunqiu* [Springs Autumns, Annals]. These classics remained such throughout the 2,000 years of various Chinese empires, and formed the cultural backbone for the appointment of imperial officials, at a certain point by means of an institutional examination system.

Even if we cannot relate the exact contents of the *Classic for Music*, the simple fact of its existence at least reveals the important role that music played in Chinese culture. Fortunately, we can add some more elements to this. Among these, we find pictures in which music was represented by a *zhong* [bell], accompanied by a *ju* [set square], a *taiyang* [sun] and a *yue* [moon]. The symbols were in the hands of a deified figure called Liu Tianjun [the sovereign of the sky Liu].[2] What relations Chinese culture created between music, the seasons, the climate, the calendar, astronomy and geometrical measurements, we shall now see. Chinese sciences evolved around discussions of bells, set squares, the Sun and the moon: real objects, then, which were visible and relatively easy to handle.

Some of the above-mentioned classics were traditionally attributed to Kong Fuzi [the master Kong, Confucius] (551 B.C.–479 B.C.). But it seems that the only text that is in any way related to the famous sage was the collection of his sayings and teachings *Lunyu [Dialogues*, or *The Analects]*. Among the most famous of these is number three of Book XIII, even if it may be a subsequent interpolation. Confucius sustained that in order to govern well, it was necessary to recreate a correct relationship between words and things and sense: "to rectify the names". Otherwise the sovereign's acts would not be successful. "If one's acts are not successful, then the rites and music are not promoted. If the rites and music are

[2]Needham 1954, first picture. *Zhuangzi* XIV and XXXIII 1982, pp. 134 and 306–309. One music teacher even became an adviser of the prince: see Sect. 3.5. Needham & Wang & Robinson 1962, p. 126. Sabattini & Santangelo 1989, pp. 120, 410ff., Lloyd & Sivin 2002, pp. 44ff.

not promoted, then the punishments are applied wrongly. If the punishments are applied wrongly, then the people do not know how to use their hands and feet." Together with three other books of Confucian inspiration (*The Great Learning, Mencius, The Doctrine of the Mean*), this, too, became a part of the essential education of the cultured Chinese man of letters. During the pre-imperial period, called "Spring, autumn" and "Fighting States" (770 B.C.–221 B.C.), as the Zhou government gradually disintegrated, young aristocrats had to learn music, together with the rites, writing, archery, how to drive a cart, and calculating: *liuyi* [the six arts].

As far as we know, we can imagine Confucius wandering from one kingdom to another, from town to town, in a China spread out around the Huanghe [Yellow River], trying to dispense wisdom and advice, which in general went unheeded. He spoke to everyone about *he* [harmony], and his desire was that the sovereigns would seek *heping* [peace]. What better way to convince them than by playing music? The sage may have plucked the *se* [a kind of large lute with 25 strings], or some other instrument.[3]

But instead, kings and feudal lords preferred to continue their *zhan* [war], which in the end was won by Qinshi Huangdi [the emperor Qin, the first]. He was clearly a powerful, astute warrior, and he proclaimed that his empire would last for *wanshi* [ten thousand generations, for ever].[4] As we relatively recently saw the foundation of a "millennial Reich" in Europe last century, which lasted for as long as 12 years from 1933 to 1945, we may suspect how long this first Huangdi may have maintained power: from 221 to 207 B.C., 14 long years. Clearly, his famous army of terracotta warriors, found at Xian not long ago, was not sufficient for him to maintain his power. The subsequent Han emperors brought a decided change in their style of government, remembering Confucius, and amid alternating fortunes, their empires lasted for just four centuries. Thus, to cut short a story which would otherwise be truly millennial, Confucius, harmony and music remained among the current values of Chinese culture.

Famous historians of the Han period, like Sima Tan (died in 110 B.C.) and his son, Sima Qian, spoke of music, together with rites, the calendar, astronomy and astrology, in their vast *Shiji* [Memories of a Historian]. Nor could music be ignored in the subsequent *Hanshu* [Books of the Han]. The Chinese sage was often presented as skilled in the art of playing a musical instrument. Poems were composed with their music. Even though during the Tang dynasty (618–907), professional musicians, who must not be men of letters, were not included among the higher ranks of the population.[5] The main task of emperors was always that of maintaining the *yuzhou* [universe] in harmony. Floods, earthquakes and other kinds

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[3] Confucius XIII, 3; 2000, p. 104; XVII, 20; 2000, p. 131. Needham 1962, p. 131.

[4] *wan* [ten thousand] also means "innumerable", like our "myriad"; *wannian* [ten thousand years] means "eternity".

[5] Sabattini & Santangelo 1989, pp. 166, 209, 270, 274, 365.

of catastrophes were interpreted as signs that they had not acted correctly, and the price to be paid was generally a loss of power.

3.2 Tuning Reed-Pipes

In the *Qian Hanshu [Books of the early Han]*, written around the first century by the Ban family, father and sons, we can already read what the Chinese theory of music was. In order to develop their mathematical theory, the Chinese preferred to reason on the basis of *lülü* [standard reed-pipes]. In the section "Lülizhi" [Annals of reed-pipes and the calendar], there was a description of the procedure of tuning pipes, which were considered as solid objects.

> Yi yue: can tianliangdi er yi shu
> [*Yi[jing]* says: 'refer both to the sky and the earth, and trust yourself to numbers']

The measurements of reed-pipes were calculated as follows.

> Tian zhishu shi yuyi, zhongyu ershiyouwu. qiyi jizhi yisan, gu zhiyi
> desan, you ershiwufen zhiliu, fan ershiwu zhi, zhongtian zhishu,
> deba shiyi, yi tiandi wu weizhi he zhong yushizhe chengzhi, wei
> babai yishi fen, ying litong, qianwubai sanshijiu suizhi zhangshu.
> [The numbers of the sky start from 1, and all together they add up to 25.[6] Take 1 in order to obtain 3,[7] write down 25, the number of the whole sky, for each of the three, and add six more, obtaining 81.[8] Take the number 5 of the sky and the earth,[9] and add it, arriving at 10, and multiply the above result by 10. This makes 810 *fen*.[10] The calendar is based on *tong*, the number of *zhang* in 1539 years.]

Meng Kang, a commentator of the second century, added here:

> shijiusui wei yizhang, yitong fan bashiyi zhang.
> [19 years make one *zhang*, every 81 *zhang* one *tong* is completed.]

The text continued.

> Huangzhong zhi shiye. ... cizhiyi, qi shier lü[lü] zhi zhoujing.
> [It [810] is also the solid of the *Huangzhong*. This means constructing the diameter for the circumference of the 12 pipes.]

Huangzhong is the name of the first note emitted by the pipe, from which the Chinese started, in order to construct the others, one by one; it literally means: "yellow bell". The bell in the hands of the sage may be interpreted as such.

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[6] This number was obtained by summing $1 + 3 + 5 + 7 + 9 = 25$. The numbers came from the *Yijing*; Needham & Wang III 1959, p. 71.

[7] The length of the circumference whose diameter is 1, taking 3 as the ratio between them [π].

[8] $3 \times 25 + 6 = 81$.

[9] $3 + 2 = 5$.

[10] 1 *fen* was equal to about one third of a centimetre.

81 was also the number that gave the relationship between the note of this first pipe and the calendar, which was based on the periodic movements of the Sun and the moon. It was underlined in the text that every 81 *zhang* corresponded to one *tong*. These were the resonance intervals to which the fundamental periods of the moon and the Sun, otherwise incommensurable, corresponded with a certain approximation: 19×81 years = 1,539 years.[11]

At this point, Meng Kang commented:

> Lü kongjing sanfen, can tianzhishu ye; wei jiufen, zhongtian zhishuye.
> [3 *fen* is the diameter for the opening of the pipe, and it also refers to the number of the sky; the circumference is 9 *fen*,[12] also the number of the whole sky.]

So in order to obtain the *Huangzhong*, the pipe was considered as a solid 90 *fen* long, with a circumference of 9. Thus 810 represented the lateral surface of the solid. Then the measurement were calculated for another pipe which would produce the note *Linzhong* [bell of the woods].

> Dizhishu shi yuer, zhongyu sanshi. Qiyi jizhi yiliang, guzhi yideer,
> fan sanshizhi zhongdi zhishu, de liushi, yidi zhongshu liu chengzhi,
> wei sanbai liushifen dangqi zhiri, linzhong zhishi.
> [The numbers of the earth start from 2, and all together they add up to 30.[13] Write down 30, the overall number of the earth, for every two, obtaining 60.[14] Multiply this by 6, which is the intermediate number of the earth: the result is 360 *fen*. This number is equal to the number of days in the periods.[15] This is the solid of the *Linzhong*.]

Meng Kang commented:

> ... Linzhong chang liucun, wei liufen. Yi wei cheng chang,
> deji sanbailiu shifenye.
> [The length of the *Linzhong* is 6 *cun*,[16] the circumference 6 *fen*. Multiply the length by the circumference, obtaining 360 *fen*.]

Another commentator, Shigu, added:

> Qiyin ji. wei shier yue wei yiqiye.
> [Periods and sounds are the bases. This means that 12 months make up the periods.]

After the sky and the earth, to calculate the dimensions of the third pipe, they considered man, who completed the universe.

> Renzheji tian shundi, xuqi chengwu.

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[11] Needham & Wang III 1959, p. 406. Lovers of numerical symbolism will find further justifications in Granet 1995.

[12] Ratio between circumference and diameter equal to 3.

[13] This number is obtained by summing $2 + 4 + 6 + 8 + 10 = 30$. See note 6.

[14] $30 \times 2 = 60$.

[15] In China, the year was divided into 24 periods of 15 days each, making a total of 360 days. They were called *jieqi* [solar terms], and indicated the atmospheric-climatic and seasonal characteristics of the relative days. See also below.

[16] 1 *cun* = 10 *fen*, about 3 cm.

[That [the number] of man follows the [numbers] of the sky and of the earth, in that order; the *qi*[17] realises their matter].

tong bagua, diao bafeng, li bazheng, zheng bajie, xie bayin,
wu bayou, jian bafang, bei bahuang, yizhong tiandizhi gong, gu baba
liushisi. qiyiji tiandizhi bian, yi tiandi wuwei zhihe zhongyu shizhe
chengzhi, wei liubai sishifen, yiying liushisi gua, dazu zhishiye.
[It connects the 8 *gua* [trigrams],[18] moves the 8 winds, manages the 8 transactions, adjusts the 8 knots,[19] harmonises the 8 sounds, causes the 8 stimuli to dance, observes the 8 directions, satisfies the 8 shortages, and takes care of all the services in the sky and on earth, therefore 8 times 8 equals 64. This means changing the sky and the earth completely. Add 5 for the sky and 5 for the earth, and multiply the total by 10; that makes 640 *fen*, because it deals with the 64 *gua* [hexagrams]. This is also the solid of the *Dazu* [big group].]

Meng Kang summarised as follows:

Dazu chang bacun, wei bafen, weiji liubai sishifenye.
[The length of the *Dazu* is 8 *cun*, the circumference 8 *fen*; their product is 640 fen.]

The Han text continued:

Shu[jing] yue: tiangong renqi daizhi.
[The *Classic for books* says: in the service of the sky, man takes his place.]

Shigu added:

...yansheng renbing tianzao huazhi gongdai er xingzhi.
[...the sage says that man receives the sky as a gift, with the task of producing transformations; let him act on its behalf.]

Returning to the most ancient text,

Tian jiandi, ren zetian, guyi wuwei zhihe chengyan,
'weitian weida, weiYao zezhi' zhixiangye.
[The sky in accord with the earth; let man imitate the sky. In this, therefore, let him add 5 and multiply. Again, let him take as his model 'Only the sky becomes great' and 'Follow only Yao'.[20]]

Here, Shigu commented:

ze, faye. 'lunyu' cheng Kongzi yue: 'dazai Yao zhiwei junye, weitian
weida, weiYao zezhi'; mei diYao neng fatian er xinghua.
['Follow', also take as your model. In the *Lunyu [Dialogues]*, Kongzi [Confucius], gives advice, and says: 'Also the *jun* [gentleman] acts like the true great Yao and follows the one and only Yao of 'Only the sky becomes great'. The good emperor Yao was capable of modelling himself on the sky, and effecting transformations'.]

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[17]The *qi* was the material and energetic substance that generated and pervaded the whole universe, a kind of material ether and cosmic breath. See below, Sect. 3.5.

[18]Lo *Yijing* governed change by means of 8 symbols called *gua*, which are formed by the combinations, three at a time, of two lines, one continuous —, which represents the Yang, and one broken – –, for the Yin. Put together, two by two, the 8 trigrams generate the 64 famous hexagrams of predictions. *Yijing* [I King] 1950; *Yijing* [I Ching] 1994.

[19]The four seasons, two equinoxes and two solstices.

[20]An archaic mythical wise emperor.

The *Qian Hanshu* settled the question.

> Diyi zhongshu chengzhe, yindao libing, ... santong xiangtong, gu Huangzhong, Linzhong, Taizu lüchang jie quancun er Wang yufenye.
> [Multiply that central number of the earth [6], the third of the Yin principles,[21] ... link all three with one another; thus the *Huangzhong*, *Linzhong* and *Tai[Da]zu* pipes each have the length of a whole number of *cun* [9, 6, 8] and lose the remaining *fen*.][22]

The page stressed the link between sky, earth and man, in the ratios of the pipes, tuned as indicated. To summarise, the *Huangzhong* [yellow bell] pipe will be 9 *cun* long and 9/10 of a *cun* in circumference; the *Linzhong* [bell of the woods] pipe will be 6 *cun* long and 6/10 of a *cun* in circumference; the *Dazu* [big group] pipe will be 8 *cun* long and 8/10 of a *cun* in circumference.

The ancient Chinese had thus invented a way of tuning their pipes, not only by following their ears, but also with the justification of a mathematical theory. It is clear from the text that in changing the measurements also of the circumference, they were following a rule; indeed, they were following the same ratio with which they changed the length. They had realised that it was not sufficient to reduce the length from 9 to 6 [by $\frac{2}{3}$] in order to obtain the *Linzhong*, because the note would be less acute than desired. They knew that they could make up for this defect by reducing the opening of the pipe. Besides this, they also knew that the adjustment had to be calculated on the basis of the same rule in accordance with which they diminished the length. Thus the *Linzhong* had to be 6 *cun* long [$9 \times \frac{2}{3} = 6$] with an opening of 6 [$9 \times \frac{2}{3} = 6$] *fen* [tenths of a *cun*]. The same procedure was followed for the *Dazu* pipe, changing the ratio from 6 to 8 [$6 \times \frac{4}{3} = 8$], both as regards the length and the circumference.

Subsequently, on the basis of all the research carried out so far, we never find anything similar in any other culture. In the West, the only scholar that I am aware of is Vincenzio Galilei (1520–1591) in the sixteenth century, who clashes for this reason with the orthodoxy of his period.[23] Still later, we find the scholars of acoustic physics, who invent complex formulas, using various different post-Cartesian symbolisms, but the final results are equivalent to those of the ancient Chinese procedures.

We need to imagine what effect the non-negligible solid dimensions of the pipe will have on the height of the sound; therefore we think of how the column of air moves. In reality, it vibrates just beyond the end of the pipe that is open at the extremity, because spherical waves are generated here. Thus the pipe behaves as if it were longer than its linear measurement l. The formula that was used as a seal for European tradition at the end of Sect. 2.1 was valid for strings, but now it needs to be modified to take into consideration geometry, which arrives, with pipes, at

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[21] *bing* also indicated the third of the 10 *tiangan* [heavenly stems] which, combined with the 12 *dizhi* [earthly branches] were used to indicate years, months, days and hours.

[22] *Hanshu* 21A, pp. 963–964. Tonietti 2003ab, pp. 238–239. Cf. Needham, Wang & Robinson IV 1962, p. 212. Robinson 1980, pp. 71–72.

[23] See Sect. 6.7, and also Conrad Henfling, Part II, Sect. 10.1.

new dimensions. It is easy to imagine that the wider the pipe opening, the more likely the vibrating air is to spill over its edges. Consequently, the actual length that determines the height or frequency ν of the note emitted depends on the length l, increased in proportion to the diameter d. For pipes, the formula thus becomes:

$$\nu \propto \frac{1}{l + cd}$$

where c is a constant generally estimated as 0.58.

According to this formula, if we want a pipe to emit a more acute note than a fundamental note obtained with the length l and the diameter d, we need to calculate, for example, in order to produce the fifth, or, in China, the *Linzhong*,

$$\frac{3}{2}\nu \propto \frac{1}{\frac{2}{3}(l + cd)} \propto \frac{1}{\frac{2}{3}l + c\frac{2}{3}d}.$$

The diameter d is thus to be reduced in the same proportion as the length l. In the Chinese text of the Han period, they started with $l = 9$ *cun* and a circumference of 3×3 *fen* [$d = 3$]. They obtained the *Linzhong* with $l = 6 = 9 \times \frac{2}{3}$ and a circumference of $6 = 3 \times 2$, that is to say, $d = 2 = 3 \times \frac{2}{3}$. Thus the ancient Chinese procedure gave the same results as these late European formulas. The latter were to arrive at the same result as the Chinese, also to produce the *Dazu*, that is to say, 8 *cun* in length and 8 *fen* in circumference, increased from 6 [by $\frac{4}{3}$].[24]

Tuning pipes undoubtedly appears to be more complex than tuning strings. The Europeans were following easier routes, and were selecting problems which better supported simplification. But the Chinese choice of pipes depended on the values present in their culture, through which they observed the world. We find them expressed in the first text that we examined. Music, or the harmony of pipes, was invented in relation to the harmony of the sky, the earth and man. Of course, the sky, with its revolutions of the Sun and the moon, fixed the first note; but the second one came from the earth, subject to changes produced by the periods of the seasons; whereas, lastly, the third note assigned to man the task of linking all three together, by acting in accordance with the instructions of the *Yijing*. The values were seen as held together by the harmony of music, and materially realised by the *qi*. By blowing into the pipes, then, the player caused all the vital energy of his breath to resonate, together with all the vital energy that pervaded the universe.

The Chinese mathematical theory of music was invented through solid pipes, because the relative cosmos was conceived of as the union of sky, earth and man, made real and material by the vital breath of the *qi*, which filled everything and made everything come into being. Sky, earth and man were seen in their manifold aspects, and above all as changeable. "... making transformations ...", "... effecting

[24]Nowadays, in the predominant idiom of our great epoch, this is called the *end-effect*. Tonietti 2003ab, pp. 239–240. Fletcher & Rossing 1991, pp. 474–477. Robinson 1980, p. 69. Chen 1999.

transformations ... ", "... changing sky and earth ... " meant imitating the good Yao: the *Yijing* was, after all, the *Classic for Changes*.

Even if observed from such a distance in space and in time, Chinese scientific culture will not seem to be sufficiently homogeneous to be summarised in a few fixed characteristics. It is only at the end that we will see what the historical context of Chinese values had filtered, and caused to prevail over the rest. As regards the theory of music, it was also sustained, with equal force, that only the length of the *lülü* needed to be adapted, leaving the opening unchanged.[25] Anyway, the *lülü* continued to represent the standard way of tuning a great variety of musical instruments made of different materials, in which, as well as the breath, also the strings, membranes and bronze vibrated. The event that the note taken as a reference was the *Huangzhong* [yellow bell] revealed the interest of the Chinese in bells. They hung them together, thus constructing big *carillons*, from which they obtained all the notes desired. Casting them out of valuable bronzes with an elliptical section, they even succeeded in obtaining two distinct notes from each bell.[26]

An even more singular instrument, which appears only in China, is the *qing* [chime-stones] (Fig. 3.1), made up of a series of L-shaped cut stones of various appropriate dimensions. In their tuning, both the two-tone bells and the *qing* presented delicate problems, far more complex than those of stringed or wind instruments.

Of course, like all musicians, they could not forego the use of their ears, and supported them with the *lülü*. The longer ones emitted *zhuo* [turbid, muddy] notes, like elderly people, and the shorter ones *qing* [clear] notes, like young people. As regards sounds in China, the language constructed a metaphor, using a symbolism that referred to a watery *continuum*, inside which the suspended particles are concentrated at the bottom. This can be seen also in the root 'water' contained in the characters. In Chinese cosmology, the same characters, *qing* and *zhuo*, were used to describe the birth of the sky, clear, and the earth, turbid, in the primordial *hundun* [chaos], which preceded the differentiation of the *qi*.[27]

To recall the Greek and Latin culture examined above, and to start to appreciate its difference, the terms used in the West to distinguish sounds derived from other parallels. The Greeks wrote that a sound could be $\beta\alpha\rho\upsilon'\varsigma$ or '$o\xi\upsilon\varsigma$, which were translated into Latin as *gravis* [deep] or *acutus* [acute], "low" or "high": heavy and attracted downwards, or elevated and penetrating.[28]

This success on obtaining harmonious sounds from solid stones, and not only from thin wires made of silk, meant that Chinese culture though that music could be produced by the earth, that harmony was also possible here, where we live. And those first three notes of the *Hanshu* became 5 or even 12, when placed in correspondence with other aspects of life and of the world.

[25] Needham & Wang & Robinson IV 1962, p. 212.

[26] Chen 1994.

[27] Needham & Wang II 1956, p. 373.

[28] Tonietti 2003b, p. 229.

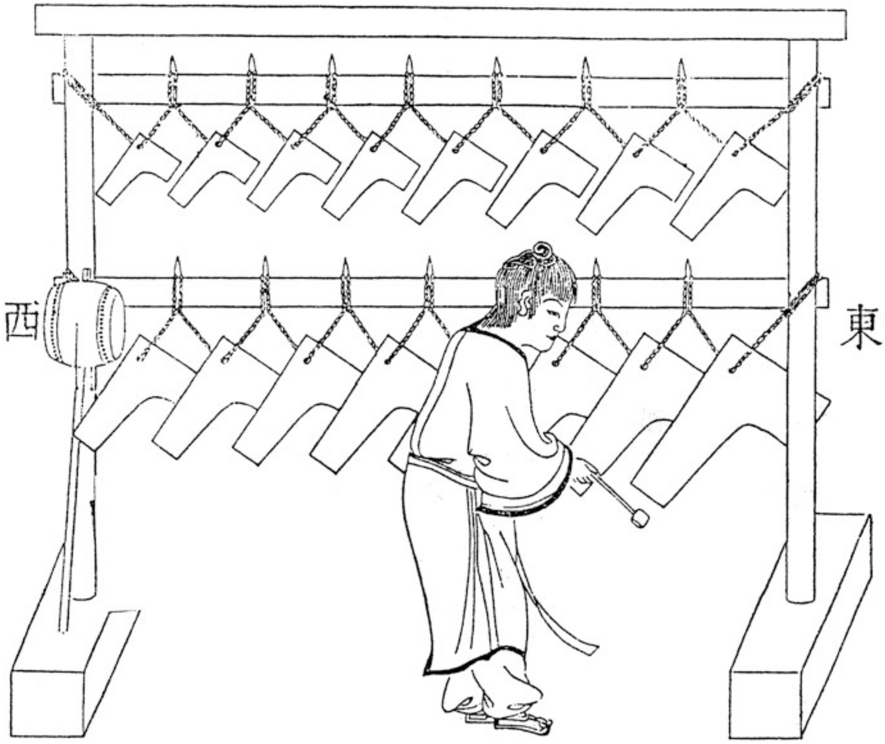

Fig. 3.1 Carillon *qing* made of chime-stones (Needham 1962, vol. IV, Fig. 304 p. 148)

An exposition of the Chinese theory of music can be found in various books, and has been the subject of various studies and commentaries.[29] Perhaps the most ancient of these was the document of the Zhou period (eleventh century B.C.–221 B.C.) *Yueji [Memories of music]*, later incorporated into the other *Yueji* (first century B.C.) of the Han period, and the subject of commentaries by subsequent historians. However, we quote the late version of Cheng Dawei (1533–1606), because this compendium of the orthodoxy, what's more, inserted into a text-book of mathematics, has been ignored by historians both of music and of mathematical sciences.[30]

In 1592, Cheng Dawei published some pages dedicated to music, among others dealing with mathematics. On the first of these, a figure with five notes stood out (Fig. 3.2). To us Europeans, it might look like a star, but in it, rather, we should

[29]Needham, Wang & Robinson IV 1962, pp. 126–228. Levis 1963. Granet 1995. Chen 1994. Chen 1996. Chen 1999. Major 1994, and others.

[30]Cheng 1592. Tonietti 2003b, pp. 228–233.

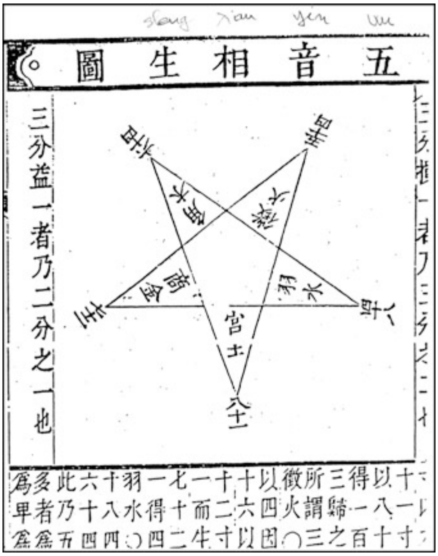

Fig. 3.2 The five Chinese notes generated from one another (Cheng Dawei 1592, fig. wu yin (5 notes))

see five musical notes distributed along a circumference, connected together in the manner indicated by the lines:

> wu yin xiang sheng
> [Five notes generated from one another.][31]

In the figure, the note *Gong* [palace] generates *Zhi*. As the value of the former is 81 and the latter 54, the operation consisted of multiplying by $\frac{2}{3}$: a downward generation. The note *Shang*, instead, was derived from the second one by multiplying by $\frac{4}{3}$, and thus it has a value of 72: an upward generation. The note *Yu* [feather] was derived from *Shang* by multiplying again by $\frac{2}{3}$, thus obtaining a value of 48, a downward generation. Finally, the last one, *Jiao* [horn] was derived from *Yu* by multiplying by $\frac{4}{3}$, and its value is 64, an upward generation. The procedure went under the name of *Sunyi* [decrease increase]. In the *Yijing*, *Sun, yi* are the characters of hexagrams 41 and 42. Hence, many cycles of phenomena are seen as "decreasing and increasing", starting, naturally, from the phases of the moon.[32]

The value 81 attributed to the starting note *Gong* was justified by starting from 9 *cun* (c. 30 cm), corresponding to the *lülü* of the *Huangzhong* [yellow bell] note, and multiplying it by itself. The numbers represented the length of the musical pipes which emitted the corresponding notes. $81 = 9 \times 9$ was the same number already considered in the *Hanshu*.

[31]Cheng 1592, p. 977

[32]*Yijing* 1950, pp. 191 e 194; *Yijing* 1994, pp. 453 and 462.

Cheng associated the five notes with the five *xing* [phases[33]] of the Chinese world: *tu* [earth], *huo* [fire], *jin* [metal], *shui* [water] and *mu* [wood]. As the lengths of the pipes decreased and increased, they were placed around a circle, as in a Zhou court. Is it purely a curious coincidence that this character *zhou* means "circumference, cycle, rotate"?

The ratio between *Gong* and *Zhi* was 3:2, like the one between *do* and *sol* [in the Greek (orthodox) system]; the ratio between *Zhi* and *Shang* was 3:4, like the one between *re* and *sol*; hence the ratio between *Gong* and *Shang*, *Shang* and *Jiao* or *Zhi* and *Yu* would be 9:8, like the one between *do* and *re*, *re* and *mi*, or *sol* and *la*. However, if we wanted to unroll the circumference along the Western musical scale, the similarities would finish here. The note *Jiao* was connected to *Gong* by the ratio $8 \times 8{:}9 \times 9$ [corresponding to the Pythagorean ditone] and cannot be again connected by the ratio 2:3, because this would produce a pipe whose length would be $64 \times \frac{2}{3} = \frac{126+2}{3} = 42 + \frac{2}{3}$. Thus this procedure could not generate a pipe whose length is $40 + \frac{1}{2}$, which would correspond to the European octave of *Gong*.

The Western octave would generally appear to be absent from Chinese musical theories. Together with the *Huangzhong* of 9 *cun*, we only find the case of a *Shaogong* [lesser palace] of $4\frac{1}{2}$ *cun*.[34] However, the orthodox position of the "decrease, increase" system, which did not include it, always remained steady, as we understand from the description of it given by Cheng Dawei, who ignored this. Obviously, it is likely that some musicians played octaves on their instruments, but they were generally excluded by theoreticians.

Huangzhong was the first note also of another broader system, which included 12 *lülü*.

> lülü xiang sheng
> [tuned pipes, generated from one another.][35]

The circle was now divided into 12 equal parts, linked together as in the following figure (Fig. 3.3).

The construction again followed the "decrease, increase" rule. Six pipes were classified as Yang: *Huangzhong* [yellow bell], *Dacu* [big cluster], *Guxi* [purification of women], *Ruibin*, *Yize* [sure rule], *Wushe* [without discharge], and six as Yin: *Dalü* [big pipe], *Jiazhong* [compressed bell], *Zhonglü* [central pipe], *Linzhong* [bell of the woods], *Nanlü* [Southern pipe], *Yingzhong* [bell that answers]. By "decreasing" the Yang pipe became Yin, and by "increasing" the Yin pipe was transformed into Yang. In the first case, the result was a downward generation, and in the second, an upward one. Thus we find a continuous process of generation, with continuous exchanges

[33]The character *xing* is sometimes translated as "element", but this is misleading, because Chinese culture represented everything in movement, exalting the possibilities of its transformation. Thus the *xing* are more like the phases of water, which is transformed into ice or steam, and can return to the liquid state.

[34]Needham & Wang & Robinson IV 1962, p. 214.

[35]Cheng 1592, p. 978.

Fig. 3.3 The 12 Chinese notes generated from one another (Cheng Dawei 1592, fig. lü lü (12 notes))

of qualities between Yang and Yin, making clear the dynamic characteristics of the procedure followed.

Each of the 12 *lülü* were assigned a *hou* [climatic season] chosen among the 24 *jieqi* [solar terms].[36] For example, *Huangzhong* corresponded to *Dongzhi* [winter solstice]; *Linzhong* recalled the *Dashu* [great heat] and so on. Furthermore, the figure indicates, together with the names of the *lülü*, the 12 *dizhi* [earthly branches][37] in their relative order. Thus the first *lülü* corresponds to the first *dizhi*, that is to say *Zi* [son] and so on. The *dizhi* were also used to indicate couples of hours during the day; for example, *Zishi* [time of the son] indicates the hours from 11.00 p.m. to 1.00 a.m., and the other *dizhi* follow the order of the hours.

The length of each pipe was calculated, starting from the first one of 9 in. Cheng immediately wrote that *lü* whose length was a whole number were obtained only in the first three cases. After that, fractions appeared.

Foregoing the beauty of a circular figure, we present the lengths of the 12 pipes, for practical reasons, in the form of a table (Table 3.1). This would be the order obtained by going round the circle anti-clockwise. It is different from the order of the generation obtained by following the lines of the star.

Cheng calculated the lengths of the pipes on the basis of the "decrease, increase" rule, obtaining the notes in the order 1, 8, 3, 10, 5, 12, 7, 2, 9, 4, 11, 6. He began by multiplying the 9 *cun* of the first pipe by $\frac{2}{3}$, and then the 6 of the second pipe by $\frac{4}{3}$, and so on, alternately. But when he arrived at the seventh pipe, *Ruibin*, and wanted to generate the following *Dalü*, the second pipe, instead of multiplying by $\frac{2}{3}$,

[36]In the Chinese calendar, the year was divided into 24 periods of 15 days.

[37]These correspond to the months.

Table 3.1 The length of Lülü, and their correlation with earthly phenomena

Lülü	Length (cùn)	(fen)	Solar terms	Hou	Earthly branches	Time
1. Huáng Zhong	9	810	22	Winter solstice	1 zǐ	23–1
2. Dà Lǔ	$8\frac{104}{243}$	$758\frac{42}{81}$	24	Great cold	2 choǔ	1–3
3. Dà Cù	8	720	2	Water and rain	3 yín	3–5
4. Jiá Zhong	$7\frac{1075}{2187}$	$674\frac{58}{243}$	4	Spring equinox	4 mǎo	5–7
5. Gu Xǐ	$7\frac{1}{9}$	640	6	Rain and corn	5 chén	7–9
6. Zhòng Lǔ	$6\frac{12974}{19683}$	$599\frac{707}{2187}$	8	Full of corn	6 sì	9–11
7. Ruí Bin	$6\frac{26}{81}$	$568\frac{8}{9}$	10	Summer solstice	7 wǔ	11–13
8. Lín Zhong	6	540	12	Great heat	8 wèi	13–15
9. Yí Zé	$5\frac{451}{729}$	$505\frac{55}{81}$	14	Limit of heat	9 shen	15–17
10. Nán Lǔ	$5\frac{1}{3}$	480	16	Autumn equinox	10 yoǔ	17–19
11. Wú Shè	$4\frac{6524}{6561}$	$449\frac{359}{729}$	18	Descent of frost	11 Xu	19–21
12. Yìng Zhong	$4\frac{20}{27}$	$426\frac{2}{3}$	20	Light snow	12 Hài	21–23

he repeated the multiplication by $\frac{4}{3}$, that is to say, again an upward generation. He justified this change as follows:

> nai sanfen yiyi zhi faci you buke xiao zhe.
> yi xia zhi yiyin shisheng zhi guyu.
> [It is impossible to explain this rule, which again increases the 3 parts by 1 [$\frac{3+1}{3}$]. Or rather, may the reason be perhaps that at the summer solstice, the Yin starts to increase?][38]

Before *Ruibin*, Yang generated Yin by "decreasing" and Yin generated Yang by "increasing"; from this point on, however, the procedure was inverted, and Yang generated Yin by means of an "increase", and Yin generated Yang by means of a "decrease".

All the pipes were assigned the same *kongwei* [circumference] of 9 *fen*. Cheng systematically multiplied all the lengths of the pipes by this number, obtaining the numbers in the second column. Unfortunately, he did not offer any explanation of the reason for these numbers. Since the height of a note emitted by a pipe depends on its diameter, as we have seen above, was Cheng perhaps trying to take this effect into account? In any case, also for him, the now 12 *lülü* had to be solid.

The *Jieqi*, which are traditionally connected in Chinese culture to the *lülü*, suggest that this model of musical theory continues to be inspired by the *qi*. The

[38]Cheng 1592, p. 982. Thus Cheng's problem did not seem to be that of maintaining all the notes inside the octave, even if he actually obtained this result. Chen 1996, tried to recover the octave also by means of the more ancient traditional system of the 12 *lülü*. It is true that the numbers calculated here limit it to the range of the octave, but this happens, as Cheng points out, in order to maintain the concordance with the increase in Yin at the summer solstice.

qi is manifested in the weather and in time, which passes inexorably. It flows in the pipes, and vibrates in the musical notes. It is the *qi* of our breath that keeps us alive. It is this energetic fluid that rules all the manifestations of the world. It is, we repeat, a kind of material ether, which gives substance to the geometrical *continuum* of the universe. The importance of the *qi* for Chinese physics and music has already been pointed out on other occasions,[39] and consequently there is no need to insist here.

In the place of heavenly bodies, which were absent from the Chinese musical model,[40] here we find the *hou* and the *qi*. Thus the music of Chinese culture was not a music of the spheres, but rather of the atmosphere.

In the dominant European tradition, a scale of 12 notes was only introduced much later. The Chinese culture, on the contrary, considered 12 notes at a very early stage, starting from the late Han empire. They entered into the system of the calendar, and were seen in relationship to the *qi*.

> Lü yi tong qi lei wu
> [The *lü* use an interconnected system whose substance is similar to that of the *qi*].[41]

As in the circle of 5 notes, also in that of the 12 *lülü*, there was no consideration of the ratio of the octave. Furthermore, Table 3.1 shows that the *Zhonglü* was not tuned in the same way as the Pythagorean fourth, the value of which, instead, was $6\frac{3}{4}$.

3.3 The Figure of the String

In the history of Chinese mathematical sciences, the most ancient book is probably the *Zhoubi suanjing [Classic for calculating the gnomon of the Zhou]*. This is a stratified text, for which it is no longer possible to fix any precise date. The subject is a matter of controversy among historians. As it speaks of the Zhou, the oldest layer may date back to their pre-imperial period. The book was probably consolidated and fixed in its present form during the Han period. Some date it to c. 100 B.C., others to c. 100 A.D. Subsequently, it was accompanied by the commentary of Zhao Shuang (third century), who started to explain it and interpret it. It was quoted in a catalogue of the Sui period (581–618). The Tang (618–906) included it among the texts of the Imperial Academy, adding the title of *jing* [classic]. The Song (960–1127) printed it in 1084; a copy of the reprint produced in 1213 still exists today in Shanghai. Without doubt, the *Zhoubi* is to be considered the most ancient printed book dealing with mathematical sciences in the world. Euclid's work, the *Elements*,

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39 Needham, Wang & Robinson IV 1962. Chen 1996. See Sect. 3.5.

40 Even if the five planets were sometimes associated with the same character as the five *xing* [phases].

41 Cheng 1592, p. 1002. Granet 1995. Needham & Wang & Robinson IV 1962. In a Taoist ritual, music had an important role. Their "Eight notes of harmony" took part in the primeval unfolding of the universe. Schipper & Wang 1986, pp. 193–194.

was printed four centuries later, in 1482.[42] Firstly, the fundamental property of right-angled triangles (seen by us from a Greek point of view in Sect. 2.4) was taken into consideration. After that, astronomical problems were discussed. Between the most ancient layer and the comments by Zhao Shuang, we find a diagram: *Xiantu* [the figure of the string]. I will give my own translation and interpretation of this, as it has long been a controversial matter.[43]

> Long ago, Duke Zhou enquired of Shang Gao: I have heard it rumoured that you are an official well versed in calculations.
>
> ∗ Duke Zhou's surname was Ji, and his first name was Dan; he was the younger brother of King Wu[44]; Shang Gao is said to have been a competent official during the period of the Zhou dynasty, also very skilful at performing calculations; Duke Zhou, who was a member of the royal household, governed with virtue and wisdom; he continued to consider himself humble in view of his low level of education, and he may have allowed all situations to rise to his authority.
> May I ask how Bao Xi[45] established the dimensions of the heavenly sphere and the degrees of the calendar in ancient times?
>
> ∗ Bao Xi, the first of the three emperors, started the book with the eight trigrams[46]; in the same way, Shang Gao skilfully led the authorities to perform correct calculations, whether of square shapes or large objects, to be placed together or far away, even at the extremes; the illustrious Bao Xi established the dimensions of the heavenly sphere and the degrees of the calendar, and devised the rules governing the variations of the *zhang* and the *bu*[47]; according to the ancients, Bao Xi, who was a member of the royal family, also spoke of the event that the criterion for the observation of the heavens above us depends on all that lies below them: the investigation of the earth below us acts as a rule for our observations.
> But there is no staircase that allows man to climb the heavens, nor is there any ruler, *chi*,[48] that can be used to measure the dimensions of the earth.
>
> ∗ Is it not true that [the sky] is too far away, and too vast, to be reached by means of a staircase? Would it not take too long, and would it not be a waste of time, because [the earth] is too large for us to measure its dimensions?
> May I ask you, where do these numbers come from?
>
> ∗ In his ignorance, [Duke Zhou] has his chance: he can ask for this to be shown to him.
> Shang Gao replied that the rule for the calculations derives from the circle and from the square.
>
> ∗ If the diameter, *jing*, of a circle is 1, the extension of the circumference will be 3; if the side, *jing*, of the square is 1, the extension of the town, *shi* [the perimeter], will be 4; if you take the circumference of the circle as the base, *gou*, and if the perimeter of the

[42]Tonietti 2006a, p. 92.

[43]*Zhoubi suanjing*. The original Chinese can also be found in Tonietti 2006a, pp. 31–40. I indicate with an ∗ the comments of Zhao Shuang.

[44]The first king of the Zhou.

[45]He usually is called Fu Xi. Bao Xi is the mythical king, to whom Chinese tradition ascribes the first discoveries and inventions. Needham & Wang 1954, v. 1, p. 163.

[46]The eight combinations, three at a time, of the continuous — and broken – – lines, which are the basis of the 64 hexagrams of the *Yijing*.

[47]*Zhang* and *bu* are the resonance periods, of 19 and 76 years respectively, which the revolutions of the Moon and the Sun approximately agree in. Needham and Wang 1959, v. III, p. 406.

[48]*chi* is the rule, similar to a foot. Now, it is one third of a meter.

square is taken as the height, *gu*, together they create a folded angle, *xie*; if these folded, *xie*, circle and square are connected with the proportion of a straight line, *jing*, the string *xian* [hypotenuse] 5 is obtained. For this reason, he says that the rule for the calculations derives from the circle and the square; the shapes of the circle and the square are those of the heavens and the earth, they are the numbers of Yin and Yang; but then, Duke Zhou asks, what does this imply as regards the heavens and the earth? Shang Gao explains the shapes of the circle and the square; in order to illustrate their image, he elaborates their rules on the basis of calculations using the odd [3] and the even [4]. By using so-called simple language to discuss them, he explains [subjects] that are distant, delicate, and truly profound!

The circle derives from the square, and the square from the set-square rectangle, *ju*.

∗ By using the square, we obtain the mathematics of the compass for the circle, and also that of the circumference and the perimeter for the square. The substance of a correct square derives from using the set-square rectangle, *ju*; a set-square-rectangle, *ju*, has its width and length.

The set-square rectangle, *ju*, derives from $9 \times 9 = 81$ [from the multiplication tables].

∗ In order to obtain the proportion of the circle and of the square, unite the numbers of the width and the length; it is necessary to multiply and to divide in accordance with the 9×9 of counting, which lies at the origin of multiplication and division.

Thus determine the properties of the set-square rectangle.

∗ This reason accounts for the way this is expressed, and goes on to provide the proportions of the base, *gou*, and the height, *gu*; this is why he says: 'determine the properties of the set-square rectangle'.

Let the width of the base *gou* be 3.

∗ This corresponds to the circumference of the circle [3]; the horizontal line which is also called the wide base, *gou*, or the width, is the width of the set-square rectangle, *ju*.

Let the extension of the height, *gu*, be 4.

∗ This corresponds to the perimeter of the square [4]: it is called the extension of the height, *gu*, or also the extension, and is the tall, thin side [of the set-square rectangle].

The straight line, *jing*, of the angle *yu* [the hypotenuse] is 5.

∗ The proportions which correspond naturally, *ziran*, to each other [are those of] the straight line, *jing*, and the right angle, *zhiyujiao*, and this line is also called the string, *xian* [hypotenuse].

Now [construct] the half set-square rectangles, *ju*, and the relative external squares, *fang*.

∗ The rule of the base, *gou*, and the height, *gu*: initially, you know two numbers, and after this, you obtain another: take the base, *gou*, and the height, *gu*, and then find the string, *xian*. Each of them is first multiplied by itself, thus becoming the square number, *shi*; having fixed the squares, the signs are changed in order to adapt them [construction]. Consequently he says: 'Now their external squares'. The squares, *shi*, of the base, *gou*, and the height, *gu*, are to be united in order to find the string, *xian*. Consequently, in the procedure towards the square of the string, *xian*, try to divide and recombine those of the base, *gou*, and the height, *gu*; the squares, changed one into the other, and not taken just equal, in a certain way can support each other. This is why he speaks of: 'half set-square rectangles, *ju*'. The art, *shu*, [also technique, method] of *gougu* [lies in this]: each is multiplied by itself, $3 \times 3 = 9$, and $4 \times 4 = 16$; adding these together, they give the square number, *shi*, 25 of the string, *xian*, multiplied by itself; by subtracting the base, *gou*, from the string, *xian*, you obtain the square 16 of the height, *gu*; by subtracting the height, *gu*, from the chord, *xian*, you obtain the square 9 of the base, *gou*.

By placing them [the half set-square rectangles] all around, *huan*, on a draughtboard, *pan*, it is possible to obtain 3, 4 and 5.

∗ The draughtboard, *pan*, is to be interpreted as the Huan kind. This board demonstrates: take these [half set-square rectangles], combine them, *bing*, all together, slanting them, *qu*, and placing them all around, *huan*, take away, *jian*, the total, *ji*, [$12/2 \times 4 = 24$] from the draughtboard, *pan* [49] and extract the square root, *kaifangchu*, of the surface, *mian*, [25] obtained. This is why he says: 'it is possible to obtain 3, 4 and 5'.

Putting together two set-square rectangles, *ju* [$2 \times 12 = 24$], and growing on, *zhang*, 25 [$24 + 25 = 49 = 7 \times 7$, the draughtboard] is called 'accumulating the set-square rectangles', *jiju*.

* The two set-square rectangles together [24] with the growing, *zhang*, the squares of the base and the height, each multiplied by itself [$3 \times 3 = 9$, $4 \times 4 = 16$; $9 + 16 = 25$] combine to form the square number [$24 + 25 = 49 = 7 \times 7$]. This will apply in ten thousands of things, but in order to do so, you must first establish that proportion.
  This is the reason why Yu gave order to these numbers, to regulate everything that lies beneath the heavens.
* Yu regulated the great waters, and organised and dredged the two rivers, the Jiang ['the long river', the Yangze] and the He ['the yellow river']; he contemplated the forms of the mountains and the rivers, set right the geographical configurations in their height and depth, eliminated the calamity of the great wave [flooding], removed the disasters caused by the confused filling [of the banks?], and led the waters to flow eastwards into the sea, and not to return back; this is how the technique of *gougu* originated.

In the elliptic manner to be expected from such an old Chinese text, this page suggests how to construct the following figure (Fig. 3.4). Here, therefore, with the aid of the diagram, he explained and demonstrated the fundamental property of the right-angled triangle. If the length of the *gou* [base] is 3, and that of the *gu* [height] is 4, then the *xian* [hypotenuse] is equal to 5. It is represented as the side of a square whose area is 25: seeing that it is the union of four semi-rectangles ($6 \times 4 = 24$) and the small square in the centre. The proof is generally valid (and not only for these particular numbers) because it is sufficient to cancel the network in the background, and not to measure the lengths, which certainly help to create the diagram, and have a ritual significance, as the text explained, but are not indispensable for the argument.

Now, then, cut out the two right-angled triangles at the top, which compose the square on the *xian* and move them, uniting them to the two at the bottom. Together with the small central square, these form the sum of the of the squares constructed on the *gou* and on the *gu*. Therefore the square on the *xian* is equal to the sum of the squares on the *gou* and on the *gu*[49] (Fig. 3.5).

Certain characteristics of the Chinese scientific culture can be inferred from the proof. The figure of the string was manipulated as if it were a material object. It was taken to pieces, which were then moved with the hands to obtain the result. Somewhat similar to the Chinese game known as *qiqiaoban* [tangram].

The figure represented a process of transformation from the initial situation, the square on the *xian*, to the final one, the squares on the *gou* and the *gu*; or vice versa. It is a proof in movement.

The figure was a part of the text, and without it (with only the characters), the text would have been highly obscure and ambiguous.

In the figure, the proof could be seen all together, at a single glance. No intermediate steps were necessary, as would have been the case with a simple description in characters. The proof can be seen directly and literally.

[49]Tonietti 2006a, p. 86.

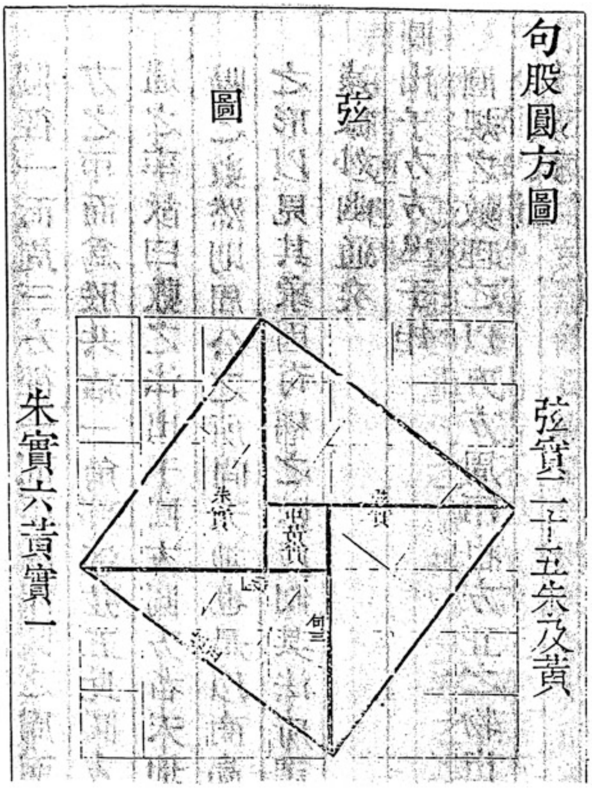

Fig. 3.4 The figure of the string (*Zhoubi suanjing*, fig. Xiantu (picture of the string); Henan jiaoyu chubanshe, Shanghai)

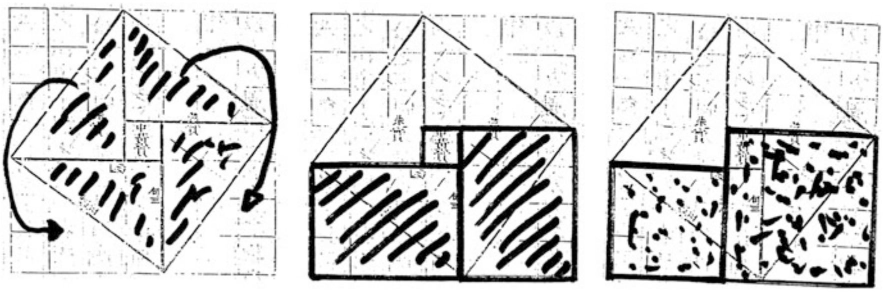

Fig. 3.5 Geometrical argument for the fundamental property of right-angles triangles (Tonietti 2006a, Fig. 3, p. 86)

Numbers, which were so important in the Chinese culture, represented measurements of length and area. Thus they are always to be taken together with geometry, and never separated from the material objects under consideration. Vice versa, geometrical figures could always be measured without any obstacles or prohibitions.

Between numbers and geometry of the circle, of the square and of the *ju* [set-square rectangle], a virtuous circle was created of reciprocal references: to be used at will, whenever expedient and useful. The Pythagorean prohibition of irrational numbers was wholly absent in China.

The page of the *Zhoubi* did NOT contain simply a rapid recipe, a rule to calculate with right-angled triangles in the various cases of need, for astronomers, metal-workers and carpenters, builders of canals, and so on. There was also an explanation of WHY that rule worked, and why it was capable of dealing with "…everything under the Sun." The reader was not only to be instructed about what was to be done, but also convinced of the reason; in a word, the *Zhoubi* offered a proof, and the Chinese way of proving was different from that of Euclid.

It also contained other typical aspects of Chinese culture, such as the *Yijing*, the close connection between sky and earth, the importance of the calendar, the interest in how to organise the empire practically and efficiently. The sky was observed from the earth: "…the criterion to observe the sky depends on what is under the sky, looking down at the earth acts as a rule for this observation." With the shadow of a pole planted in the ground, and the shadow of the meridian or gnomon, the movement of the Sun was studied.

In the *Zhoubi*, there was an indication of the correct procedure to follow in order to arrive at a useful result. That is, they showed how to construct the result, with a diagram and with characters. This was done in a concrete manner, avoiding an abstract presentation of affirmations, but at the same time, sustaining their general validity.

Between the most ancient layer and the comments of Zhao Shuang, we may observe an evolution in the terms used for the reasoning. The "straight line" [hypotenuse] that closes the right angle was originally called *jing* [straight line] and then *xian* [string, also the string of musical instruments, or a bow].

In archaic times, *ju* meant "a carpenter's set-square". This was visible in the hands of Bao Xi in pictures of him. It was also visible in the hands of the sage Liu Tianjun, together with the bell. The *ju* appears to be closed by a small triangle. In the *Zhoubi*, the *ju* was used to construct rectangles and triangles, and later evolved into a right-angled set-square. With the passing of time, *juxing* [in the shape of a right-angled set-square] subsequently assumed the geometrical meaning of a "rectangle".

Gougu meant a right-angled triangle. In China, the sides [catheti] that meet at the right angle cannot be exchanged, because *gou* (which measured 3) was always the shorter, seeing that it represented the sky, whereas *gu* (which measured 4) remained the longer, symbolising the earth. Sky and earth were united by the *xian*. On the basis of these numbers, 3, 4, 5, we have seen that the lengths of the *lülü* [standard musical pipes] were fixed.

Given the lengths of the *gou* and the *gu*, and the condition that the angle was a right angle, the proportion of the *xian* was established *xiran* [in a natural manner].

Shi meant both the square of a number, and the geometrical figure of a square.

Initially, everything was justified by the purpose of obtaining "…the dimensions of the heavenly sphere and the degrees of the calendar."

With the passing of time, the book was to create a tradition for problems concerning right-angled triangles, the *gougu*, establishing a regular procedure for similar cases, and suggesting a characteristic way of demonstrating the results: here, it was indicated as *jiju* [accumulating right-angled set-squares]. Subsequently it was to be defined in other ways. Liu Hui (III sec.), who we shall meet in Sect. 3.4 below, summed it up as follows:

lingchu ru xiangbu
[[One] is made to go out, and [the other] to come in; they compensate for each other.]

Xu Chunfang (a twentieth century historian) has called it *yanduan* [developing the parts]. Nowadays, others speak of the "Out-In complementary principle". It presents itself as a proof in movement, in which a figure is disassembled and reassembled, changing its shape and maintaining the areas.[50]

When European scholars came into contact with the *Zhoubi*, many of them, too many, denied that it contained a 'true' demonstration. They took as an absolute, universal criterion not only the model of Euclid, but, even worse, in a totally anachronistic manner, the axiomatics of proofs formalised in the twentieth century.[51] I will return in Sect. 3.6 to the reasons why this page of the *Zhoubi* was treated so badly, with the due exceptions; there, the necessary questions connected with the different historical contexts and the prejudices of relative scholars will be discussed.

3.4 Calculating in Nine Ways

Together with the *Zhoubi*, the other ancient book about Chinese mathematical sciences was the one called *Jiuzhang suanshu [The art of calculating in nine chapters]*. This, too, probably contained pre-imperial layers, but it was consolidated during the Han period and was the subject of a commentary by the most famous ancient Chinese mathematician: Liu Hui (third century). He subsequently developed Chap. 11, dedicated to right-angled triangles, in another original book of his, *Haidao suanjing [The classic for calculating the island in the sea]*.[52]

The text included 246 problems for which it supplied the solutions calculated, often explicitly, but not always, by means of appropriate rules. The first chapter dealt with measuring fields to be cultivated. There were 38 examples, accompanied by procedures to calculate the areas, and to work with fractions. The fields had the shape of rectangles, triangles, trapezia, circles, segments and annuli. Sometimes, Liu Hui used procedures similar to those seen for the figure of the string.

In the second chapter, *lü* [proportions] were used to calculate the yield of cereals like millet, corn or rice, when they were hulled. The prices of these, and other

[50] Needham 1954, p. 164. Tonietti 2006a, pp. 108–116.

[51] Tonietti 2006a, pp. 45–116.

[52] *Jiuzhang suanshu. Neuf chapitres* 2004, pp. 43–46. Martzloff 1988.

agricultural products, were also evaluated. In the third chapter, cereals were shared out and taxes were distributed in a similar proportional manner. Chapter 4 returned to the subject of the dimensions of fields, not only by means of divisions, but also through the extraction of square roots. At the end of this chapter, the diameter was calculated for a sphere whose volume is known, and for this calculation, Liu Hui followed his own method, arriving at a conclusion equivalent to that of the usual formula.

The practical problems of the fifth chapter regarded the movement of various kinds of earth, to construct walls, dams and canals. Thus it was necessary to calculate the volume of solids of widely varying shapes. Furthermore, calculations were made for the number of people to be employed in the work, bearing in mind the distance they would have to cover, and the volume of the baskets carried. In the sixth chapter, there was a consideration of how to impose taxes fairly, bearing in mind transport, customs, and so on. The problems of Chap. 7 were various in nature, but were dealt with by means of a common procedure, called in Chinese *Ying buzu* [excess and deficit], known in the West also as a "false position".

The *Fangcheng* [measuring according to the square] made it possible, in Chap. 8, to solve more complex problems, with several unknowns, and many variables. In our post-Cartesian, post-algebraic and post-symbolic age, the method followed in China can be compared with the systems of linear equations solved by means of matrices of numbers arranged in the shape of squares and rectangles, like the Chinese character *fang*. A typical example of a problem was the following one:

> Let us suppose that
> 1 functionary, 5 officials and 10 footmen eat 10 chicken.
> 10 functionaries, 1 official and 5 footmen eat 8 chicken.
> 5 functionaries, 10 officials and 1 footman eat 6 chicken.
> The question is, how many chicken would one functionary, one official and one footman eat? Answer: one functionary would eat $\frac{45}{122}$ of a chicken, one official would eat $\frac{41}{122}$ of a chicken, and one footman would eat $\frac{97}{122}$ of a chicken.[53]

The ninth and last chapter presented the procedure of the problems of the *gougu*, already seen in the *Zhoubi*. This had dealt with the heavenly bodies using right-angled triangles. Now, in the *Nine chapters*, the problems presented were connected rather with the earth: lianas twisted around trees, reeds in ponds, sticks, ropes, walls, doors, bamboo canes, towns, people, mountains, wells in various positions, for which it was necessary to calculate lengths, distances, heights, depths. The only purely geometrical problems, obviously preparatory for the others, were numbers 14 and 15, where either a square or a circle had to be inscribed inside a right-angled triangle.[54]

[53] It was curious that the footman got the biggest part. Was it because he toiled hardest? Or was his status higher? Perhaps the wings, considered to be the greatest delicacy in China, were always the prerogative of imperial functionaries.

[54] *Jiuzhang. Nine chapters* 1999. *Neuf Chapitres* 2004.

On the basis of the preface by Liu Hui, we can confirm and enrich those characteristic origins of mathematical sciences on which Chinese scholars constructed their dominant orthodox tradition. Thus we again find the *Yijing* paraphrased and quoted, with its hexagrams, and an insistence on the world in continuous transformation, in which everything is united to everything else. Sages, like Bao Xi, possessed the ability to "see" all this. Here, in the *Nine chapters*, numbers, represented by the 9×9 of the multiplication tables, and intended also as negative numbers, seem to take the first place.

The [yellow] emperor "regulated the calendar, tuned the *lülü* [standard pipes], using them to study the origins of the *Dao* [Way], also taking as his model the *qi* of the two *yi* [aspects], *jing* [bright] and *wei* [deep] and the four *xiang* [images]."[55]

The "cruel" first emperor, Qinshi, is said to have "burnt the books" which he did not like, those by Confucius. But while the book about music was truly lost, others survived, including those dealing with mathematics. There will be a need to return to the subject of this famous bonfire of Chinese books, and the excessive use that historians sometimes make of it.

In order to achieve the *li* [texture, explanation], it was interesting that Liu Hui wrote about

> ... jieti yongtu ...
> [... disassembling the bodies, using figures ...].

This would seem to be the same attitude as that of Zhao Shuang towards the *Zhoubi*: "I designed the figures in the light of the book."[56] But, while the other commentator, following the most ancient layer, had actually left us the figure of the preceding section (Fig. 3.4), no figures remained of the *Nine chapters* and of Liu Hui. Were they all lost? Or did scholars know perfectly well all about the already famous figure of the string? In effect, it seems to me that the words "disassembling the bodies, using figures" refer to it all too clearly to be ignored. Furthermore, in his commentary on the 11th chapter, dedicated to the *gougu*, Liu Hui often mentioned figures coloured vermilion and blue-green, or red and yellow, although these are missing in the editions. It is from here that we have taken his formula, quoted above, of the procedure for the proof (of the fundamental property of right-angled triangles) which he also used in problems 14 and 15. The colours attributed to the diagrams not only made the argument more effective, ma also reinforced the feeling that it was something that belonged to the reality and the variety of this world.

Zhao Shuang clearly affirmed that there was a link between the most ancient text and the figures. Also Liu Hui often flaunted them. Consequently, we are forced to advance hypotheses about why so few of them are extant. Was it perhaps a habit of scholars to draw their figures themselves? Or have we lost above all those figures which are related to the most ancient layers? In ancient China, editorial conventions

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[55] The two *yi* were Yin and Yang, represented visually by the broken – – and continuous — lines. The four *xiang* were their arrangements two by two.

[56] Tonietti 2006a, p. 63. *Neuf Chapitres* 2004, pp. 126–127 and 665.

do not seem to have been the same as contemporary ones, where figures are quoted in the text, and every book used must rigorously be indicated. Even in the West, the habit of giving detailed bibliographies only appeared in recent academic times. Does anybody really think that Newton did not know about Kepler because he hardly ever quoted him? I believe that the same is true for the figures of the Chinese texts under examination.[57]

In commenting on problem 18 (a complicated calculation of prices with five unknowns for cereals and legumes) in Chap. 8, Liu Hui proposed a variant on the usual *fangcheng* procedure. In so doing, he compared himself to the butcher of the *Zhuangzi [Master Zhuang]*,[58] who quartered an ox, rhythmically and with a musical harmony, following the empty spaces, without touching the bones. Thus his knife remained sharp for many years. Analogously, the Chinese scholar succeeded in disassembling the *li* of the procedure, avoiding the need for too much *shu* [calculating]. If the result is sought with *heshen* [the mind in harmony], thus saving the *ren* [cutting edge of the knife], it is obtained more speedily and with fewer mistakes. Otherwise, the calculations would be

> . . . sijiao zhudiao se zhilei.
> [. . . similar in that case to tuning the *se* [the ancient lute] with the pegs glued up.][59]

Here, we not only again find the music and harmony of Confucius, but also the attitude that transpired from the references to one of the most famous Taoist books. Following the *Dao* [way] on the *Zhuangzi*, meant allowing things to find a solution by themselves, spontaneously, without forcing them: *wuwei* [not doing anything]. Instead of using the chopper with ignorant violence, indifferent to the *li* of the animal, that musical butcher allowed the meat to open up by itself, limiting his work to simply detaching it.[60]

We shall see in Chap. 9 that Francis Bacon (1560–1626), the famous philosopher of the modern experimental method, proposed a general idea of nature, to which his attitude was quite different, far less *he* [in harmony]. His approach was not a gracious "disassembly", but a bloodthirsty "dissecare naturam" [dissecting nature].

The problems presented and solved in the *Nine chapters* had been proposed because they were in various ways necessary for the life of the empire. The cases were described from a practical point of view, and almost always dealt with the

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[57] *Neuf Chapitres* 2004, pp. 704–705, 709, 719, 726–729, 745. *Nine chapters* 1999, p. 459. It is disconcerting that some historians have tried to cut the link, explicitly stated by Zhao Shuang, between the ancient layers and the figures; Cullen 1996; Karine Chemla in *Neuf Chapitres* 2004, pp. 673ff.

[58] *Zhuangzi* 1982, pp. 33–34.

[59] *Neuf Chapitres* 2004, pp. 651–652.

[60] The proximity of Liu Hui to the *Dao* has not been completely neglected, but somewhat underestimated, both by the *Nine chapters* translation, and by the *Neuf Chapitres* version, where the *Dao* is often translated as *méthode*, as if we were talking about Descartes, and the *li* as *structure*, as if we were still in the period of structuralism. But cf. Guo Shuchun in *Neuf Chapitres* 2004, pp. 61–62 and Martzloff 1988, p. 126.

calculation of numbers. Chinese mathematical sciences, as they would appear if we only read this book, would thus seem to be dominated by the need to calculate "ten thousand things" in all the possible ways, so as to be able to use them. In our great epoch, historians have not failed to point out similarities, if not actual anticipations, with present-day calculation procedures linked to the computer: the term "algorithm" is widely used to characterise Chinese mathematics.

This is the opinion even of Joseph Needham (1900–1995), the scholar to whom we will always be grateful for giving us a patient historical study on the Chinese sciences, admirable and impressive.[61] Though animated by the best intentions of understanding and respecting Chinese scientific culture without subjugating it to that of the West, he ended up by writing, "Now algebra was dominant in Chinese mathematics as far back as we can trace it ...". For him, "...the genius of Chinese mathematics lay rather in the direction of algebra." "...while geometry was characteristic of Greek, was algebra characteristic of Chinese mathematics?"[62]

It is certainly true that the Chinese loved and love to count everything: the "five classics", the "four books", the "three obediences", the "four virtues", the "three religions", the "nine schools", the "four directions", the "four seas", the "five flavours", the "five mountains", the "five colours", the "five organs", the "six animals", the "six *qi*" (wind, cold, summer heat, dampness, dryness, fire), the "six relationships" (of kinship), the "six arts" (rites, music, archery, cart-driving, calligraphy, mathematics), the "seven openings" (of the head), the "seven emotions", the "eight directions", the "eight characters", the "four modernisations", the "band of the four", the "hundred families", the "wall of the ten thousand li" and the list could go on, up to the "10,000 things" and "10,000 years" *Wansui!*, which means Cheers!.

Without any doubt, in this *Zhongguo* [Country in the centre, the Chinese name for China], numbers had, and have, the most varied and variable symbolic meanings,[63] as we have also seen in the *Zhoubi*. The *Yijing* started from 50 stems of *Achillea millefolium*, and by dividing and counting arrived at the numbers 6, 7, 8, 9, which indicate one of the 64 hexagrams.[64]

The units of measurement for the Han empire were established in accordance with the *lülü*, the calendar and the *Nine chapters*, declared a senior functionary of the Treasury. "The *lülü* and the calendar are based on the *Huangzhong*,[65] the

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[61] Needham & many others 1954–2004.

[62] Needham & Wang II 1956, p. 292; III 1959, pp. 112, 91, 23. This rhetorical question must receive a negative answer from me. Engelfriet 1998, p. 444. *Nine chapters* 1999, pp. vii, 27 and *passim*. Also Karine Chemla, in *Neuf Chapitres* 2004, pp. 104ff., makes a widespread use of the term "algorithm". As a result, she was then forced to give a lengthy explanation that these calculations are not only algorithms. Then wouldn't it have been better to choose another word, less Arabic, less computer-friendly, less modern, less technical and more Chinese, that is to say, even geometrical-material?

[63] Granet 1995.

[64] Tonietti 2006a, pp. 226–228.

[65] As the *Huangzhong pipe* had a fixed length and diameter, it became the unit of measurement also for the volume of various kinds of goods, such as cereals.

uniformity of lengths, weights and sizes, and the adjustment of the Sun, the moon and the five planets are based on the *Art of calculating in nine chapters*, in order that everything inside the bounds of the seas will be in harmony."[66]

But now we have arrived at the right point to understand how the Chinese conceived of numbers, and used them, in a different way from the prevailing Western tradition. As the centuries passed, Chinese tradition gave birth to new ways of writing numbers, simpler and more suitable for calculations than the Greek and Latin ones. They indicated the value of a number by its position in the orderly succession of figures; these used nine symbols, to which another one was added to indicate an empty place: zero. Without going into the useless and harmful controversy about who invented it first, a Chinese text of the thirteenth century, *Lülü chengshu [Complete book of the lülü]* (once again, our standard pipes), for example, indicated it by means of a small square.[67]

Negative numbers were used for intermediate passages in calculations, for example, in the "procedure of the positive and the negative" in the *Nine chapters*. Lastly, numbers were represented by rods, which could be manipulated with the fingers, like the stems of the *Yijing*, to carry out calculations more easily. Fractions were commonly used, and not just ratios, to which the Greeks limited themselves. The *Dayan* [big development] principle became famous in the West under the name of "Chinese theorem [sic!] of the remainder", the procedure to calculate simultaneous congruences in indeterminate analysis. Also its name derived from the *Yijing*, where it indicated the initial pile of 50 stems. It was inspired by the calculation of the astronomic cycles necessary for the calendar. Carl Friedrich Gauss (1777–1855) was the Western mathematician of the nineteenth century who took the most interest in this subject for analogous reasons.[68]

Above all, the Chinese had an idea of the extraction of the square root *kaifangchu* [divide by opening the square], which united the operation of finding the side "by opening the square" to that of division between numbers. And they knew that in general, they would never arrive at the end of the calculation. Therefore Liu Hui called it *bujin* [no termination].[69] The procedure was geometrical, and as the name stresses, it consisted in *kai* [opening] the square, breaking it down subsequently into surfaces whose colour was yellow, blue-green or vermilion. For cubic roots, a cube was disassembled.

They had also invented a way to extract the roots of any order, and consequently they were able to determine unknowns even in problems with any power. This procedure may be compared with the methods of Ruffini or Horner, introduced at the beginning of the nineteenth century to solve an algebraic equation of any degree.

[66]*Neuf Chapitres* 2004, p. 57.

[67]Martzloff 1988, pp. 159–193.

[68]Needham & Wang III 1959, pp. 119–122. Martzloff 1988, pp. 145 and 296–305.

[69]*Neuf Chapitres*, pp. 718–719.

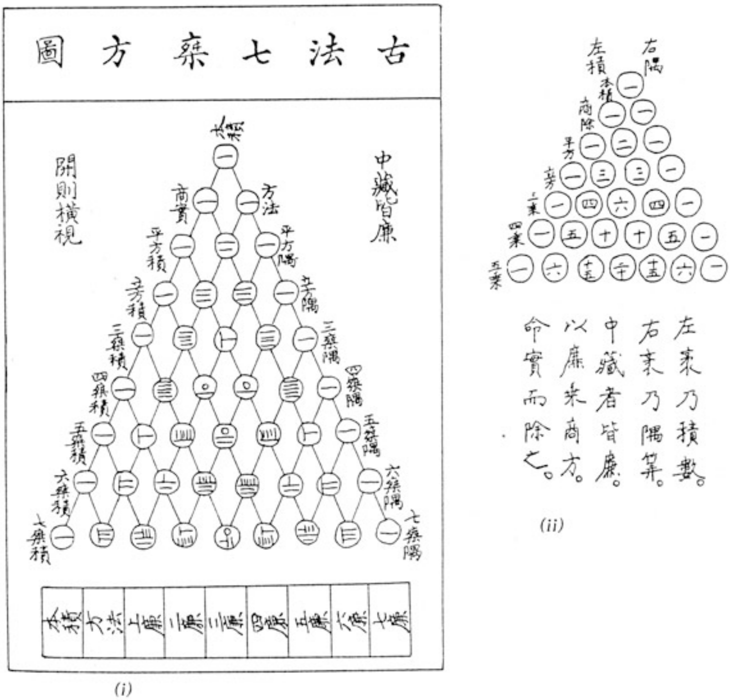

Fig. 3.6 The Chinese coefficients of the binomial in the shape of a triangle (Martzloff 1997, p. 231)

This was the context in which also coefficients of binomials appeared, known in the West thanks to the triangle of Pascal (Fig. 3.6).[70]

Zu Chongzhi (429–500) obtained for the *lü* [ratio] between the circumference and the diameter of the circle [π] a value of 3.14159292, correct up to the sixth decimal figure.[71]

In the close relationship between numbers and the things to be measured, in particular geometrical figures, with the idea that the extraction of the square root was not such a very different procedure from division, we do not find any trace here in Chinese mathematical texts of things that could not be measured. Everything produced numbers, some finite, others *bujin* [not finite], and all these could be used for their practical purposes without any limitation. The Graeco-Latin distinction between commensurable ratios, which could be expressed numerically by pairs of whole numbers, and incommensurable numbers, for which that was not allowed,

[70] Needham & Wang III 1959, pp. 126–128. Martzloff 1988, pp. 210–231.

[71] Needham and Wang III 1959, p. 101. Martzloff 1988, p. 262.

was completely foreign to Chinese culture. Here, then, the typical, centuries-long Western prohibition of numbers without a *logos* was never suffered.

At this point, it is legitimate to think that the above-mentioned ability in calculations derived precisely from this characteristic. However, it would be misleading and partial, to say the least, to reduce Chinese mathematics to numbers, or, worse still, to numbers constrained to Pythagorean symbology.[72] Let us repeat, then, here, from the exergue, what we read in chapter XLII of the *Daodejing* [Classic of the way and of virtue].

> dao sheng yi, yi sheng er, er sheng san, san sheng wanwu.
> wanwu fu yin er bao yang. chong qi, yiwei he.
> [The tao generates 1, 1 generates 2, 2 generates 3, 3 generates 10,000 things. The 10,000 things bring the Yin and embrace the Yang. Thanks to the *qi*, they then become harmony.][73]

Thus numbers were generated by means of a process, a way that entailed the alternation of Yin and Yang. Among the 10,000 things of the world, in spite of the variety and the continual transformations, relationship were produced in harmony. For this, the ancient Chinese thanked the *qi*. We shall see that in the Europe of the seventeenth century,[74] Leibniz postulated a 'pre-established harmony', pre-established by a transcendent divinity, thus misinterpreting Chinese mathematics.

All in all, the classical Chinese people thought of their numbers as forming a *continuum*. And they saw them as intimately connected with geometry. The *Daodejing* tells us that the reason lies in the *qi*.

3.5 The Qi

Guo Xiang (fourth century) wrote, in his comments on the *Zhuangzi*:

> Bude yi weiwu, gu zigu wuwei you zhi shi er chang cun ye
> [It is not possible for the same [something] to become nothing, and so from ancient times, there has never been an [initial] moment of existence, but in fact it exists continually].

Inspired by Taoism, therefore, he believed that the world had always existed, and represented it as a continuous process without any beginning.[75]

Zhu Xi (1131–1200) was a famous philosopher of the Song period (960–1279) who was considered a neo-Confucian, because he had enriched with aspects of Taoism imperial orthodoxy, after this had become more open and pluralistic under the Tang empire (618–907). He has left us the *Zhuzi quanshu [Complete books of the Master Zhu]*, where he wrote the following sentences: "Someone asked about the relation of *li* to number. The philosopher said: 'Just as the existence of *qi* follows

[72]Granet 1995.

[73]*Daodejing* 1973, p. 218; my translation is different from the one printed on p. 107.

[74]See Part II, Sect. 10.1.

[75]Quoted in Graham 1990, p. 347. Graham 1999, p. 235.

from the existence of *li*, so the existence of numbers follows from the existence of *qi*. Numbers, in fact, are simply the distinction of objects by delimitation'."[76]

Thus Zhu Xi represented numbers as cuts in the *continuum* which contains them and generates them, and from which they are extracted for practical purposes. The *continuum* of the *qi* included *all* numbers without distinction and made it possible to measure things with them *all*, and *all* the geometrical figures that appeared equally immersed in it, and could move in it freely. For anyone who had this image of the world as a *continuum* filled up with *qi*, it must have been unthinkable and absurd that if the value of the side of a square was 1, the diagonal could not be measured by another number. The Pythagoreans would have been refuted by these scholars (if they had known of them, obviously), and Zeno of Elea would have suffered the same fate, with his paradoxes involving movement. The idea of a *continuum* full of *qi* came to a culture that desired to understand the world, leaving it in movement and in transformation. And for this purpose they used not only whole numbers and fractions, but also roots and all the other types of ratios between numbers. And all they needed was included in the *qi* of the universe. We are at a distance of twice 10,000 *li*[77] from the Pythagorean sects, who, with their religious transcendence, had turned numbers out of time and space.

In his *Shuduyan [Developing numbers and magnitudes]*, Fang Zhongtong (1633–1698) sustained[78] the need for a return to the *Jiuzhang suanshu*. The *Jiuzhang*, in turn, were to be reconnected to the *Zhoubi suanjing*. For him, therefore, everything started with the *gougu*, and the figure of the string. The book related the origin of numbers in a way that ought to help change the misleading commonplace about the essentially numerical nature of Chinese mathematics.

> jiushu chuyu gougu, gougu chuyu Hetu, gu Hetu wei shu zhi yuan
> [The nine numbers come from the *gougu*, the *gougu* comes from the *Hetu* [Figure of the Yellow River], thus the *Hetu* is the source of numbers.]

For Fang, then, numbers sprang from the geometry of the right-angled triangle, which, in turn, went back to a mythical figure quoted also in the *Yijing*. According to Chinese tradition, *Hetu* is a figure that arose from the Yellow River, which represented the numbers from 1 to 9 by means of black and white balls. This *Hetu* was usually accompanied by the *Luoshu* [Book of the (River) Luo], which was another similar figure, but with the balls arranged differently, that had arisen from the River Luo. These two figures have often been interpreted in the West as magic squares, because the sum of the numbers, both horizontally and vertically and diagonally, is the same, at least in the *Luoshu*.[79] It would be a mistake, however, to reduce the figures to the numbers represented in them, as it would be to reduce

[76]Needham & Wang II 1956, p. 484. Unfortunately, in his comment, Needham did not see much more in these numbers than "the sterile Pythagorean numerological symbols". Cf. Graham 1990, pp. 421ff. Graham 1999, pp. 392 and 430.

[77]The *li* corresponded to about half a kilometre, or a third of a mile.

[78]When Western sciences had already penetrated into China. Tonietti 2006a, Chap. 4, pp. 179–181.

[79]Granet 1995, pp. 131–133; Needham & Wang III 1959, pp. 55–57.

numbers to geometry. Rather, as can be seen, there is an inextricable web connecting them all together. The myth of the origins made knowledge derive from the water of the river. And here again, we have a *continuum* that also derives from the *qi*.[80]

In the great encyclopaedia *Shuli jingyun [Chosen deposit of the reasons for numbers]* (eighteenth century) of the Qing [Manchu] period (1644–1912), the first part was reserved for the history of the origins, under the title *Shuli benyuan [Original sources of the reasons for numbers]*. Here, too, everything is said to have started in China from the *Luoshu* [Book of the (River) Luo], and from the *Hetu* [Figure of the Yellow River]. Also the words of Shang Gao in the *Zhuobi* now entered explicitly into the genealogy. It is interesting to see how the origin of numbers was represented.

> lun qili shewei Jihe zhi xing er ming suoyi li suan zhi gu
> [To discuss its *li* [reasons], to establish [what] serves as the *xing* [form] for the *Jihe* [how much it is, geometry], to make clear and thus establish the reasons for calculating].

For the good of the country, then, it was necessary:

> shi li yu shu xie.
> [to apply the *li* [reasons] with the *shu* [numbers] in a harmonious manner].

Here, too, then, calculation found its reasons in the *li* and in the geometrical shapes with which it harmonised.[81]

We immediately met the *qi*, when the *lülü* were connected with the *jieqi* [solar terms] of the calendar and the climate. The *lülü* enjoyed harmonious relationships among them, thanks to the substance of the *qi* that vibrated in them. Numbers were obtained from the *qi*, when objects were differentiated. The *qi* appears to be a characteristic of Chinese culture that is often encountered, and has always been the subject of discussion and various studies. Like the *Dao* [Tao], Yin and Yang, the *qi* is found in all the 100 Chinese schools of thought, and is present in the studies, even of scholars who were very distant from one another. It cannot be defined, because it would be limited, like the Tao, and it cannot be translated.

It is my (perhaps personal) conviction that the closest Western scientific theory to the *qi* was the Theory of General Relativity, completed by Albert Einstein in 1914–1915. It is only here that we can find the characteristic combination of matter, energy, field, space and time (which are generally kept separate in many studies) in a *continuum* of substance, which makes everything capable of being connected to everything else in accordance with a universal harmony. The *qi* is a kind of energetic ether, provided that this also has a material substance.[82] In the Chinese language, the character *qi* is a component of the words that indicate the climate, the atmosphere, breath, people's humour, and energy fluids, like gases and relative machines.

Among the innumerable presences of the *qi* in Chinese culture, the only problem is choosing, but we are forced to accept our spatio-temporal limitations. Luckily,

[80]Tonietti 2006a, pp. 226–239.

[81]*Shuli jingyun*, pp. 2–3, *Zhongguo* ... III, p. 12. Tonietti 2006a, pp. 195–196.

[82]Tonietti 2003a. Tonietti 2006a, p. 230.

others have already given the subject due consideration. We offer a partial review, with the aim of understanding better and identifying the variants.[83]

The *qi* sprang from the *Daodejing* [Classic of the way and of virtue], where we have already seen that it harmonises numbers. “In concentrating the *qi* to make it yield the most, do you know how to behave like a newborn? .... When the gates of heaven open and close, can you play the female role? [...] If the *xin* [heart-mind] places obligations on the *qi*, strainings are created.”[84]

The Master Zhuang invoked it, to justify himself when he started singing on the death of his wife [will anyone sing so much when I die?]. “...there was a time when form did not exist yet. Not only did form not exist, but at that time, the *qi* was not present, either. In the stirring of the amorphous mass, something changed, and the *qi* was born. As the *qi* altered, it created form. As form changed, life was created. After further transformations, now we have arrived at death.” “Do not listen with your ears, but with your *xin* [heart-mind]. Instead of listening with your *xin*, feel with your *qi*. The ears limit themselves to listening, while the *xin* limits itself to representing itself. The *qi* is the tenuous and it conforms to things. Only the *Dao* [Tao, the way] accumulates the tenuous.” “Man is born from the concentration of the *qi*. Concentration is considered life, dissipation death .... A single *qi* pervades the whole world”. “Thus the sky and the earth are the biggest of the forms, while Yin and Yang are the greatest among the *qi*.”[85]

The *qi* was also found in the *Chongxu zhenjing [True classic of the deep void]* (fifth century B.C.), attributed to Liezi, where it enables the body of the sage to be in contact with the entire universe. The *qi* could be regulated by music, as was related by a Taoist apologist. By means of the notes, it was possible to change and choose the seasons, making plants blossom or freezing rivers.[86] For the School of the Confucians, people could be directed towards good or evil by regulating the *qi*. All things possessed it, however. By adding *sheng* [life] to it, living creatures are obtained; with *zhi* [perception] as well, we have animals, and finally, with *yi* [justice], we arrive at man. Others considered form, as well, but they all started from the *qi*.[87] While it was their breath that brought the *qi* for living creatures, it was water that did so for earth, according to the *Guanzi[Master Guan]* (fourth century B.C.). “Now water is the blood and *qi* of the earth; flowing and communicating, as if in sinews and veins. Therefore we say that ‘water is the preparatory raw material of all

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[83]The best modern studies, together with the volumes of Needham and others II and IV, are to be found in Graham 1990, pp. 8, 421–426ff., and Graham 1999, pp. 133–138 and 456–458. Also Lloyd & Sivin 2002 *passim*.

[84]*Daodejing* X and LV; [*Tao Te Ching* 1973, pp. 46 and 128, 186 and 231]. Graham 1999, pp. 306–307.

[85]*Zhuangzi* XVIII, IV, XXII and XXV; 1982, pp. 158, 39–40, 197–198 and 244. Graham 1999, pp. 238, 269 and 451.

[86]*Lieh-tzu [Liezi, Master Lie]* 1994, pp. 52 and 74–75.

[87]Needham & Wang II 1956, pp. 20ff. Graham 1999, pp. 350ff.

things.' ... Human beings are made of water. The seminal essence of man, and the *qi* of the woman unite, and water flows, forming a new shape."[88]

In the *Lüshi chunqiu [Springs and autumns of Master Lü]* (third century B.C.), the creation and succession of the early reigning dynasties became a game of cycles governed by the various prevailing *qi*, indicated by the succession of the five phases (earth, wood, metal, fire and water), the five colours (yellow, green, white, red and black), the four seasons (–, spring, autumn, summer and winter) and the four directions (centre, east, west, south and north). "The yellow Emperor said: 'the *qi* of the earth has won'. As the *qi* of the earth had won, he honoured the colour yellow, and acted by taking the earth as his norm." The *qi* Yang grew, at the expense of the *qi* Yin (which decreased) until the summer solstice of the fifth month, only to diminish subsequently in favour of the *qi* Yin (which increased), until the winter solstice of the 11th month.[89]

In the *Huainanzi [Masters of Huainan]* (second century B.C.), one of the most ancient cosmogonic texts, the origin of the world by means of the *qi* was related. "The Tao began with Emptiness and this Emptiness produced the universe. The universe produced the *qi*, and this was like a stream swirling between banks. The pure *qi*, being tenuous and loosely dispersed, made the heavens; the heavy muddy *qi*, being condensed and inert, made the earth. ... The combined essences of heaven and earth became the Yin and the Yang, and four special forms of the Yin and the Yang made the four seasons, while the dispersed essence of the four seasons made all creatures ...". "[...] anything that shines emits *qi* and therefore fire and the Sun project an image outwardly. Anything that is dark keeps back the *qi*, and thus water and the moon attract an image inwardly. Anything that emits *qi* 'makes into'. Anything that keeps back the *qi* 'is transformed from'; thus Yang 'makes into' and Yin 'is transformed from' ". Atmospheric phenomena, such as the wind, rain, thunder, lightning, fog, snow and the like were explained in terms of the *qi*.[90]

In the book about medicine from the Han period, *Huangdi neijing suwen [Classic of the yellow Emperor on the interior [of the body], simple questions]*, the floating of the earth in the cosmos was interpreted in the following way: "The great *qi* keeps it raised aloft. The *zao* [dryness] hardens it, the *shu* [heat] steams it, the wind moves it, the *shi* [dampness] soaks it, the cold hardens it, and fire warms it. Thus the wind and the cold are below, the dryness and the heat are above, and the damp *qi* is in the middle, while fire wanders, and moves between. Thus there are six *ru* [entries] which bring things into visibility out of the void, and make them undergo change."

Good health depended on the cycles of the seasons, which were represented as governed by the *qi*. "The three summer months are called 'prosper and achieve'. The

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[88]Needham & Wang II 1956, pp. 42ff. Graham 1999, pp. 488–489.

[89]Graham 1999, pp. 452 and 481. Compare this with the decrease and increase of musical pipes in Sect. 3.2.

[90]Needham & Wang II 1956, p. 371. Graham 1999, pp. 456–458. However, the dualistic, structuralistic classification of the *qi* proposed by Graham leads astray, and does not do justice to the dominant Chinese culture, as I will explain in Sect. 3.6 below.

qi of sky and earth mix together, and the host of creatures flourishes and becomes mature ... do not expose yourself too much to the sun ... shed your *qi*. [...] The three autumn months are called 'control and calm'. The *qi* of the sky is windy, the *qi* of the earth is bright ... Keep the *qi* of autumn under restraint. ... preserve *qi* pure in your lungs. [...] The three winter months are called 'close and store'. ... do not allow it to dissipate through your skin, because it would rapidly subtract all the *qi*." Yang Xiong (53 B.C.–18 A.D.) and his *Taixuan [Supreme mystery]*, together with the Yin and Yang and the five phases, made the *qi* play the leading role in his organic synthesis from cosmology to medicine.[91]

In the *Dadai liji [Summaries of the rites of Dai the Elder]* (first century), fire and the Sun emitted *qi* as light, while the earth and water absorbed it. The former were Yang, and the latter Yin. Then the calendar and our *lülü* [standard pipes] arrived. "The *lülü* [pitch-pipes] are in the domain of the Yin, but they govern Yang proceedings. The calendar comes from the domain of the Yang, but it governs Yin proceedings. The *lülü* and the calendar give each other a mutual order, so closely that one could not insert a hair between them. The sages established the five rites ... the five mournings ... They made music for the five-holed pipe, to encourage the *qi* of the people. They put together the five tastes, ... the five colours, gave names to the five cereals, ... the five sacrificial animals ...".[92]

Zhang Heng (first century) said that the *qi* held the sky up, whereas the earth was supported by water. Interpreted as a *gangqi* [impetuous *qi*], according to others, he was capable of explaining the movements of the heavenly bodies.[93]

In his *Lunheng [Dialogues on the weigh-beam]*, Wang Chong (c. 27–100) united sky and man.

> Tianren tongdao
> [Heaven and Man, the same Tao.]

He explained the processes of transformation by the multiple phases that the *qi* passed through. "The *Yijing [Classic for Change]* commentators say that prior to the differentiation of the original *qi*, there was a *hundun* [chaotic mass]. And the Confucian books speak of a wild medley, and of two undifferentiated *qi*. When it came to separation and differentiation, the *qing* [pure] formed heaven, and the *zhuo* [turbid] ones formed earth ...".[94]

> Shuining wei bing, qining wei ren
> [As water turns into ice, so the *qi* crystallises to form the human body.]

"That from which man is born is the two *qi* of the Yin and the Yang. The Yin *qi* produces his bones and flesh; the Yang *qi* his *jingshen* [vital spirit]. As long as he is alive, the Yin and the Yang *qi* are in good order ...". The *qi* of the *Dayang*

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[91] Needham, Wang & Robinson IV 1962, p. 64. Graham 1999, pp. 483–484. Lloyd & Sivin 2002, pp. 269–271.

[92] Needham & Wang II 1956, pp. 267ff.

[93] Needham & Wang III 1959, pp. 217 and 222–223.

[94] The characters *qing* and *zhuo* also distinguish acute sounds from deep ones; see Sect. 3.2 above.

[sun] had to mix with the bodily Yin *qi* to generate man, otherwise only ghosts and simulacra would have been obtained. Likewise, in the egg, the liquid Yin *qi* had to unite with the Yang *qi* of heat, for the chick to be hatched. In the cosmic generation, the *tian* [sky] soaked everything with the *qi* of the five phases, thus bringing them into conflict among themselves. Shouldn't it have used a single *qi*, then, to make its creatures, if it had intended them to love one another, without making war on each other?[95]

With the shining *qi* of the moon and the Sun, which periodically died down, the Han scholar even tried to explain eclipses. He was more successful in understanding, again by means of the *qi*, the water-steam-clouds-rain cycle, and lightning. "The genesis of thunder is one particular *qi* [kind of energy] and one particular kind of sound. [...] thunder is the explosion of the *qi* of the solar Yang principle." In a book of the Han period, he wrote:

> Kunlunshan youshui, shuiqi shang zheng weixia
> [In the Kunlun mountains, there is water; the *qi* of the water rises, evaporates and becomes clouds.]

With coal and feathers, they even tried to detect the damp or dry *qi*.[96]

It was written in the *Huashu [Books of transformation]* (Tang period): "The *xu* [void] is transformed into *shen* [magical power]. The magical power is transformed into *qi*. *Qi* is transformed into material things. Material things and *qi xiangcheng* [ride on one another], and thus sound is formed. [...] Sound leads [back again] to *qi*; *qi* leads back to magical power; magical power leads back to the void. [But] the void has in it power. The power has in it *qi*. *Qi* has in it sound. One leads back to the other, which has the former within itself. Even the tiny noises of mosquitoes and flies would be able to reach everywhere. [...] *Qi* follows sound and sound follows *qi*. When *qi* is in motion, sound comes forth, and when sound comes forth, *qi zhen* [is shaken]."[97]

Here we have found a good, clear, emblematic representation of those characteristics of Chinese scientific culture on which we shall dwell in the following section. The Chinese world was seen as in continual transformation. It changed, following reciprocally inclusive cycles which were linked in their resonance. Through the game of the *qi*, perturbations, however small they were, were able to propagate everywhere.[98]

In his *Zhengmeng [Correcting the ignorant]*, Zhang Zai (1020–1076), wrote: "In the great void, *qi* is alternately condensed and dissipated, just as ice is formed or dissolves in water. When one knows that the great void is full of *qi*, one realises that there is no such thing as nothingness ... How shallow were the disputes of the

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[95] Needham & Wang II 1956, pp. 368ff.

[96] Needham & Wang III 1959, pp. 411–413, 467–471, 480–481.

[97] Needham, Wang & Robinson IV 1962, pp. 206–208.

[98] Also some of the contemporary sciences would seem to have rediscovered these characteristics, which certain modern-day philosophers termed "complexity". The famous "butterfly effect", which exemplifies the complexity of the earth's atmosphere, is well represented by the mosquitoes and the flies of the Tang period. Tonietti 2002a; Tonietti 2003a. Cf. also Jullien 1998, p. 78.

philosophers of old about the difference between existence and non-existence; they were far from comprehending the great science of *li* [pattern-principles]." Following the *qi*, the scholar Song arrived at a description of how sound was formed. "The formation of sound is due to the *xiangya* [mutual grinding] between material things, and *qi*. The grinding between two *qi* gives rise to noises such as echoes in a valley, or the sound of thunder. The grinding between two material things gives sounds such as the striking of drumsticks on the drum. The grinding of a material thing on the *qi* gives sounds such as the swishing of feathered fans or flying arrows. The grinding of *qi* on a material thing gives sounds such as the blowing of the reeds of a *sheng* [mouth-organ]."[99]

In the famous *Mengqi bitan [Essays from the pond of dream]* (1086), Shen Gua (1031–1095) described the transformations of the five phases from one into another, as due to the *qi*.[100]

Zhu Xi took the theme up in his *Zhuzi quanshu [Complete books of the Master Zhu]*. "Heaven and earth were initially nothing but the *qi* of Yin and Yang. This single *qi* was in motion, grinding to and fro, and after the grinding had become very rapid, there was squeezed out a great quantity of sediment. ... This sediment was the sediment of *qi*, and it is the earth. Therefore it is said that the purer and the lighter parts became the sky, and the grosser and more turbid ones earth."[101] The neo-Confucian scholar also sustained, "Speaking in terms of the *qi* as one, both men and other things are generated by receiving this *qi*. Speaking in the terms of the coarse or the fine, men receive *qi* which is well adjusted and permeable, other things receive *qi* which is ill adjusted and impeding. [...] Therefore the highest in knowledge, who know from birth, are so constituted that the *qi* is clear, bright, pure, choice, and without any trace of the dull and murky ...".[102]

Unlike the case of numbers, where he ignored their immersion in the *continuum*, Joseph Needham recognized the importance of the *qi* for those physical phenomena particularly studied by the Chinese. They showed a special attention for continuous movements, and often made reference to water, to account for influences with no visible or material intermediaries.

The astronomer Liu Zhi (third century) interpreted the connection between the light of the Sun and the moon in the following way: "They respond to each other, in spite of the vast space that separates them. When you throw a stone into the water, [the ripples] spread out after one another: the *qi* of water propagates." For him, the same thing happened to the *qi* of light. The *qi* filled everything, and everything moved, therefore, in waves, gathering rhythm and integrating into the harmony of the world. To Chinese scholars, the world appeared to be a general increase and decrease, a continual oscillation of *qi* or of Yin and Yang: *Sun* [decrease] and *yi* [increase], as established by the two hexagrams of the *Yijing*, as observed in the

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[99] Needham, Wang & Robinson IV 1962, pp. 33, 205–206 and 146–147.

[100] Needham & Wang II 1956, p. 267.

[101] Needham & Wang II 1956, pp. 368–374.

[102] Graham 1990, p. 424.

phases of the moon, as in the notes emitted by the *lülü*. Liezi considered continuity as the main *li* [reason] behind the world.[103]

In the *Guanzi[Master Guan]* (fourth century B.C.), the *qi* realised the unity of the universe. "But that will not be because of their show of force, instead of sending forth your *jing* [essence] and your *qi* to the utmost degree. What unifies the *qi*, so that it can change, is called the essence; what unifies [human] affairs, so that they can undergo change, is called wisdom." "Consequently, you cannot restrain this *qi* by force … […] … Thus the sage shows moderation with appetising savours and is timely in his movements and his pauses, he guides and compensates for the fluctuations of the six *qi*, denying himself all excesses with music or women. His limbs are not acquainted with any corrupt gesture, and lying words never proceed from his lips. […] The magical *qi* which is in his *xin* [heart-mind] comes and goes. So minute as not to contain anything smaller inside, and so vast as not to find anything larger outside. The reason why we lose it is that we damage ourselves in anxiety. If the *xin* can remain unmoved, the *Dao* [Tao, the way] will set itself up by itself … […] … The *jing* [essence] is the essence of the *qi*. When the *qi* is on the Tao, it vitalises; he who has become vital can imagine; he who can imagine knows; he who knows limits himself. The *xin* of everybody is formed in this way; if knowledge goes too far, life is wasted. […] Concentrate the *qi* and become equal to the *shen* [demonic spirit]. All the 10,000 things will be there at your disposal. […] To control anger, nothing is equal to the *Shijing [Classic for odes]*, to free oneself from thoughts, nothing is equal to music … ".[104]

During the Tang period, attempts were made to account for tides by means of the original *qi* which expands and contracts.[105] Earthquakes and a famous seismograph were described in terms of the *qi*. It is particularly interesting that Zhou Mi (thirteenth century) stated that he did not understand the reasons for them, since they were not regular like the heavenly movements. "But earthquakes come from [unpredictable and unmeasurable] *buce* collisions of the Yang and the Yin. Take the case of the body of a man; the blood and the *qi* are sometimes in accord and sometimes in opposition, hence the flesh responds … If the *qi* reaches [the vital point], he moves; … [When the seismograph] is said to have been placed in the capital, far away from the place where the earthquake occurred. How could the collision of the two *qi* make the bronze dragons vomit forth the balls?"[106]

For the great Ming pharmacopoeia, Li Shizhen described together with Yin, Yang, sky, earth and man, the various types of fire that were in different ways produced by the Sun, friction, the human body, petroleum, gas and even by "a lot of

[103] *Lieh-tzu* 1994, p. 72. Needham & Wang III 1959, p. 415. Needham, Wang & Robinson IV 1962, p. 28. It is particularly curious that Joseph Needham saw in all this the idea of action at a distance, when, on the contrary, it was the *qi* that made it possible to maintain bodies in contact, as the notion of the electro-magnetic and gravitational field was to do in Europe.

[104] Needham, Wang & Robinson IV 1962, p. 30. Graham 1999, pp. 159 and 131–138.

[105] Needham & Wang III 1959, pp. 490ff.

[106] Needham & Wang III 1959, p. 634.

qi of gold and silver".[107] "Stone is the kernel of the *qi*, and the bone of the earth." Minerals were born from the various ways of coagulations, followed by the Yang and Yin *qi* of the earth. For example, after a centuries-long series of transformations, the "*qi* of the Great Unit", in other words gold, appeared. The analogy with the human body was developed. "Now if the *qi* of the earth can get through [the veins], then the water and the earth will be fragrant and flourishing . . . and all men and things will be pure and wise . . . But if the *qi* of the earth is stopped up, then the water and the earth and natural products will be bitter, cold and withered . . . and all men and things will be evil and foolish . . . ".[108]

The *qi* was a continual point of reference for those Chinese scholars who tried to explain the surprising effects of sounds and music on people and on the world. In the *Zuozhuan [Comment of Zuo]* on the Confucian *Chunqiu [Springs and autumns]* (fifth and third century B.C.), the six *qi* were Yin, Yang, wind, rain, dark and light. "Heaven and earth give rise to the six *qi* and make use of the five *xing* [phases]. The *qi* give rise to the five flavours, and emit the five colours, and are manifested in the five notes of music. When the [responses of humans] are excessive, they lead to confusion, and people lose their proper nature." "Their excesses generate the five illnesses. . . . They create the nine songs, the eight winds, the seven sounds and the six *lülü* to sustain the five notes."[109]

Xunzi (third century B.C.), as a loyal follower of Confucius, could not have avoided justifying music, and made the first official defence of it against the adversaries of the rival schools. He exploited the event that the same character could be pronounced either as *yue* [music and dance] or as *le* [joy and amusement], and began, "Music gives joy, and is what a real man inevitably refuses to give up." But if he was not guided, the enjoyment would lead men to disorder. Therefore the ancient kings established the *Shijing [Classic for odes]* as a guide. ". . . so that sounds would be pleasant without being licentious, and the variations, complexity and wealth of the musical network and rhythms would be such as to inspire the *xin* [heart-mind] of man towards good . . . This is the secret of music established by the ancient kings: why does Mozi [Master Mo, (fourth century B.C.)[110]] condemn it?" That secret depended on the *qi*. "Every time that corrupt sounds stimulate man, the *qi* reacts in a discordant manner, and this dissonance generates disorder. When man is stimulated by virtuous sounds, the *qi* reacts harmoniously, and the harmony gives rise to order." In ceremonies in public or in the family, various instruments were played, such as bells, drums, harps, chime-stone cymbals and flutes. "That which sounds limpid and bright in music is modelled on the sky; that which sounds broad and spacious is modelled on the earth. The gaze that descends and rises and turning

[107]Needham, Wang & Robinson IV 1962, pp. 64–65.

[108]Needham & Wang III 1959, pp. 637ff. 650.

[109]*Chunqiu* X, year 25 (517 B.C.) 2, *Primavera e autunno [Spring and Autumn]* 1984, p. 766. Needham, Wang & Robinson IV 1962, p. 134. Graham 1999, pp. 446–447. Lloyd & Sivin 2002, p. 255.

[110]See below, Sect. 3.6.

around hark back to the cycle of the four seasons. [...] The ear hears more distinctly and the eye sees more clearly. The blood and the *qi* are quiet and in harmony. [Music and rites] modify and substitute customs, until the whole world is pacified ... [...] ... music is the unalterable element to create harmony, while rites are indispensable to define models. Music unites what is similar; rites distinguish what is different. Unity based on music and rites runs through the *xin* [hearts-minds] of men."[111]

As it pervaded all the 10,000 things of sky, earth and sea, including man, the *qi* created agitation and movement. Depending on the means used, a perturbation was propagated more or less far: more in water than in mud, even further in the *qi*. In the *Chunqiu fanlu [Abundant dew of springs and autumns]* (Han period), when the *qi* of people was discordant with heaven and earth, disorders and disasters would take place. "The transforming *qi* is much softer even than water, and the ruler of men ever acts upon all things without cease. But the *qi* of social confusion is constantly conflicting with the transforming of Heaven and Earth, with the result that there is now no government."

How the *qi* acted in men was explained in the *Guoyu [Discourses of states]* (layers of the Zhou, Qin and Han periods). "Sounds and tastes generate *qi*. When *qi* is present in the mouth, it makes speech, and when in the eye, intelligent perception. Speech enables us to refer to things in accepted terms. Intelligent perception enables us to take action at the right times. Using terms, we thereby perfect our government."

In the *Yueji [Memories of music]* (Han period), music resumed its central function. "The *qi* of Earth ascends above; the *qi* of Heaven descends from on high. The Yang and Yin come into contact; Heaven and Earth shake together. Their drumming is in the shock and rumble of thunder; their excited beating of wings is in wind and rain; their shifting round is in the four seasons; their warming is in the Sun and moon. Thus the hundred species procreate and flourish."

> zhicize, yuezhe tiandi zhihe ye.
> [Music thus [realises] the harmony of heaven and earth, relating this to us.][112]

Sima Zheng (eighth century) discussed the *lülü* as channels of the *qi*.

> lüzhe suoyi tongqi
> [Those *lülü* therefore canalise *qi*].[113]

We have already seen[114] that of the 12 *lülü*, six are Yin and six Yang. The tradition already began in the *Zhouli [Rites of the Zhou]* (Han period), with the *Liji [Reports of the rites]* (Han period) and the *Qian Hanshu [Books of the former Han]* (second century) from which we started. Here we also read that "The *qi* of Heaven and Earth combine and produce wind. The windy *qi* of Heaven and Earth correct the 12 *lülü*." In the fourth century, a commentator added, "The *qi* associated with wind

[111]Graham 1999, pp. 355–358.

[112]Needham, Wang & Robinson IV 1962, pp. 203–205.

[113]Needham, Wang & Robinson IV 1962, p. 134.

[114]Above, Sect. 3.2.

being correct, the *qi* for each of the 12 months *ying* [echos, causes a sympathetic reaction]; the *lülü* never go astray in their serial order."[115]

Furthermore, it was written in the *Hou Hanshu [Books of the latter Han]* (fifth century) that "Tubes are cut [from bamboos] to make *lülü* [pitch-pipes]. One blows these in order to examine their tones, and set them forth [on the ground] in order to make manifest the *qi*." This idea produced a rite that is surprising (for us Westerners), in which can be tested the real, deep, by no means formal conviction of this culture in the *qi* and in its way of maintaining all the phenomena of the world united: *houqi* [watching for the *qi*]. Considering the procedure followed in its performance, it was also called "blowing of the ashes". By means of the *lülü*, the impalpable, yet substantial matter of the *qi* became not only audible, but even visible.

A room was constructed, well protected from the winds, with triple walls and a bare earth floor. Here the 12 pipes (generally made of jade) were interred almost in a vertical position, arranged in a circle in accordance with the points of the compass, because the *lülü* were also linked with these, and not only with the months of the year.[116] The mouths of the pipes sticking up from the ground were filled with a very fine ash, obtained by burning the interior of reeds coming from a suitable place. When they arrived at a certain month or a precise moment of the year, the *qi* blown from the inside of the earth would escape from the mouth of the corresponding *lülü*, thus blowing the ash out. For example, the *qi* should emanate from the *Huangzhong* [yellow bell] at midnight on the winter solstice (Table 3.1).

In the sixth century, a commentator of the *Yueling [Ordinances of the months]* (Zhou period) specified that the openings of the pipes were to be covered with silk gauze. "When the *qi* [of that month] arrives, it blows the ashes [of the relative pitch-pipe] and thus moves the gauze. A small movement means that the *qi* are harmonious. A large movement is an indication that the ruler is weak, his ministers strong, and that they are monopolising the government. Non-movement of the gauze is an indication that the ruler is overbearing and tyrannical." To increase the precision of the performance, Xin Dufang (sixth century) constructed 24 revolving fans, with which he attempted to measure the 24 different *qi* of the year. As soon as the *qi* of the period became active, only the relative fan moved, and the others remained still. Among other things, Xin also invented a seismograph.

Shen Gua was so convinced of his beloved *qi* that he criticised, in his *Mengqi bitan*, the explanations of other scholars as regards the "blowing of the ashes". His own explanation actually sounds more complete, because it agreed with the different lengths of the interred pipes. "At the winter solstice, the Yang *qi* stops at a point nine *cun* from the surface of the ground; and inasmuch as it is only the *Huangzhong* tube that reaches to such [a depth], it is therefore *Huangzhong* that responds." And so on for the other *lülü* of different lengths which are reached on each occasion by the *qi* in movement at the appropriate moment. "The case is like someone who uses a needle

[115] Needham, Wang & Robinson IV 1962, p. 136, 187.

[116] Bodde 1959, p. 29.

to probe into the channels [of the *qi* in the human body]: these *qi*, in compliance with the needle, will then issue forth." Thus acupuncture controlled the *qi* of the body, while the *lülü* controlled that of the earth.[117] This also seems to be the best moment to recall that this book of the eleventh century became famous for its clear accounts of the magnetic compass and printing with movable characters. Wasn't the movement of the compass needle towards the north yet another manifestation of the *qi*?

Guo Po (276–324) explained the attraction of the magnet and of amber by means of the *qi*. "The lodestone *xi* [breathes, attracts] iron, and amber collects mustard-seeds."

> qi you qiantong, shuoyi minghui, wuzhi xianggan.
> [The *qi* has an invisible penetratingness, rapidly effecting a mysterious contact, according to the mutual responses of material things.]

The idea that the earth was crossed by currents of *qi* and that these could orientate a magnetic body became the Chinese form of geomancy known as *fengshui* [wind water]. Traditionally, great importance was given to this in the orientation of houses. The "spoon" shape of the magnetic body, left as free as possible to move, recalled by analogy that of the heavenly Plough. The magnetic compass was created on earth thanks to the *qi*, long before anyone used it on the seas.[118]

Cai Yuanding (1135–1198) made the link between the *qi* and the height of notes even closer. "When a *lülü* [pitch-pipe] is long, its tone is low and its *qi* arrives early; but if overly long, it makes no tone at all, and the *qi* does not respond. When a *lülü* [pitch-pipe] is short, its tone is high, and its *qi* arrives late; but if overly short, it makes no tone at all, nor does the *qi* respond. [...] When its tone is harmonious and its *qi* responds, the *Huangzhong* is really a *Huangzhong* indeed!"[119]

Thus it was believed that the *qi* could be regulated by means of musical notes. And attempts were made to influence the atmospheric conditions, depending on the circumstances. The notes *Jiazhong* [compressed bell] and *Wushe* [without discharge] corresponded to the winds of the north, the winds of the south corresponded to the *Guxi* [purification of women] and *Nanlü* [southern pitch-pipe].[120]

In connection with this subject, there was no lack of literature dealing with military campaigns. The *Zuozhuang* established that when the *qi* of the *Nanlü* did not issue freely, the note meant a great massacre in battle. In the second century, the comment was added: "As the pitch-pipes are also the tubes [for] 'observing the *qi*', the emanation is called *feng* [wind]. This is why we talk of the *ge* [song] and the *feng* [wind]".[121] It is thus possible to understand why also the *Sunzi bingfa [Rules for war*

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117 Bodde 1959, pp. 18, 21, 26. Lu & Needham 1984.

118 Needham & Wang & Robinson IV 1962, pp. 233, 239, 243. Needham & Wang III 1959, pp. 232–233.

119 Bodde 1959 pp. 30–31. Needham & Wang & Robinson IV 1962, pp. 186ff.

120 Needham, Wang & Robinson IV 1962, p. 136.

121 Needham, Wang & Robinson IV 1962, pp. 136–137. I have modified a few words in their translation.

of Master Sun] (sixth and fifth centuries B.C.) had spoken of the five musical notes and of "energy", as well as how to direct battles by means of gongs and drums.[122] This was a book that became famous in the West, as well; it went back to the pre-imperial periods of the "Springs and autumns" and the "Fighting States", and must have inspired above all the School of the Legalists.

Sima Zheng, the scholar of the Tang period that we have seen above, had a reason of his own to deal with the *qi*. "Over every enemy in battle array there exists a *qise* [vapour-colour of *qi*]. If the *qi* is strong, the sound is strong. If the note is strong, his host is unyielding. The *lülü* [pitch-pipe] is [the instrument] by which one canalises *qi*, and may thus foreknow good or evil fortune." The prince of the Ming period, Zhu Zaiyu (1536–1611), who we shall meet in the second part,[123] made the tuning of the *lülü* depend on the state of mind of the player: young, strong people did not bring the same *qi* as elderly people or children. When King Wu of the Zhou made war on the previous Shang sovereigns, "... he blew the tubes and listened to the sounds. From the first month of spring [i.e. the longest *lülü*] to the last month of winter, a *qi* of bloody slaughter [was formed by their] joint action, and the ensuing sound gave prominence to the *Gong* [Palace] note." The episode was related in these terms in the *Shiji [Historical records]* by Sima Qian (Han period).[124]

In Chinese culture, the sound of bells is thought to excite people to war, because it increases the *qi*. Bells appeared to be capable, like the *lülü* pipes, but unlike strings, of containing a good quantity of *qi*, since they are concave and empty.[125] This explains why the majority of the 12 notes are indicated either as *zhong* [bells] or as *lü*. For a similar reason, the famous Chinese art of casting bronze was particularly associated with carillons of bells.[126]

The practice of "watching for the *qi*" may prompt, and has prompted, various comments. It brings to light their ideas of the sciences and of their history. Was it just a magic rite, without anything truly scientific? Or on the contrary, did it contain some aspects of scientific investigation into a presumed phenomenon, which in the end proved to be non-existent? So, in the late Ming era, it was questioned and abandoned by some. Might an experimental physicist of today conclude that the *qi* does not exist at all, and that it was only an imaginative Chinese superstition? And yet some texts described the construction of the room for the experiment in such detail that it might be called in modern times a laboratory to avoid any disturbing effects. They even improved their instruments, measuring the *qi* with gauze and whisks. With an equally modern approach, as philosophers panted with defining the scientific enterprise, the Chinese scholars took precautions against the possibility that no ash was blown out of the pipe at the right moment: it was all the oppressive

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[122]Sunzi 1988, pp. 91–93 and 103.

[123]See Part II, Sect. 8.1.

[124]Needham, Wang & Robinson IV 1962, pp. 138–140.

[125]Needham, Wang & Robinson IV 1962, pp. 153 and 171.

[126]Needham, Wang & Robinson IV 1962, pp. 194–199. Chen 1994.

emperor's fault, because he blocked the circulation of *qi*, or else the reeds which were to be burnt to obtain the impalpable ash had been picked in the wrong places.

Paradoxically, the procedure followed would appear to be the most modern, and "Western" one that Chinese scientific culture created, in the spirit of future scientific experiments, but only to verify one of the least "Western", and most characteristic phenomena that it invented. If we had misunderstood the *qi* as the "ether" of Western scientific tradition, we would be tempted to conclude that the *qi* should rightly go the same way as "ether", which was removed from physics manuals during last century. But leaving aside the infinite diatribes on the latter among historians of science, "ether" was to return into Western physics in other guises, such as "metric tensor $g_{\mu\nu}$" or "cosmic radiation 3K". We would therefore suggest, without any great hopes of being listened to, that this Chinese *qi* should not be the object of discrimination, and should be treated in the same manner. Anyway, those interred pipes would seem to be more suitable to measure the *qi* of water, whereas the *qi* diffused by the *lülü* would seem to be rather that which is present in the atmosphere and in breath. In any case, if we interpreted it as energy, the *qi* could be measured in many other ways. Whereas if we compared it to gravitational waves, even these may perhaps be found in the future.

Chinese culture was the only one, to the best of my knowledge, that used the musical notes of the *lülü* as the basic system of measurement for all the other units of length, weight and capacity. Once the *Huangzhong* had been correctly tuned, the *Qian Hanshu* provided not only the length of 90 *fen* [1 *fen* = 0.33 cm], but even the number of the 1,200 grains of millet that it had to contain, and the weight of 12 *zhu* [$\frac{1}{24}$ of 50 grams].[127]

The most famous follower of Confucius was Mencius. His book the *Mengzi [Master Mencius]* (Zhou period), was so successful that it became one of the Four indispensable books, together with the Five [six] classics for every good imperial functionary. Here, too, a moral conduct was achieved by administering the *qi* correctly. "–'I know how to speak, and I am capable of taking care of my *qi haoran* [great justice]'. –'May I ask, what do you mean by *haoran qi*?' –'The subject is difficult to explain. It is the *qi* in its supreme state of vastness and firmness. If it is nourished with rectitude, and you do not interfere with it, it fills the space between heaven and earth. It becomes that type of *qi* which brings together justice and the Dao [Tao, the way]. Without these, it would suffer from hunger. It is generated by accumulating a correct behaviour, and it cannot be grasped by behaving correctly [only] sporadically.' " It could be read in his books, thanks to the *qi*, that the ears and numbers chosen for the sounds of music were in harmony, because they extended their possibilities.[128]

Following the *qi*, we have discovered the reasons why the *lülü* became an orthodox system of rules based on numbers. The last, but definitely not the least of the reasons: the scheme seemed, to the eyes of the dominant Confucian school,

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[127] Needham, Wang & Robinson IV 1962, pp. 199–202.

[128] Graham 1999, p. 170. Needham, Wang & Robinson IV 1962, p. 183.

a good way to make sure that the undeniable pleasures of music would not corrupt the desired morality.

But in practice, would musicians be satisfied with those rules? In reality, some tried to increase the number of notes that could be played, both with bells and chime-stone cymbals and with flutes and the strings of instruments played with a bow. Composers and players of music loved planning together and passing (modulating) from one mode to another, from one scale to another. But it was not easy to do that in the orthodox system. Contrary to every rule that disturbed the spontaneous flow of the world, the heretical sceptic of Master Zhuang exhorted: "Destroy the six *lülü*, snap the flute and smash the lute, deafen the ears of Shi Kuang and everyone will maintain the sharpness of his own hearing." Shi Kuang was a renowned musician, mentioned many times in the *Chunqiu [Springs and autumns]*, who, significantly, had been assigned the task of warning the prince in accordance with the teachings of Confucius. He also interpreted birds' songs for military expeditions. "I often sing on the pipe of the winds from the north and on that of the winds from the south. The pipe of the winds from the south cannot compete, and echoes with the cries of many dead."[129]

Then, numbers were not always respected, as they preferred to follow their ears, as in the *Huainanzi [Masters of Huainan]* (Han period). "This [simplification into round numbers] was clearly done with some reference to the ear, and not merely as a mathematical convenience, . . . ". During the Han period, we find a certain Jing Fang, who obtained a variety of as many as 60 notes from increases and decreases in the never-ending cycle of the 12 *lülü*. Others, like the scholar Wang Bo in the Song period, tried to introduce the octave into the orthodox imperial system, which in general did not allow for it.[130] However, the most convenient idea, that is to say, convenient for music and musicians, of dividing the *qi* into equal parts, either by the ear or by means of radical numbers, had to wait for the fall of the Ming Empire.[131]

> Zhi buzhi shang buzhi zhi bing
> Knowing that you don't know is superior, not knowing that you know is a mistake.[132]
>
> Daodejing, LXXI.

> Great intelligence embraces,
> Little intelligence discriminates.
>
> Zhuangzi, II.

> Mathematics is a MULTICOLOURED *mixture*
> of test techniques. And this is the basis of
> its multiple applicability and its importance.
>
> Ludwig Wittgenstein

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[129] *Zhuangzi* X 1982, p. 87. *Chunqiu* IX, year 14 (559 B.C.) c; year 18 (555 B.C.) 4 and 6; year 26 (547 B.C.) a; year 30 (543 B.C.) b; X, year 8 (534 B.C.) a; year 9 (533 B.C.) b, or *Spring and autumn* 1984, pp. 485–486, 497, 499, 541, 580 and 678. Levis 1963, pp. 63–80.

[130] Needham, Wang & Robinson IV 1962, pp. 218–220.

[131] See Part II, Sect. 8.1.

[132] Or: "First knowing [but] you don't know [and] not knowing [but] you know, [that is] a mistake". Various interpretations of the text are possible. Cf. *Tao te ching* 1973, p. 156; Hansen 1983, p. 67; Needham & Harbsmeier VII 1998, p. 256.

3.6 Rules, Relationships and Movements

In the end, even though we searched for some common characteristics in the historical context, Greek and Latin scientific culture did not appear to be very homogeneous, nor, even less, could they be reduced to a few basic concepts. Obviously, there was a dominant orthodoxy, but there were also some heretics that were different. Consequently, we represented it as subject in time and in space to numerous events and transformations. Likewise, even considered at a distance, it does not appear that the Chinese scientific culture can be reduced to a few general characteristics. Even more so in a country which, though governed for thousands of years by imperial dynasties with similar bureaucratic apparatuses, does not speak the same language even today, does not eat the same food, and does not suffer the same climate. In Europe, it is not hard to distinguish the Finns from the Sicilians, and the same is true of the Chinese. During the periods of the "Springs and autumns" and the "Fighting States" (770–221 B.C.), "a hundred schools" of thought were in contention. "A hundred rivers" continued to flow into the Yellow River and the sea. "A hundred flowers" blossomed before the Chinese empire attempted to cultivate even one of them (without success?).

And yet, also in China, the historical context appears to have been capable of selecting certain dominant, recurring aspects, together with Confucian orthodoxy. We shall start with the language, or rather, with its peculiar form of writing, by means of characters, which the imperial bureaucratic apparatus had gradually rendered relatively stable and common in all the country. Then we shall summarise the events of this chapter around the world, represented as an organism modelled on the examples of the earth, and thus seen in continual transformation. Rules and lines of conduct could be given for it, provided they were not too strict, as could be observed among living creatures and men. This world was thus not static, but it moved, and it did so in the space-time *continuum*, whether this was full of an impalpable energetic substance like *qi*, or one that was more dense like water.

Lastly, following other scholars, we shall summarise the main differences between the Chinese scientific culture and that of the Greek-Latin world.[133] Whether we prefer to call it a hypothesis or a thesis, I have presented here strong, well-documented arguments in order to sustain that Chinese culture, like that of the Graeco-Latin world, had elaborated its own characteristics sciences. These, placed in comparison, have not proved to be either inferior or superior, or either to follow or to precede any other culture. They have simply proved to be different. Jean-Pierre Vernant had already said this with an unrivalled clarity and concision: "Greek culture is no more the measure of Chinese culture than the opposite is true. The

[133] Here we offer a summary, also clarifying, enriching, reorganising and correcting, wherever necessary, Chap. 6 of Tonietti 2006a. Occasionally, the study may overlap with those of Lloyd & Sivin 2002, and of Jullien 2004 and 1998.

Chinese did not go less far than the Greeks, they went somewhere else."[134] Now we have shown that also for the sciences.

3.6.1 Characters and Literary Discourse

I am convinced that a language that is not linear, but pictorial and ideogrammic, like Chinese, had its weight in favouring representations that were different from those of Greek and Latin, where alphabetical languages are used. In the text of the *Zhoubi*, numbers referred to figures, and figures to numbers. The former made it possible to understand the latter, and the latter were a help to understand the properties of the former. The argument was developed around a circle, and as this moved, it became possible to use it in 10,000 different cases. The *Zhoubi* appears to us as a page written in the same characters as other literary texts. The passage from the *jing* [straight line] character to the *xian* character for the hypotenuse probably only indicated the evolution of the language, and did not represent the search for a special technical term. Actually, *xian* also means the chord of the arc and the string of musical instruments.[135]

With their choice of characters, Chinese scholars appear not to desire to distinguish their possible different uses when they discussed of mathematics or other subjects. In general, they did not manifest any particular interest in the invention of a symbolism that would allow them a certain detachment from current literary discourse. The famous *zheng ming* of Confucius: "It is necessary to rectify the names" was an invitation to make the names agree with the sense, so that the sense could successfully guide behaviour.[136] The only ones who had tried (with what results?) to coin new technical terms for the sciences would seem to be the followers of Master Mo.[137] But for the most part, the invention of new terminology was condemned.[138]

When Westerners had already come on to the scene, a mathematician of success like Mei Wending (1633–1721), rectified some terms of mathematics in order to avoid confusion, using the same characters of the language. He, too,

[134]Vernant 1974, p. 92.

[135]"Their beginning and their end are like a circle, whose order has no end"; *Zhuangzi* XXVII, 1982, p. 256. Above, Sect. 3.3. Archery was not only one of the arts that a gentleman had to learn, together with music. It also had to serve as an example to follow in order to educate. Mencius wrote: "The virtuous man draws [the bowstring], but without darting it". That is to say, the pupil should be encouraged, but left to follow his own way spontaneously. "How similar the Tao of heaven is to the act of drawing a bow!"; *Daodejing* LXXVII; 1973, p. 165. Jullien 2004, pp. 309ff.

[136]Confucius XIII, 3; 2000, pp. 104–105. Cf. Hansen 1956, pp. 72–82. Jullien 2004, pp. 225–235 and 264.

[137]Hansen 1983, p. 109. In order to achieve the maximum of clarity, the Mohists renounced the elegance of literary style. Needham & Robinson 2004, pp. 101–103.

[138]Graham 1999, p. 359.

followed the tradition of designating some mathematical objects by means of terms indicating concrete things. For example, he wrote "canine teeth" for the angles of an icosahedron.[139]

As well as facilitating open circular arguments, and helping scientific discourses to keep both feet on the ground, and to be comprehensible for all educated people, some Sinologists have underlined a third important characteristic of the Chinese language which is pertinent here: the lack of a verb "to be", that is to say, of a character that can express a suitable equivalent. The character *you* [there is, exists, not distinguished from the verb 'to have'] expresses possession, and refers above all to *wu* [material things]. Surrogates, recent among other things, such as *shi* [right, yes, this] are weak, compared with the peremptory "is/are" of old Europe, rich in cultural harmonics. Instead, in the Country at the centre, discourses abound in *wei* [to do, to act, to become], a character also used in the place of what would be expressed in Europe by the verb "to be".[140]

François Jullien has underlined the event that the inclination of Chinese culture to prefer organic correlations has remained impressed in the language. For example, these can be seen in the terms *dongxi* [east-west] to indicate a generic "thing", or *shanshui* [mountains-waters] for "charming panoramas"; we have already met another case, which will immediately be found again: *yuzhou* [space-time] for "universe". They also say:

> huwen jianyi
> [the ones with the others the writings, [and] you will see [their] correctness.]

Kenneth Robinson and Joseph Needham have interpreted the same characteristics, e.g. *shensuo* [lengthen-shorten], as a different way of abstract terms, in this case "elasticity", which are created in great quantities in the Indo-European languages. However, it seems to me, rather, a way of avoiding abstractions, and remaining anchored to the sensible aspects of things. The two authors bring arguments to show, to anybody who needs them, that Chinese was capable of expressing anything that it wanted to express, including numbers, technical details or scientific reasoning. But, surprisingly, we also read today that for the *Zhuobi* men of letters would be satisfied with presenting a simple "example", in order not to destroy the Tao by proofs, passage after passage. "There was, therefore, in China a considerable barrier to the

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[139]Martzloff 1981b, pp. 173–178 and pp. 301–302. Needham & Robinson 2004, pp. 172–175. Even Harbsmeier – in Needham and Harbsmeier VII 1998, p. 234 – had to recognize "... a natural and strong gravitational force towards the non-abstract down-to-earth use of words." The insistence of Karine Chemla on assigning a technical meaning to characters in classical Chinese mathematical texts sounds anachronistic, because it derives from modern symbolic habits; cfr. *Neuf Chapitres*, pp. 99ff.

[140]Hansen 1983. Graham 1990. In Part II, Sect. 8.2, we shall recall the problems of the most famous translation from Latin into Chinese, that of Euclid's *Elements*. Tonietti 2006a, Chap. 4.

formulation of mathematical proofs, due to the social attitude of the educated."[141] On the contrary, we have seen that Chinese books offered *different proofs* from those of Euclid, which were not, are not, and will not be the only ones possible, even in Europe. Whether it depended on the Tao, on language or on culture in general, the reader will be able to make his own opinion through the following pages.

Chinese words, which do not distinguish in general between verbs, nouns and adjectives, express better a reality in movement. In order to render a Chinese text more faithfully, then, we need to abound with the verbs, and limit the number of nouns, especially if abstract. I prefer to leave to the following Sect. 3.6.4, dedicated to the *continuum*, a discussion of the characteristic presence in the Chinese language of classifiers, because it will become clearer and more meaningful there.

> Two philosophers were arguing and one said to the other:
> "There are two types of people, those who like dichotomies
> and those who don't". The other replied:
> "That's nonsense!"
>
> Evelyn Fox Keller.

3.6.2 A Living Organism on Earth

In China, the world was not divided between heaven and earth, because they were both part of the same unitary cosmos. Both shared the same *fa* [rules] and the same *li* [reasons]. In the *Zhoubi*,we read that the former appeared to be round, and the latter square. But the two forms were considered as welded together, like the *gou* and the *gu* with the *xian*, like 3, 4, and 5, like the flat bottom and the curved shell of a tortoise. Joseph Needham started out on his long study of Chinese sciences with the idea that in them, the world was represented as a living organism.[142] The Chinese scholar did not forget that we live on the earth, and we observe the sky from here. Therefore, in order to study it, it needs to be brought back to earth.[143]

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[141]Jullien 2004, p. 429. Needham & Robinson 2004, pp.108–110, 141 and *passim*. Joseph Needham (at the age of 93) even affirmed "...certain disadvantages of the Chinese script ...". Compared with the advantages of printing with the alphabet, the Chinese "were hamstrung by the complexities of Chinese characters." Needham and Robinson 2004, pp. 210, 227 and 230. In this aspect, Needham ended up by becoming too similar (was it his Christian attitude?) to the Jesuit Matteo Ricci; see Part II, Sect. 8.2. The serious Eurocentric stereotype, about the lack of proofs among the Chinese, is discussed and criticised also by Marc Elvin. But even he does not succeed in doing so in a satisfactory manner, with his contaminated style typical of one who presumes to decide who was "superior" or "inferior", "ahead" or "behind" in the supposed race towards a "progress" which is as imaginary as it is senseless; Needham and Robinson 2004, pp. xxxi–xxxii. Chinese poets' anchoring their verses to space-time was stressed by Turner 1986. Cf. Fenollosa and Pound 1919.

[142]Needham & Wang II, 1956.

[143]Lloyd 1994, 160; Tonietti 2003b, p. 233.

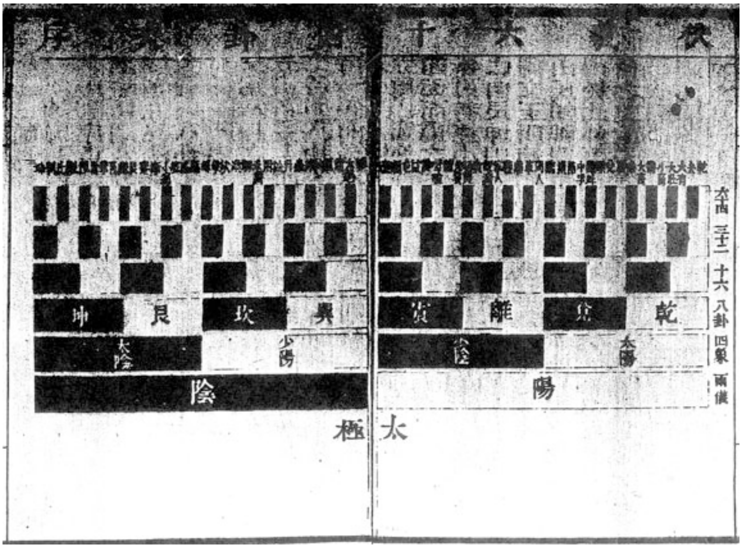

Fig. 3.7 How Yang and Yin penetrate each other in the "Extreme limit" (Needham 1956, vol. II, plate 21)

The figure of the string, as an argument for the fundamental property of the *gougu*, did effectively remain on earth. In the Chinese culture, they trusted the eyes that observe and the hands that shift and move the pieces of the figure. The sky could be seen and measured with the gnomon. Indeed, it could be understood by watching a shadow on the ground. The Chinese scholar remained down here, because he put his trust principally in the earth. Only the school of the Mohists criticised knowledge by means of the senses, and in particular by sight.[144]

Chinese scholars tried to arrive at knowledge by studying the links between things. They believed that the world was not white on one side and black on the other. A characteristic figure is the one called *taiji*. This can be literally translated as "the greatest limit", "the extreme limit" (Fig. 3.7).

This represented the interplay of the two principles that generate the world, *Yin* and *Yang*. It is clear that these were inseparable, and that their mixture, taking the process beyond all limits, became a sort of emulsion like a fractal. It is well known that Yin represents female, cold, damp, shadow, the moon ... whereas Yang refers to male, hot, dry, the Sun ... Another more popular modern figure is circular in shape, and is reproduced almost everywhere, as in the editions of the *Yijing*. Also in these, it is important to note the presence of the small black circle in the white area, and

[144]Graham 1999, pp. 188–189; Needham & Harbsmeier VII 1998, p. 254.

vice versa, a white circle in the black part; these reproduce on a small scale the initial circle. And so on, beyond all limits.[145]

Unfortunately, dualism is so deeply rooted in the way of thinking of Westerners, that even Yin and Yang are often considered in this light. The most famous precedent of this serious misunderstanding of Chinese culture (and of the *dao*) is the one found in Leibniz.[146] As regards numbers, a similar mistake was made by our valiant Needham, who was misled by the myth of a single universal science, even though the Taoist philosophies, which he knew very well and appreciated, should have led him elsewhere.[147]

The character *xin* means both heart and mind. In this case, the search for the link is no longer necessary, because the language has made them identical. In the Chinese culture, the *zhen* [the true] is not separated from the real, and so it remains on the earth. To arrive at the *veritas* of the Greek and Latin world, it would be necessary to write *zhenli*, but the *li* [reasons] would still maintain with the *zhen* reality as the authentic place that decides it. In the same way, *cuowu* [the false] is not opposed to the true, but would be more suitable to render the idea of a wrong or bad action. I becomes a moral judgement on the actions of human beings, an expression regarding the quality of things like food: *cuowu* is its harmful result.

In commenting on the *Jiuzhang [Nine chapters]*, Liu Hui criticised the astronomer and mathematician, Zhang Heng (first century).

> Suiyou wenci, si luandao poyi.
> [Although he writes in a classical style, [Zhang Heng] confuses the Dao [the way, the procedure] and damages what is right.]

How is it possible not to perceive, even here, for a mathematical calculation, the echo of a judgement that is more moral than technical?[148]

The absence of *veritas* in Chinese culture has already been effectively explained by Jacques Gernet: "The concept of a transcendent, unchangeable truth is foreign

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[145] *Yijing* 1950 [I King], p. 39; 1995 [I Ching], pp. 67–72. Needham & Robinson 2004, p. 90.

[146] Part II, Sect. 10.1.

[147] Needham & Wang III 1959, pp. 139–141; II 1956, pp. 339–345. Joseph Needham knew very well that Chinese culture had generally (not only with Taoism) remained distant from the dualistic transcendence typical of the West, at least until recent times. We can read of several cases in his books, which we, too, have appreciated. And yet he would have liked, thanks to his ecumenical idea of science, to succeed in finding a compatibility (in a Confucian or Taoist style?) between Christianity, Marxism and Taoism. The modern reader may judge by himself to what extent this was only the personal experience, and wishful thinking, of the professor educated at Cambridge in the Thirties, or, on the contrary, what an ironic twist of history was to concede in the China of the twenty-first century. But that conviction was so strong and deeply rooted in him that, unfortunately, it closed his eyes sometimes in front of St. John the Evangelist, Plato, the Legalists, Francis Bacon, or the quantum mechanics of Bohr and Heisenberg, with all its defects of nuclear war technology. Needham and Robinson 2004, pp. 84–94 e 232.

[148] On the contrary, with her different translation and interpretation, Karine Chemla blots out the moral judgement, and reduces it, anachronistically, to a criticism of errors of calculation; cf. *Neuf Chapitres*, pp. 62 and 382–383. Lloyd & Sivin 2002, p. 52.

to Chinese thought". "…Chinese thought has never separated the sensible from the rational, …nor has it ever admitted the existence of a world of eternal truth separate from that of appearances and transitory realities".[149] Thus, in the Country in the centre, even judgements which would be taken (or rather, misinterpreted) as technical and factual judgements in the West, were inevitably linked with morals, and with the good or bad result of human behaviour.

Among the "one hundred schools", one could surely be found which sustained a different position: that of Master Mo. There, on the contrary, the desire was to distinguish, to make a clear-cut separation, instead of connecting.

buke liang buke ye.
[It is not possible that both [are] impossible.][150]

In the evolution of Western sciences, the principle was indicated by the Latin name of "tertium non datur". In other words, there is no way out of the straightforward alternative between true and false. But the rival schools of the Confucians and the Taoists criticised this, and succeeded in overcoming it.

The *Zhuangzi* sustained: "How could the Tao be obscured to the point that there should be a distinction between the true and the false? How could the word be blurred to the point that there should be a distinction between affirmation and negation? …That which is possible is also impossible and the impossible is also possible. Adopting the affirmation is adopting the negation. […] The appearance of good and evil alters the notion of the Tao. […] The word is not sure. It is from the word that all the distinctions established by man come. […] The word that distinguishes does not arrive at the truth. …Knowing that there are things that cannot be known, that is the supreme knowledge."[151]

Discriminatory distinctions are to be found only in one passage with a Legalistic tone, quoted by *Lüshi Chunqiu [Springs and autumns of Master Lü]*, a text considered syncretic.[152] Among Mohists, Confucians and Taoists, disputes took place about the possibility, or otherwise, of discriminating between 'right' and 'wrong'.[153] It is interesting, however, that even the Mohists used the character *bian* with the meaning "to reason, to dispute, to debate", and not the similar, more stringent homophone, *bian* (drawn differently) "to differentiate, to distinguish, to discriminate". Thus, even those who were the fiercest supporters of the possibility

[149]Gernet 1984, p. 72 and *passim*; pp. 219ff. Hansen 1983, p. 124. Graham 1999, pp. 24–27, 31–32, 227, above all 264–271. Cf. Needham & Harbsmeier VII 1998, pp. 193–196. Jullien 2004, pp. 157 and 229.

[150]Graham 1990, p. 335. Hansen 1983, p. 121. Graham 1999, p. 226.

[151]*Zhuangzi* II, 1982, pp. 23–28. See also in VII the apologue on Indistinctness.

[152]Needham & Harbsmeier VII 1998, pp. 226 and 234.

[153]Needham & Wang II 1956, p. 180 and *passim*; Graham 1999, p. 43 and *passim*.

of “debating” what was ‘right’ and what was ‘wrong’, left the subjects of the dispute present in the term used.[154]

In the *Daodejing*, the exaltation of “non-discrimination” went so far as to connect “nothing” with “something”.

> youwu xiangsheng
> [nothing and existence are born from one another,]

“difficult and easy complete each other, long and short make up for each other, high and low are determined by one another, sounds and voices are in harmony with each other, former and latter form the series with each other.”[155] The famous *wuwei* [non-doing] represented another expression of the *youwu*. Is it really necessary to underline again that the conception was facilitated by the characters *wu* [nothing] and *you* [something], which referred to real things on the earth, subject to transformation?[156]

All the living creatures on the earth were a part of the same world-organism, including human beings, guided by their moral values. And this was the reason why the Chinese scholar could not have imagined any of his activities, including mathematical sciences, as independent of ethical behaviour. In this, with their behaviour, people, starting above all with the emperor, became jointly responsible for cosmic harmony.[157] The Confucian *ren* [benevolence, *pietas*], continued to guide the good scholar, also in the mathematical, astronomical and musical disciplines. Consequently, he did not conceive of any distinction between “man” and “nature”; as a result, nature, too, was charged with the relative moral values.[158] The *dao*, the procedure for obtaining the result, also depended on the *de* [virtue], as in the title of the Taoist classic, *Daodejing*. The fact that the links between sky, earth and man were real and effective was guaranteed by the unique universal *qi* which pervaded everything and made everything sympathetic.

3.6.3 Rules, Models in Movement and Values

> Dao ke dao, fei chang dao; ming ke ming, fei chang ming.
> Wu ming tian di zhi shi; you ming wan wu zhi mu.
> [The way, the right way [is] not the unalterable way; the description, the right description [is] not the unalterable description; the beginning of heaven and earth is undescribable; the mother of the 10,000 things is describable.]

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[154] Hansen 1956, pp. 82–88 and Chap. 4. Apart from the inevitable shifts in the meaning of characters, this subtlety escaped the attention of Graham 1990, pp. 335ff., who, however, corrected himself in Graham 1999, p. 43. On the contrary, it is overlooked by Lloyd & Sivin 2002, pp. 61ff.

[155] *Daodejing* II, *Tao te ching* 1973, pp. 31 and 178.

[156] Graham 1990, pp. 345–346.

[157] Graham 1999, p. 485. Lloyd & Sivin 2002, p. 78. Jullien 2004, pp. 172–173.

[158] Sun 2005, pp. 53–55.

The Chinese scholar who took his inspiration from the beginning of the *Daodejing*, would believe that the most appropriate description for a changeable world was a changeable description. He thought that studying the generation of things had a sense, but not trying to find out the beginning of heaven and earth.[159]

In the *Yijing [Classic for changes]*, the procedure does not finish in just one hexagram, but this may be transformed into another one. The Yang associated with 9 is transformed into Yin, while the Yin of 6 becomes Yang. The result, as represented by the two different hexagrams, appears to be unstable. Thus it is not fixed, but rather a process, a tension between two different situations.[160] What is to be done in the uncertainty of the conduct to be followed? What decision should be taken? The *Yijing* helps the enquirer to reflect, so that in the end, he or she can choose. It is not expected to reveal a destiny already established and unchangeable, learnt by means of a deterministic procedure. On the contrary, it is recognized that the world where we live is unstable and in continual transformation: it appears to be dominated by chance and by uncertainty. Why ever should we search for stable, eternal elements in it, on whose rules it would depend? In the *Yijing*, Chinese culture mimes its instability, rather, through a procedure that includes counts that depend on chance. In counting the stems, numbers are created which are necessary to know the uncertainty of the world.

In China, everything was represented as in movement and in transformation; certainty was achieved with the ability to shift from one thing to another, depending on the aims and convenience. The proof of the *Zhoubi* was obtained by moving pieces of figures with simplicity and in reality.[161] In a book of 1921, Liang Shuming wrote: "The Chinese have never discussed questions that derive from a static, unchanging reality. [Chinese] metaphysics has only dealt with change, and never with static, unchangeable reality."[162] Referring to the seismograph, a scholar of the eighteenth century had criticised it, affirming:

> *youding buneng ce wuding*
> [. . . [it] remains fixed, it cannot measure that which is not fixed.][163]

Enough has already been written, from various points of view, on the difficulty of finding a good equivalent of the European (also scientific) law in Chinese culture.[164] The philosophers-cum-ministers who inspired the guiding principles followed by the first Emperor, Qinshi, to subdue a large part of China, were called *fajia*. In the West, this name has been commonly translated as Legalists, because they would have liked to regulate everything with precision and intransigence, systematically

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[159] *Daodejing* I; 1973, p. 177; translation different from the one on p. 27. *Zhuangzi* XXVII; 1982, p. 256. Cf. Hansen 1983, p. 71. Graham 1999, p. 299. Cf. Jullien 2004, pp. 320 and 376–380.

[160] *Yijing*, any edition. Jullien 1998, p. 61.

[161] See above, Sect. 3.3. Needham & Robinson 2004, pp. 182–183.

[162] Gernet 1984, pp. 260, 263ff.

[163] Needham & Wang III 1959, p. 634. Jullien 1998, pp. 66ff.

[164] Needham & Wang II 1956, Chap. 18. Graham 1999, pp. 375–381 and 397.

and rigidly excluding everything that did not conform. But they fell with the first Emperor. The subsequent Han dynasty refused the methods of the Legalists, preferring *ren*, the sympathetic *pietas* recommended by Confucians. Since then, a dominant orthodoxy of this type has existed in Chinese culture.

In brief, people (in other words, unfortunately, men, in China,) were preferred to laws, and were considered as more important than abstract general principles. Consequently, the possibility was favoured of adjusting the rules to single cases, considering them as easily adaptable depending on the circumstances. Responsibility remained with the people who decide or choose, and was not passed on to (or rather, covered by) a superior, transcendent entity. The obvious limitation of the Chinese judicial system lay in the greater possibility of corruption. If the judge was good, and intelligent, the sentence would be more equitable, but if he was bad, or bribable, the result would be particularly unfair. In any case, anybody in power would have various means available to decide trials.

In the Chinese culture, the event that it was people who did things and therefore they bore the responsibility, was clearly shown by the use of the term *jia* [family] to indicate even schools of thought. The important person in these was the teacher, to be followed also in his moral behaviour, from whom the adepts were almost considered to be blood-descendants.[165]

For all these reasons, the Confucian man of letters could not believe that laws with universal claims were the best instrument to understand the multiple forms and continual transformations of the world in which we live. They would become too tight an attire, a rigid suit of armour that would hamper necessary movements.

The character *fa* was not often included in the titles of ancient books on mathematics or astronomy. On the basis of a rapid statistical review of the rich sample of books contained in the bibliography of volumes III and IV compiled by Needham, we find about 20 titles. Most of them refer to books that gave *suanfa*, that is to say, "rules for calculations", including the one from which we took the pages explaining the *lülü*.

In the *Jiuzhang suanshu [The art of calculating in nine chapters]*, there are very few examples of *fa*. In chapter one, problem 32, we find:

> *ran shifu cifa*
> [However, for generations, this rule has been taught,]

which was used to calculate the ratio, 3 to 1, between circumference and diameter.

Or:

> *fangcheng fa*
> [...*fangcheng* rule ...]

[165]Lloyd & Sivin 2002, pp. 52–55.

to solve systems of equations by arranging the numbers of the coefficients in a square. Only those heretics of the Mohists used to speak frequently about *fa*.[166] In the title of the book, *fa* was not used, but *shu*, that is to say, the "art", "craftwork" or "ability" of calculating by hand, using rods arranged in a certain orderly sequence on a table. The term *shu* was continually used in the text.[167]

Fa were also provided for the calendar, as well as for measuring fields, for water-clocks and for buildings. In any case, *fa* did not have a meaning similar to 'law', because it referred rather to a 'model', a 'way', a procedure to obtain something. It would thus appear to be a less rigid and absolute term, seeing that different models can be invented. For this reason, I prefer to translate it as 'rules for'. *Fa*, we shall see,[168] was to become the favourite term of the Jesuits for their books on mathematics and astronomy, which were presented as full of *xinfa* [new rules], though for them, these had now become laws.

Due to the use that the Legalists made of it, the character *fa* had not to be always popular in China. Chinese scholars preferred to use the character *li*, which means 'texture, weave, reason'. For example, it forms the word *wuli* 'the reasons for material things' that is to say, for us Westerners, 'physics'. Can this discipline be interpreted as 'the laws of matter'? Also in this case, the translation would be forced, because the idea of man following a spontaneous order would remain also in the *li*: *li dongxi* means "putting things in order". *Li* is always accompanied by the spatial image of the frame on which carpets are woven.[169]

In a culture that took living bodies as its model, where numbers were represented by wooden rods in movement on the table, also the mathematician Liu Hui took his inspiration for the *li* (far more often present than *fa*) from the butcher of the *Zhuangzi*.[170] Here, the *li* meant the spatial organisation of the body to be cut up. Thus, in order to render it better outside classical Chinese, we should avoid "laws", "principles" or "structures", and be content with "organs" and "organisations".[171]

For the geometry of the *Zhoubi* as for the pipes of music, for the calendar of astronomy as for the cycles of time and the seasons, Chinese scholars searched, not for laws, but for models. For them to be convincing, they had to be manifested in ways that were clearly visible and sensible. Preference was attributed to *xiang* [image] models, linked with phenomena.[172] They trusted appearances because they thought that behind the mask, they would find exactly what they saw. They

[166] *Neuf Chapitres* 2004, pp. 179–180, 635–636 and 918–919. But here *fa* has been translated by the overly Cartesian term of *méthode*. Graham 1999, pp. 199–200.

[167] *Neuf Chapitres* 2004, *passim* and 986–987. Now Karine Chemla's translation of *procédure* renders the idea appropriately.

[168] Part II, Sect. 8.2. Cf. Jullien 2004, pp. 273–276, 383ff.

[169] Gernet 1984, pp. 219–226. Graham 1990, pp. 420–435. Graham 1999, 391–394. Cf. Needham & Harbsmeier VII 1998, pp. 238–240. Also Jullien 2004, pp. 192 and 261.

[170] See above, Sect. 3.4.

[171] *Neuf Chapitres* 2004, pp. 950–951.

[172] Graham has quoted the "Great comment" of the *Yijing*; Graham 1999, pp. 497 and 499.

even related it in an apologue.[173] The world appears to us to change continually because it is really continually transformed. It would be vain to search for another representation of it. Where could it be hidden? What advantages would we obtain from inventing universal, absolute *fa* and *li*? It is useless to study the sky as independent from the earth, or the earth as governed by *fa* and *li* different from those of man. Man remained at the centre of things, and related to them.

For all these reasons, in Chinese scientific culture, nobody in general would have tried to detach *fa* and *li* from the literary language, maybe in an impossible search for absolute symbologies. The literal meaning of a term remained pregnant with other possible senses, through which the ambiguity maintained by the text would succeed in expressing the richness of its links with the other aspects of a world in continual evolution. And in the end, it would be the environment constructed by people that would give the text a comprehensible meaning, precisely because it is changeable like them in history. In the Country at the centre, there was to the letter the cult of history: history was consigned to two classic works, the *Shujing [Classic for [historical] books]* and the *Chunqiu [Springs and autumns]*, a history which *necessarily* (or rather, following the Dao) continually had to be rewritten.

Hence, we are forced to conclude that in general, we would not find any attempts to render the facts independent of the values, to put it briefly, in accordance with the current habits of certain Western philosophies. Here, in Chinese culture, the values that guided the behaviour of scholars, as they investigated the figures of geometry, the sounds of the pipes, that calculations of roots, the winds or the *qi*, were not hidden, but were rather presented as an integral part of the explanations. Even when commenting on poems, the Chinese man of letters avoided interpreting them in a symbolic, transcendent sense; he preferred to find in them, among the various figurative meanings hidden between the lines, the historical, political and moral values congenial to him. And he was so reluctant to lose himself in abstract symbols that he used the term *qixiang* [image of the *qi*, atmosphere] for the scene portrayed in the poems.[174]

3.6.4 The Geometry of the Continuum in Language

The harmony of music has guided us from the beginning to recognize the background against which Chinese scholars have generally set their affairs, including the naturalistic and mathematical ones. It was made up of the material *continuum* of the *qi*, impalpable and at the same time fraught with consequences. The event that we have dedicated our attention, and a whole section to it, should save us from misunderstanding Chinese scientific culture, as reduced to numbers. We may thus re-establish the balance with an equally important and interesting geometry, starting

[173] Tonietti 2006a, p. 239.

[174] Sivin 1986. Graham 1999, pp. 444, 480 and 485–486. Jullien 2004, pp. 193ff., 204, 216–217.

from the original kind contained in the "Figure of the string". That blending and confusing together numbers and geometry was typical of the organic Chinese culture of links, but it has sometimes been obscured by European scholars, subject to their totally different mathematical education.[175] And yet there are some historians of mathematical sciences who are more attentive to these geometrical aspects.[176]

Here, we would now like to call attention to that characteristic of the Chinese language in which the conception of the world as a *continuum* has remained impressed. Luckily for me, some Sinologists had already considered it to be important, and what's more, in a way that is suitable for this study. Also the linguist Chad Hansen has understood perfectly the geometrical-continual characteristic of Chinese culture. "…the world is a collection of overlapping and inter-penetrating stuffs or substances. A name (…) denotes (…) some substance. The mind is not regarded as an internal picturing mechanism which represents the individual objects in the world, but as a faculty that discriminates the boundaries of the substances or stuffs referred to by the names."[177]

To speak of a person, a cat, a book, or a horse, the Chinese say *yige ren*, *yizhi mao*, *yiben shu*, *yipi ma*. That is to say, they use the so-called classifiers *ge, zhi, ben, pi* between the number one and the thing to be counted. Also in English, we do not say "a water", but "a glass of water": *yibei shui*. Within its undifferentiated *continuum*, water presents itself to us as something that can only be counted if it is separated into different glasses or bottles. Well, by using classifiers, the Chinese language reveals that people, cats, books, horses and innumerable other objects are considered in the same way as the *continuum* of water. To be able to count them, they need to be separated into their respective *continua*, each by means of a different instrument, which depends, roughly, on the shape and the size of the object. Are we not justified, then, in thinking that there is a trace of a culture that conceived of its world as a *continuum*, even in the language?

Sinologists and linguists like Chad Hansen and Angus Graham have actually written about "mass nouns", as distinct from "count nouns". "…Classical Chinese nouns function like the mass nouns rather than the count nouns of Indo-European languages."[178]

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[175]Granet 1995. Needham & Wang III 1959. Graham 1999.

[176]Martzloff 1981a, p. 40. Martzloff 1981b, pp. 155 and 324. Jami 1988a. Siu 2000, p. 162. Wu 2001, p. 84.

[177]Hansen 1983, p. 30.

[178]Hansen 1956, pp. 32–35; Graham 1990, pp. 196–197. Unfortunately, in one of the last volumes of the work edited by Needham, Christoph Harbsmeier rejected the conclusions presented by Hansen and Graham as an "…untenable mass noun hypothesis …". But, in contrast with what is also my own conviction, backed up by the present essay, this Norwegian logician-cum-linguist did not offer any stringent argument, and referred to other studies to be carried out (and never actually performed?). Rather, he reveals, once again, how misleading the idea was of doubtlessly original Chinese inventors in mathematics and logic, but of the presumed single universal science, instead of appreciating their characteristic differences. Needham & Harbsmeier VII 1998, pp. 311–321. Tonietti 2006a, pp. 236–238.

It is precisely the insistent and general reference by many Chinese scholars to the *qi* that is (organically?) linked with the numerous collective nouns found in the language of the Empire at the centre. Bearing in mind, therefore, everything recalled in the previous section, pending the decision of some Sinologist or Chinese scholar to dedicate an entire book, more exhaustive, together with Hansen and Graham, I shall continue to relate Chinese scientific culture as a culture of the *qi* and of the *continuum*, which has remained impressed even in its language.

3.7 Between Tao and Logos

In Europe, it is natural that historians that speak about Greece are more common than those that study China. But among the former, those who, at least by way of contrast, have considered the latter, are very few. On the contrary, for the latter, the comparison with with their own Greek and Latin tradition has appeared, vice versa, to be inevitable and almost compulsory. It is surprising, then, that the differences between the two cultures have been seriously underestimated and less clearly understood by the latter, whereas we shall discover among the former those rare scholars with whom we can feel a greater harmony. In order to be able to hold a debate with them, we shall here compare the characters studied in the second chapter with those of this third chapter. It is highly unlikely that any direct meeting between them ever took place in historical reality, and in any case it would be impossible to narrate, as no documentary traces of such meetings are extant. In this connection, in spite of some well-known adventurous journeys like that of Marco Polo, we encounter significant events only starting from the late sixteenth century, as we shall see in Part II.[179]

Euclid reasoned by lining up one proposition after another in a straight line, starting from definitions and postulates. The more closely linked the passages and the more surely the extremes were fixed, the better his arguments held.[180] When it arrived on paper, is it not true that what was described in a language that wrote its sounds in a linear succession would also tend more naturally to assume a linear form? We find a completely different development in the *Zhoubi*, where, on the contrary, the proof follows a bizarre circular course, without reducing numbers to geometry and geometry to a few basic elements.

Greek needed to fasten on to something absolutely immobile in order to feel sure. In the Chinese text, on the contrary, the pieces of a figure were shifted inside a world in movement. Being convinced by Euclid, but not by the Chinese, would seem to me a question, if not of personal taste, perhaps of psychology, undoubtedly of culture. Unfortunately, however, some European scholars, including some all too authoritative, have refused the argument of the *Zhoubi*, as if it were not a true

[179] See Part II, Sect. 8.2.

[180] Cf. Lloyd & Sivin 2002, pp. 154, 166.

demonstration, because it is not modelled on Euclid.[181] Why should what is mobile become, for this reason, unreliable and superficial? Where could it come from this fear toward a world that changes, and where things are in movement? Can music perhaps help us to answer these questions? And yet, without moving, life is not possible; the immobility of a body is not a very positive sign. Cai Yong (second century) presented even the heavenly movements as irregular. "The motions of the Sun, moon, and planets vary in speed and divergence from the mean; they cannot be treated as uniform. When the technical experts trace them through computation, they can do no more than accord with their own times."[182]

Euclid tried to define geometrical figures, and started to detach scientific discourse from common language. He used words to which he attributed meanings different from the usual ones. In this way, Euclid tried to make them suitable for arguments that intended to avoid the inevitable ambiguities of language, and reach an absolute precision. But on an approximate, changing earth, this appeared to be impossible. In European culture, symbolism was to develop much more than in China (or anywhere else), because the explanations, answers to problems and arguments were sought by invoking elements that were not directly present, but invisible and transcendent. Symbols became indispensable because they made it possible to bring these elements closer, and use them. Symbols are letters, words, images created *ad hoc* in order to succeed in formulating discourses about the imaginary world of ideas, above us, separate, and not accessible in any other way from the world where we live. Thus an attempt is made to impose a discipline on this world, to dictate laws and regulate it by means of the symbols of the other world.[183]

Even though indications in this direction are rather scarce in Euclid, there was already an abundance of fundamental reasons in the Pythagorean and Platonic schools. Eventually, century after century, in Europe, there was, amid alternating fortunes, a blooming development of symbolism in mathematics and physics, starting from the seventeenth century. Chinese mathematical sciences, on the contrary, did not develop any symbolism, because they remained attached to the environment of the functionaries of the Empire, who were essentially selected for their literary ability to comment on the classical books of Confucius. Above all, they maintained their geometrical arguments and their calculations on the earth.

Angus Graham has linked the availability of a verb like *esse* [to be] to the relative capacity to treat the abstract concepts of European philosophies as real: "...such an immaterial entity more truly *is*, is more real, than the phenomena perceived by the senses." In China, on the contrary, the verb *esse* does not exist, and *you* refers to material things. "The verb 'to be' allows us to conceive of immaterial

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[181] Tonietti 2006a, Chaps. 2 and 3. Needham & Robinson 2004, p. xxxi.

[182] Lloyd & Sivin 2002, p. 192.

[183] The event that, unlike that of the Greeks, Chinese culture was, in general, neither symbolic, nor allegorical, nor transcendental, has also been explained by François Jullien through the collections of poems, the *Lunyu [Dialogues]* of Confucius, the *Daodejing* and the *Zhuangzi*; Jullien 2004, pp. 186ff., 194, 198, 211, 251ff., 313, 333–334, 350, 369–372.

'entities' detached from the material, for example, God before the Creation. But if the immaterial is a Nothing which complements Something, it cannot be isolated; the immanence of the Tao in the universe is not an accident of Chinese thought, it is inherent in the functions of the words *you* [to have, to exist, there is] and *wu* [nothing, not to exist]."[184]

The Greeks separated heaven from earth, soul from body, eternal ideas from ephemeral phenomena, the only universal truth from deceptive appearances, and so on. We have seen theirs was a dualistic culture. Whereas Chinese scholars did not spend a large part of their time, like their Western colleagues, in dividing everything into two parts. Greek and Latin culture, on the contrary, favoured transcendental, dualistic explanations. Here, books are full of truths and falsehoods, good and evil, soul and body, friendship and enmity. Innumerable *aut aut* alternatives can be found between finite or infinite, created or eternal, atomistic or continuous, immobile or in movement. Western philosophies of sciences theorised the distinction between primary and secondary qualities. In order to allow greater freedom of movement and transformation, Confucians and Taoists did not want to distinguish or discriminate, whereas Plato and Aristotle did nothing else. Aristotle did not limit himself to classifying animals and human beings between male and female, but also theorised that the separation was a good thing (except in one negligible case), because in this way, the male could dedicate himself to higher functions in the hierarchy. The distinction between free citizens and slaves had a very significant corollary, which helps us to understand the cult of dualism practised here. Not only did the separation lead to the inferiority of the slave, but the theory completely ignored the event that the real life of the former was closely correlated with the latter.[185]

In time, the Greek and Latin culture were to come into contact, and to be pervaded and conquered, by the Jewish-Christian one, derived from the Classic of the West, the *Bible* [The Book], the only book that has had the same influence on European culture, for thousands of years, as the Five Classics and the Four Books have had on Chinese culture. With the spread of the *Bible* in the Western world which was the heir of the Greeks and the Romans, the distinction between good and evil, soul and body, true and false was to become law: the absolute dogma of the whole of the dominant European culture. The Greek *logos* [discourse] was interpreted as the God of the Christians [the Word].[186]

With the *qi* and the *dao* [Tao], progress was made by mixing, blending, confusing, bringing into contact, considering the real world as connected by means of countless links, which were not to be broken.[187] The preferences in favour of a transcendent dualism in Western sciences had their roots not only in Greek philosophy, but also in the *Bible*. The differences between Chinese sciences and

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[184] Graham 1990, pp. 344 and 346. Graham 1999, pp. 303–304. Likewise, Gernet 1984, pp. 259ff. And lastly, Jullien 2004, pp. 257–283, 344ff.

[185] Lloyd & Sivin 2002, pp. 128, 203, 181–182.

[186] *Genesis*, **3**, 5. John 1, 1. Tonietti 2006a, p. 208. Jullien 2004, p. 425.

[187] Lloyd & Sivin 2002, pp. 196ff.

those of the West partly derived from there. We all know that not infrequently, the evolution of sciences in Europe encountered religious problems. We shall come back to this subject in Part II, when Western sciences had been brought to China, in the wake of Christianity.[188]

What instruments had been invented by the Greek and Latin characters to divide the True from the False, Good from Evil, the good from the bad, friends from enemies, compatriots from foreigners and so on? Unchangeable, eternal laws. This culture introduced all kinds of them. α'$\iota\tau\iota'\alpha$ in Greek meant both the cause (of an effect) and the blame and accusation in legal trials. The term was used both by legislators and in tribunals for legal cases, and by philosophers, doctors and scholars of $\phi\upsilon\sigma\iota\varsigma$ [nature], to indicate the laws and causes. Latin and Italian maintain both the legal and the naturalistic meaning of the Greek (like English).

Having eliminated Jupiter, Poseidon or Juno as the agents responsible for phenomena and illnesses, now it was necessary to separate the human world from the natural world, in order to make the laws of *physis* impersonal, and take away the responsibilities from the inventors. Deductive proofs, starting from sure bases in the style of Euclid, were thought by Ptolemy and by Galen to be the best way to reach results that were certain and precise, and thus free from debatable human uncertainties, to which no objections could be made: here is the truth; there is nothing more to be said! And yet Plato and Aristotle called such a demonstration '$\alpha\pi o\delta\epsilon\iota\xi\iota\varsigma$, which was again a term also used in tribunals, to mean "evidence" shown, or exhibited.[189] Language continued to betray what it was now preferable to hide. Responsibility had been taken away from anthropomorphic divinities, without any personal assumption of it, but assigning it to a concept, or an idea, that of nature or a mathematical law a part of a world that was not earthly, and at the same time divine as well, because it was separated from the human.

First, the laws had religious origins, also in the *Bible*, and then the Greek-Roman *lex* arrived, and so on. On the contrary, in the Empire at the centre, no divine laws were introduced, but rather human rules, suitable for certain limited purposes, and models in movement were proposed, to include, all together, sky, earth and man, with all the relative responsibilities and the inevitable imprecisions.

In the Pythagorean schools, whole numbers were the general principle followed in order to overcome the instability of the world, which for them was only apparent. For this reason, numbers were imagined as transcending the earth: they had been raised to the heaven of an esoteric mysticism outside space and time. European numerological traditions should not be confused, therefore, with the position assigned to numbers in the Chinese culture. Here, we have seen them spring from the Tao in the earth, among all the other things, and as cuts in the *continuum* of the *qi*. Again following Hansen, "This 'cutting up things' view contrasts strongly with the traditional Platonic philosophical picture of objects, which are understood

[188]See Part II, Sect. 8.2. Tonietti 2006a, Chap. 4. Cf. Gernet 1982.

[189]Lloyd & Sivin 2002, pp. 161–165 and 168–173.

as individuals, or particulars, which instantiate or 'have' properties (universals)."[190] Unfortunately, even renowned Sinologists have looked at calculations performed by the Chinese through Pythagorean spectacles. Having taken the wrong road, they also arrived at a misunderstanding of the mathematics of the pipes for music.[191]

In the light of the results achieved by Chinese culture, also in the mathematical sciences, and having understood better its differences compared with the Greek and Latin tradition, what questions should we ask ourselves? Those of Marcel Granet? "Can a language which suggests, rather than defining, be fit for expressing scientific thought, for its diffusion, for its teaching? A language made for poetry, composed of images, instead of concepts, is not only without any instruments of analysis. It does not even succeed in forming a rich heritage, with all the work of abstraction that every generation has succeeded in accomplishing".[192] Even the highly renowned French Sinologist sustained that the Chinese language had always facilitated the concrete expression of things, instead of abstract ideas, had often sought explanations by means of comparisons by analogy, rather than practising (Cartesian) distinctions, had undoubtedly offered representations with intuitive visible images, in contrast with ideal (Platonic) concepts, had imagined a world in continual movement, dominated by the rhythm of songs and dances, seeing the rules of social order and of natural events united in it.

But then he again asked the same question, which unfortunately sounds rhetorical. "All in all, as long as thought is orientated towards the particular, as long as Time, for example, is considered to be a group of durations of a particular nature, and Space is thought to be composed of heterogeneous extensions, as long as language, a collection of singular images, confirms this orientation, and as long as the world appears to be a whole of particular aspects and mobile images, what dominance can the principles of contradiction or causality assume without which it does not appear to be possible to practise, or express, scientific thought?"[193]

Here I have offered (sufficiently documented?) arguments to give the opposite answer. Chinese scientific culture has been, and may still continue to be misunderstood, if two paths are followed which do not lead to the inside. If we accept the particular characteristics of this culture which make it different, we should not transform the differences into inferiority or exclusion. On the contrary, these should be maintained and appreciated, not only for music, poetry or cooking, but also in the sciences.

The other misleading route is to recognize, undoubtedly, and fully enhance the contributions of the Chinese, like Needham and his other colleagues, only to flatten them out, subsequently, in a single universal science, proceeding triumphantly towards eternal truths.[194] These defects were not avoided even by Graham, when he

[190]Hansen 1983, p. 30.

[191]Granet 1995, pp. 111, 149–150 and 156–186. Tonietti 2006a, p. 228.

[192]Granet 1953, p. 154. Cf. Needham & Harbsmeier VII 1998, p. 23.

[193]Granet 1953 *passim* and p. 155.

[194]Needham & Harbsmeier VII 1998, pp. 4, 254–266.

ventured on to the unfamiliar terrain, for him, of science. He admitted that Chinese culture makes no distinction between facts and values. Would he accordingly have taken for granted that their alleged independence, as sustained by certain Western scholars, was a necessary condition for the sciences? Then how could Chinese culture have produced them?[195]

But as we have just related some scientific results obtained by the Chinese, showing their particular characteristics, the event that they do not hide their relative values should not become a reason for exclusion. The argument should thus be turned upside down, into the simple: not even sciences succeed in distinguishing facts from values, either in China, or in the West. The relative scientific differences thus stem partly from relative differences in values.[196]

In the Chinese cultural environment, scientific knowledge did not assume the style of absolute, transcendent, dualistic laws, represented in a linear symbolic language, but rather that of complex, relative and earthly models, expressed in the characters currently used by men of letters.[197] Although they are almost always considered and presented as absolute, indisputable and independent by the people who invent them, even mathematical sciences share values. In the comparison made here between China and Greece, we have found a particular confirmation of this. I could say that in the "Figure of the string" different values may be observed from those present in the "Theorem of Pythagoras" and particularly distant historical circumstances have been studied.

Also Jacques Gernet wrote about the "…radical differences in the traditional ideas of mathematical activity …" … "…the *li* is the immanent reason in a universe made up of the combination and alternation of contraries …".[198] The Greeks stood up straight on the earth, but their eyes contemplated the sky. The Chinese remained bent over, looking at the earth. "If you say that it is necessary to lift up your head to examine the sky, then the beings that possess life and feeling will lose everything that makes up their roots."[199]

In Greece they elaborated the theory of the four elements, in China the *qi*, seen as "…a synthesis in which heaven, earth, society, and human body all interacted to form a single resonant universe." In Greece, they discussed as if it were always a question of (dualistic) contrasts between ideal concepts. In China, disputes took

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[195]Graham 1999, pp. 440, 480, 485–486. Tonietti 2006a, pp. 231–234.

[196]My oldest personal memories go back to when it was sustained that sciences were not 'neutral', but rather 'historically and socially' influenced, in essays like those of Ciccotti, Cini, De Maria, Donini & Jona-Lasinio 1976 or Donini & Tonietti 1977. However, there have undoubtedly been others, before, during and afterwards, who have sustained similar theses, but with scarce subsequent developments. One of the few who continued to move in this direction was Marcello Cini 2001. See also Lloyd & Sivin 2002, p. 174.

[197]Needham & Harbsmeier VII 1998, p. 408. Karine Chemla 1990 has studied how certain mathematical texts followed a "parallel" style, particularly widespread among classical literary texts.

[198]Gernet 1984, pp. 257–266. Gernet 1989, pp. 329 and 332.

[199]Quoted in Gernet 1984, p. 169.

place, on the contrary, between people, with all their baggage (morals included). Furthermore, here attempts were made to cover them up and avoid them, in the desire to arrive at all costs at harmony and consensus, whereas in Greece, comparisons were sought, and attacks were made openly, in public and with a pleased way, on the ideas of the adversary.[200]

In an attempt to penetrate through appearances, Greek philosophers investigated the hidden principles, the foundations and the causes; the Chinese correlated the visible and audible aspects, without bringing their senses into doubt. The former studied an idea called ϕ'$\upsilon\sigma\iota\varsigma$ [nature], the latter did not even have any equivalent character. The modern use of *ziran* as referring also to "nature" began only in the nineteenth century, when Western "natural" sciences had already been present in China for some time. In his commentary on the *Zhoubi [The gnomon of the Zhou]* Zhao Shuang (third century) used *ziran* in the sense of "Proportions which [in the right-angled triangle] correspond to each other naturally", meaning, "spontaneously, inevitably, not forced". The character *zi* means "by itself".[201]

The majority of Greeks though of living bodies as structures of organs and tissues traversed by various humours; the Chinese expressed themselves differently. In the *Huangdi neijing lingshu [Classic of the yellow Emperor on the interior [of the body], the divine pivot]*, "The subject of discourse, briefly put, is the free travel and inward and outward movement of the *shenqi* [divine *qi*]. It is not skin, flesh, sinews, or bones." Surgery was not practised, in general, before the third century. We can quite understand that cutting the body would have meant dramatically interrupting the flow of the *qi*. "Man is given life by the *qi* of heaven and earth and grows to maturity, following the norms of the four seasons." In the *Lüshi Chunqiu [Springs and autumns of Master Lü]*, "When illness lasts and pathology develops, it is because the essential *qi* has become static."

The stars were grouped together in the Chinese sky in accordance with the hierarchies of the imperial palace, with the Pole Star called the Sovereign of the North. In Europe, everybody knows about the heroes and the myths projected on to the sky by the Greeks.[202]

> Hellenic thinkers fundamentally redefined rare words, or coined new ones, to take the initiative away from their opponents. Elements $\sigma\tau o\iota\chi\epsilon\tilde{\iota}\alpha$, nature $\phi\upsilon\sigma\iota\varsigma$, and substance or reality o'$\upsilon\sigma'\iota\alpha$ are examples (...). Chinese cosmologists instead adapted or combined familiar words to fit new technical contexts, which their old meanings still influenced.[203]

Table 3.2 summarises the differences between China and Greece as regards their concepts of music.

However, all these differences between the two cultures might risk assuming a dualistic, Western form. Now, therefore, we must show that the comparison is

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[200] Lloyd & Sivin 2002, pp. 241, 53, 61–68, 129.

[201] Lloyd & Sivin 2002, pp. 158–165, 200ff. Above, Sect. 3.3 and Tonietti 2006a, pp. 36–37. Cf. Jullien 1998, p. 105.

[202] Lloyd & Sivin 2002, pp. 219–220 and 223–224.

[203] Lloyd & Sivin 2002, p. 238.

Table 3.2 Comparison between Chinese, and Greek theories of music

China	Greece
Lülü [pipes]	Monochord
qing [clear] zhuo [turbid]	High acute low deep
Five phases	Seven heavenly bodies
Circle court	Scale
No octave	Octave
Notes are generated in movement	Static immobile
Fractions	Ratios
Lengths	Numbers abstractions
12 lülü	Seven notes
12 dizhi [12 earthly branches] qi [24 seasonal terms] hou	Seven heavenly bodies
Music of the earthly atmosphere	Music of the heavenly spheres

more complicated, because neither China nor Greece should be reduced to the most orthodox scholars.

The school of Master Mo sustained the need to discriminate between true and false. There, a logic with two values was cultivated, inspired by legal questions, and preferentially, debates were about *fa*, where the character now shifted towards the "laws". More than linking together sensible aspects, the Mohists investigated "causes".[204] Between opposites, these scholars imagined clear-cut, insoluble conflicts. Thus they did not search for either the benevolent harmony of the Confucians, or the sceptical nuances of the Taoists. In the *Zhuangzi*, it was written that the Mohists had disputed against music, preferring economy. According to this text, they expressed their condemnation for war, and preached universal love.[205] In spite of this, according to others, the followers dedicated themselves also to the construction of war machines. But in the Empire at the centre, the Mohists were kept at the margins by the Confucians of success.[206]

In Greece, a character like Aristoxenus had been excluded from the Platonic-Pythagorean orthodoxy for music. The continuous idea of the world resisted only outside the circles of mathematical disciplines, e.g. among Aristotelians and Stoics. Among the latter, the $\pi\nu\epsilon\tilde{\upsilon}\mu\alpha$ [puff, breath, air] is at least partly similar to the *qi*. In Greece, there was not only the *logos*, but also the $\mu\tilde{\eta}\tau\iota\varsigma$, that is to say, the astuteness of the mythical Ulysses combined with the practical ability of craftsmen. However,

[204] Needham & Harbsmeier VII 1998, pp. 286–287ff. Graham 1999, pp. 185–229. Lloyd & Sivin 2002, p. 159.

[205] *Zhuangzi* XXXIII 1982, p. 308. Lloyd & Sivin 2002, p. 213.

[206] "…military engineering, an almost exclusive activity of the Mohist school."; Graham 1999, pp. 450 and 504. Jullien 2004, pp. 292–293.

it was underestimated and put to one side with respect to the other dominant rational concept.[207]

In brief, the orthodox and the heretics were selected in different ways in the Graeco-Latin and Chinese contexts. Thus the Mohists would have been more successful in the West, because they were similar to the orthodox here. Between Greece and China, we can also find some similarities, but the different selective filters made scholars orthodox in Greece and heretics in China. Vice versa, those who dominated in China, like the advocates of the *qi* and the *continuum*, would have found little space among the mathematicians in Greece.

And yet, also the internal distinction between orthodox and heretics would be too sweeping, it would change in time, and should thus be taken 'beyond all limits'. European scholars of mathematical sciences would have been variously placed, changing into orthodox or heretics depending on the historical context and alternating fortunes. Surfaced Lucretius, Leonardo da Vinci, Simon Stevin and so on, down to Luitzen Egbertus Jan Brouwer (1881–1966), Ludwig Wittgenstein (1889–1951), Albert Einstein (1879–1955) and René Thom (1923–2002).[208] Some of their positions might have enjoyed greater fortune in China.

In 1923, Einstein would have like to go to Beijing from Japan. "Beijing is so close and yet I cannot fulfil my long-cherished wish [to visit it], you can imagine how frustrated I am." Hu Danian tells us that Relativity aroused the enthusiasm of the educated Chinese public, and was rapidly absorbed after 1917. "…the absence of the tradition of classical physics in China seemed to have helped the Chinese to absorb the theory of relativity within a short period of time and virtually without controversy."[209] In this, I believe that a positive role was also played by the presence of a long tradition linked with that pervasive energetic fluid called *qi*. Let me recall that to indicate "universe" or "cosmos", the Chinese use the expression *yuzhou*, whose characters literally mean "space time", and the restricted and general theories of Relativity offered them a new conception also of *space-time*.

Various reasons can be found for how and why the selection took place, and among these, we should not underestimate either chance or the heterogenesis of purposes (or in other words, the stupidity of obtaining completely different effects from those desired). Sometimes, however, we are faced with what we are tempted to call, anachronistically, undoubtedly, an explicit scientific policy. As a reaction to a military defeat in 258 B.C., suffered by his king, Qin (the future victorious first Emperor), Lü Buwei gathered a group of scholars and sages: he invited them to compose the *Lüshi Chunqiu [Springs and autumns of Master Lü]* (third century B.C.). A similar case, with comparable military repercussions, was that of the

[207] Needham & Wang III 1959, pp. 626 and 636–637. Needham, Wang & Robinson IV 1962, p. 12. But also in this case, it would be better to underline the differences between *qi* and the *pneuma*, as did Lloyd & Sivin 2002, pp. 8–9. Jullien 1998, p. 12.

[208] See Sects. 2.8, 6.4 and Part II, Sects. 8.1, 12.4. Tonietti 2006a, pp. 212 and 230–235.

[209] Hu 2005, pp. 90 and 110.

Huainanzi [The Master of Huainan] (second century B.C.) attributed directly to relative prince Liu An.

When the Han had centralised decisions in the imperial palaces, everyone can imagine how much easier it became to conform anything to orthodoxy. For the calendar, the rites, or to justify decisions taken, the "... state's uses of mathematical astronomy shaped it." Whereas "... the Mohist lineages dwindled and died out; ..." In some cases, to maintain Confucian orthodoxy, a way was followed that was not at all Confucian. "The therapy he prescribed was for the emperor to have large numbers of his own flesh and blood executed", including the previous rival prince, Liu An.[210]

But we should not believe that, from the beginning with autocratic decisions, the Chinese impeded discussions and selected scholars, while those the Greeks would have left free to confront each other, until the better man won, that is to say, in their *logos*, the truth. Because also in Chinese culture, we may observe a wide variety of different positions among the followers of Confucius, Mozi, Laozi and a hundred other school-families, who held countless debates on every subject (also scientific).[211] Vice versa, also the bitter disputes to win the argument between the followers of Pythagoras, Heraclitus, Parmenides, Plato, Aristotle, Aristoxenus, Euclid, Claudius Ptolemy and a hundred others had all been influenced and shaped in various ways: by all these Archytases, Alexanders, Ptolemies I and II, Gerons, as well as by the anonymous *agorà* [assemblies] where thousands of people expressed their political choices.

Thus the differences described here, clearly visible in the results of the selection, undoubtedly would depend on the absolute decisions taken by the Han emperors, for example, in favour of Confucianism. But it did not happen that on the other side of the world, there was a Euclid or a Ptolemy, fairly free to move around Alexandria, between the Library and the Museum founded by Alexander's generals, without ever being influenced in the slightest.

On the contrary, we know that one of the Stoics tried to incriminate Aristarchus of Samos for daring to move the Earth from the centre of the universe, putting the Sun in its place. And seeing the discredit that his theory encountered in his times, we may fear that it really happened, even if we hope that philosopher had not acquired enough power to put his threat into practice. But wouldn't that have been more likely to happen if Plato had succeeded in setting up his Republic of philosophers in Athens or at Siracusa, as well as being protected by Archytas?[212] On the basis of studying similarities and differences, in ancient times between Greece and China, the latter, far more visible, were the result of historical contexts, which in both cases thus acted as a filter, discriminating in various ways between orthodox and heretics, using their different means for different purposes.

[210] Lloyd & Sivin 2002, pp. 29ff., 38, 48, 52, 55, 63–68, 76–77, 243.

[211] Hansen 1983. Graham 1999.

[212] Lloyd & Sivin 2002, p. 171.

Innumerable disputes, both theoretical and practical, were held in Greece and in Rome, around the various monarchies, oligarchies, tyrannies, empires, limited democracies, rare citizens' republics and totally absent anarchies. Sometimes philosophers even took them as models for their own cosmologies. In China, the alternative between simple kings and princes disappeared with the emperors. The first of these was a lover of war, inclined to impose his inflexible laws above all else. The following emperors, on the contrary, maintained order successfully with the 'benevolence' of the 'just mean', guaranteeing harmony in the union of sky, earth and man. These were different hierarchic strategies in order to maintain power. The second system abounded in ministers, advisers and even music masters, capable of influencing the political line of the Emperor; the former system did not admit any interference, and there were no ministers at all.

One nuance that should not be overlooked derives from the opposite style followed by scholars in sustaining their arguments. In Greece, they were presented as a way to contrast categorically those of the adversary, with the aim of gaining greater visibility and publicity, from which they could sometimes obtain not only prestige, but even financial advantages. Especially at Athens, discussions were held everywhere, and on every topic, in public assemblies, whether the decisions to be taken regarded war policies or support for cases in the tribunal. The practice had been extended also to philosophers, medical doctors and scholars. "Much Greek philosophy and science thus seems haunted by the law court – by Greek law court, that is, where there were no specialised judges, no juries limited to a mere dozen people, but where the dicast could number thousands of ordinary citizens acting as both judge and jury. Nobody who has a philosophical or scientific idea to propose in any culture can fail to want to make the most of it. But a distinctive Greek feature was the need to win, against all comers, even in science, a zero-sum game, in which your winning entails the opposition losing."

On the contrary, the art of rounding off sharp edges, to leave the way open to the harmony of mediation, without ignoring the problems, but looking for hidden implicit solutions, was particularly fostered in China. And yet, continuing to follow the Tao 'beyond all limits', even the Graeco-Latin world recorded the attempts of a neo-Platonic, Simplicius (sixth century), to reconcile Plato with Aristotle, or a Galen who did the same with Plato and Hippocrates. In Greece, philosophers felt that they were authorised to follow their own private interests, of their caste or their school, without any sense of shame. In China, everything had to appear to be done for the common good, even if it was really all a trick.[213]

It should not seem to be paradoxical, but rather inevitable, that in the end, the very Greeks who were lovers of everlasting, heated discussions, invoked the independence of the truth from themselves, although up to that moment, they had fought each other with thrusts of *logos*. They did this astutely, in order to take away the arms of rhetoric and dialectic from the adversary, after using them incessantly with great ability.

[213] Lloyd & Sivin 2002, pp. 180–183, 118–138, 246.

The explicit conquest of the truth in the dialogue-dispute between earthly people was hidden behind the implicit mask of divine, disembodied, eternal ideas. In this, Euclid was unrivalled for centuries. At the opposite extreme, those who intoned and painted ambiguous polymorphic Chinese characters, in a submissive, and at times subservient manner, continued to maintain the results of the disputes in contact with the people who were walking along the *dao*. Here in the Country at the centre, what was implicit in the discussion had become explicit again in human decision-taking. What was at stake now should be how much hypocrisy to tolerate facing a reality of things in continuous movement. We shall come back to this subject when the confrontation, for the moment only imagined, had to become a dramatic, real, historical clash.[214]

Among the reasons capable of facilitating and aiding the development of one direction of studies rather than another, we again have to include military requirements for making war. Not that there was a lack of them in China, especially in the crucial period between the "Springs and autumns" and "Fighting States", but subsequently the followers of Confucius declared that they were putting them aside, and concentrating on harmony.

In the *Chunqiu [Springs and autumns]*, the wars continually narrated from East to West were truly deplorable: "Zhou Xu trusts in arms, and relies on cruelty; he who trusts in arms remains without the crowds, he who relies on cruelty remains without relatives. It is difficult to succeed when the population rebels, and relatives abandon you. Arms are like fire: if you do not extinguish it, it will burn you." "War is the bane of populations – . . . – an insect that devours their resources is the greatest calamity for small states. When someone wants to make it stop, even if we say that it is impossible, we must express our agreement." But then, now and again, some realistic details slipped within. "Without that subjection, they would be arrogant, and due to their arrogance, disorders would break out. When disorders arise, [states] are undoubtedly destroyed: this is how they come to an end. The sky has produced five *cai* [materials] and the population uses all of them, without being able to forego any of them. Who can suppress arms? Arms have imposed their presence for a long time: they are what is used to keep in subjection those who do not observe the laws and do not shine for elaborate virtue. Thanks to them, saints flourish and the turbulent perish. The law that regulates perishing and flourishing, surviving and dying, darkening and shining, is dictated by arms. When you want to suppress them, are you not deceiving?"[215]

Some historians have, quite reasonably, identified the five *cai* [materials] as the five materials, with which arms were made. These five then became the precursors of the very famous five *xing* [phases] which Chinese books were full of. We have already recalled that it was a military defeat that led Lü Buwei to make his famous collection of studies mentioned above. But his sovereign, Qinshi Huangdi, did

[214]See Part II, Sect. 8.2.

[215]*Chunqiu* I, year 4 (719 B.C.) 3; IX, year 27 (546 B.C.) 2 and c; *Primavera e autunno [Spring and autumn]* 1984, pp. 11, 553 and 559.

not show any gratitude, and our merchant-cum-scholar was compelled to commit suicide. A successor of Confucius like Xunzi (third century B.C.) left us a work in which he discussed with an army commander the use of military force. In the *Guanzi[Master Guan]*, a chapter entitled "The five officials" (perhaps third century B.C.) discussed how to win the war, following a strategy not very different from the famous art of deception sustained by Sunzi. The followers of Master Mo had given an ambiguous image of themselves: how to be contrary to war and military conquests, but all the same engaged in developing the relative techniques? Did they perhaps admit so-called 'defensive' warfare?

Some scholars of the Han period received positions in the army, above all as civilians appointed to lead the soldiers. The Emperor Han Wudi (140–87 B.C.) followed a policy of conquests, and expanded the Chinese territories considerably, continuing, however, to centralise them. His aims, therefore, were not so different from those of Qinshi, and actually he was nicknamed "Martial". Yet he was the one who established which Confucian classics were to be studied. Putting them next to the names of the territories captured, historical accounts related that our Emperor Martial invited to his court technicians who were experts in hundreds of arts, including a substantial number of astronomers-cum-astrologers. We are therefore justified in suspecting that in this way, he sustained his military ambitions. He must have learnt the relative strategy from his relation, and rival for power, Liu An, who was put to death by him. In the book already mentioned several times, *Huainanzi [Master of Huainan]*, the defeated prince dedicated the 15th chapter to the military arts.[216]

Should we conclude from this, as Sunzi had suggested, that also the ostensibly 'pacific' imperial policy had become the art of warfare pursued by other means? And as this is based on deceiving the enemy, should politics, consequently, also be interpreted as the art of lying, saying one thing in order to do another? And where do the teachings of Confucius end up? But judging negatively the traditional (according to the well-known stereotype) duplicity of the Chinese would again betray the assumption of the direct, naked, Western, dualistic truth as right. On the contrary, we should rather recognize the Chinese ability to move along ways that are indirect, transversal, tortuous, circular, subtle, dissimulated, hidden and also effective.[217] Let us again place this tortuous procedure of the *Zhoubi [The gnomon of the Zhou]* in comparison with the straight lines drawn by Euclid, but without any hierarchy. In China, it would have been indelicate to enter directly into a room. The screens

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[216]Lloyd & Sivin 2002, pp. 257–258, 262, 66, 208, 213, 37–38, 207, 297; Sabattini & Santangelo 1989, pp. 144ff., 163.

[217]On the hypocrisy of the Confucians, see *Zhuangzi* XXIX, 1956, pp. 274–281. Jullien 2004; Jullien 1998. Joseph Needham loved to say that in the Western culture, a confrontation proceeded directly by means of a "slap in the face". In the Chinese culture, on the contrary, dialogues continued amid smiles, while the participants prepared "a stab in the back"; Needham and Robinson 2004, p. 233. Let's look on the bright side: our future will undoubtedly be full of dialogues and smiles, but also stabs in the back. Or is this a stereotype as well? We need to interrogate the *Yijing*.

placed after the door indicated an oblique course to be followed. The evil demons would go straight in, and as a result would not succeed in passing. A poet of the Jin period (third to fifth centuries) wrote: "Short cuts are very useful, but only tortuous pathways arouse fascination and wonder." The Chinese constructed their gardens, in general, in accordance with this rule.[218]

Anyway, like the generals of Sunzi, also the imperial Confucian generals had to "perform many calculations before the clash". Though it was perhaps true that the Han had "... transformed the ideal of the ruler once and for all from a military strong man into a ritualist ... ", this had undoubtedly been done because "... ritual is more effective for the purpose than force, although from time to time, he also relapsed into coercion and violence."[219]

In Greece, from Heraclitus to Plato, from Heron to Archimedes, without forgetting the medical studies performed by Galen, we have noted a much more explicit presence of the problems of warfare in arguments and in the sciences. In the second chapter, we have already re-read the pages from the *Republic* written by Plato on the training of the soldiers who were the guards of the State, by means of mathematics and music that was not lascivious, or the pages about the feats of Archimedes during the siege of Siracusa.[220] Here I may add that a follower of Parmenides like Melissus of Samos (fifth century B.C.) made a name for himself also as a politician and a general, by defeating the Athenian fleet in his capacity as admiral.

The Ptolemies, who were the sovereigns at Alexandria in Egypt, did not develop their institutions for scholars, thinking only of the glory or the truth. "Philo of Byzantium [II century] reports that they also supported engineers, but [sic] that falls into a different category, for their research – into catapults – had military applications." We have already recalled also Vitruvius (first century A.D.) for his work with the Roman army.[221]

The political, legal and philosophical confrontation in Greece, the $'\alpha\gamma\omega'\nu$ [competition, debate, discourse, trial] with the *logos*, might be compared with that other $'\alpha\gamma\omega'\nu$, the same word used to indicate the battle conducted in wars by the phalanxes of soldiers, shield against shield, lance against lance. In that case, the clash was decided by the greater number of armed troops, with the well-known exceptions. In the former case, the Pythagorean religion based on whole numbers that provided the general foundation of the world, or the attention that Plato showed for their mathematics, would seem to find a certain elementary practical justification. Thus, even that typical Graeco-Pythagorean conflict between numbers and geometry would acquire, for its part, military harmonics. In China, victory was not made to depend on the quantity of armed troops, because even a small number, if they carefully prepared favourable circumstances, had the *de* [virtue] to succeed,

[218]Chen 1990, p. 44. Jullien 2004, p. 398.

[219]Sunzi I, 23; Sunzi 1988, p. 69. Lloyd & Sivin 2002, p. 236.

[220]See above, Sects. 2.3 and 2.7.

[221]Lloyd & Sivin 2002, pp. 91, 98 and 101. See above, Sect. 2.9.

as the last element of a patient story. That is to say, confrontation was seen in a qualitative way, where one small factor produced a great effect. War was thus seen as a "catastrophe", also in the technical and mathematical meaning given to the word by René Thom.[222]

In China, war was conducted as a process to be prepared and followed without forcing things, but waiting for the right moment.

> Shengren yi wuwei dai youde
> [Following 'non-doing', the sage waits to have the virtue [capacity]].

In Greece and the West, the face-to-face clash and military action suffered from the inevitable distance between practice and theory, between tactics and strategy.[223] In China, the hope was always to win without fighting; in Greece and the West, the fight never led to a definitive victory. In China, war was linked and subordinated to politics; in the West, politics was subordinated to war.

> Arms are instruments of ruin, and not instruments of a noble man. He uses them against his will, and gives priority to calm and rest. Even if he is victorious, he does not find it gratifying. The common man considers necessary that which is not necessary. For this reason he makes use of arms. He who loves arms tries to satisfy his own desires. He who puts his trust in arms will perish. [. . .] The perfect man . . . is like running water, like great purity that expands. You know only the tip of the hair, and you ignore great peace.[224]

In a moment of unjustified optimism, however, we finish with a page about music, which, in its pre-imperial antiquity, allows us to conclude by returning to where we started. "The end as the beginning", was written in a book on diplomacy of the fourth century.[225] The way, the Tao, stretches out, without ever stopping, along pleasant, winding, slow-moving paths which seem to turn back on themselves, and not straight, rapid, arrogant motorways, projected in the *hubris* of conquest. The text is a good representation of Chinese scientific culture, through its characteristic organic links.

In the Confucian *Chunqiu [Springs and autumns]* with the comment of Zuo, a doctor diagnosed the illness of a noble as due to his sexual behaviour. "Take as your norm the music of the previous kings, with which they regulated all activities: for this reason, there are the rules of the five notes – . . . – the *adagio* and the *presto* follow each other from the beginning to the end, and they stop when the melody is complete. After the stop of the five notes, it is not allowed to touch [anything else]. Therefore the sage does not listen to the exaggerated music of disorderly hands, which obstructs the heart-mind and the ear, giving them a joy which makes them forget the balanced harmony. . . . It is the same also for activities: they are put aside when they arrive at the limit of disorder, otherwise they provoke illnesses. The sage approaches lutes and guitars in accordance with the rules of convenience, not to

[222]Tonietti 2002a. Jullien 1998, pp. 163–164.

[223]Jullien 2004, pp. 48ff. Jullien 1998, pp. 83–84 and 228.

[224]*Daodejing* XXXI, 1973, p. 86. *Zhuangzi* XXXII, 1982, p. 301.

[225]Jullien 1998, p. 86.

give joy to his heart-mind. The sky has six *qi*, which, in their descent, produce five flavours, revealing themselves form the five colours, and making themselves perceived, form the five notes. When these become excessive, they give rise to the six illnesses. The six *qi* are: Yin, Yang, wind, rain, dark and light. Separating, they form the four seasons; putting themselves in order, they form the five crucial points of the yearly circle. When they are excessive, they become harmful: Yin gives free course to illnesses connected with cold, Yang to those connected with heat, the wind to those of the extremities, rain to those of the stomach, dark to those connected with agitation of the senses, and light to those of the heart-mind. Female creatures belong [both] to Yang and to the time of darkness. If you lose all restrictions, you give rise to heat and internal agitation . . . "[226]

Confucius included "the harmonious pleasure of rites and music" among the "three advantageous pleasures", but if it was an "end to itself", it would become disadvantageous, like "the pleasure of feasts and banquets".[227] This Chinese sage thus considered music moral, provided that it was balanced, and integrated into the order of the State.

There were not only Greeks or Chinese on the earth, but also other important scientific cultures, in both a written and an unwritten form.

> My music begins with fear, which brought you unhappiness;
> It continues in abandon, which recommended docility to you;
> It finishes in the unravelling of the whole soul, which led you to stupidity.
> The state of stupidity provokes the experience of the Tao.
> The Tao can sustain you and accompany you everywhere and for ever.
>
> Zhuangzi XIV.

> Zaohua zhong shenxiu
> YinYang, ge, hun xiao
> [Magic and beautiful feels the nature with love
> YinYang, she gathers, dusk and dawn]
>
> Du Fu

[226]*Chunqiu* X, year 1 (541 B.C.) g; *Primavera e autunno [Spring and autumn]* 1984, pp. 632–633. Cf. Lloyd & Sivin 2002, pp. 256–257. But Sivin admits that he does not understand why women were also considered Yang, and perhaps he does not notice the non-dualistic interpretation. And yet, previously, on p. 199 of his book, he had gone so far as to explain that a young woman could act as Yang with an elderly man, as probably happened in the case examined by the doctor.

Like men, women could become both Yang and Yin, depending on the circumstances, as they were a mixture 'beyond all limits'.

[227]Confucius XVI, 5; 2000, pp. 125–126.

Chapter 4
In the Sanskrit of the Sacred Indian Texts[1]

Having abandoned attachment to the fruits of action,
ever content, depending on nothing, though engaged
in karma [action], verily he does not do anything.
Bhagavad Gita, 4,20

Do your allotted work, but renounce its fruits
-be detached and work- have no desire for reward and work.
Gandhi, "The Message of the Gita"

Passage, O soul, to India!
Eclaircise the myths Asiatic – the primitive fables.
Not you alone, proud truths of the world!
Not you alone, ye facts of modern science!
But myths and fables of eld – Asia's, Africa's fables,
The far-darting beams of the spirit! the unloos'd dreams!
The deep diving bibles and legends,
The daring plots of the poets – the elder religions;
Walt Whitman

4.1 Roots in the Sacred Books

The deepest layers of Greek sciences emerge from the texts of philosophers. From these, century after century, the books of specialists like Euclid derived. In the more ancient classical China, we have seen that books were written or commentated by imperial functionaries, with the aim of practical results for the administration, such as the calendar; in them we find the more interesting results for mathematical

[1]Chapter elaborated together with Giacomo Benedetti, who is responsible for various direct translations from Sanskrit into Italian.

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sciences. On the contrary, for India, traces of scientific procedures and reasonings are to be sought in the original texts of the Hindu religious traditions, that is to say, in the *Veda [Wisdom]* and their commentaries.

Four of these exist, which go back to 2,000–1,500 B.C.: the *Rig Veda [Veda of hymns]*, the *Sama Veda [Veda of melodies]*, the *Yajur Veda [Veda of sacrificial rites]* and the *Atharva Veda [Veda of the Atharva, or magic formulas]*. To these we may add the *Vedanta [End of the Veda]* composed by the *Upaniṣad [Session nearby, or esoteric doctrine]* and the *Brahmasutra [Aphorisms of Brahma]*. Whichever of the innumerable divinities the rite was addressed to, whatever the purpose was, whatever the modalities were, they had to be described with the maximum precision and accuracy, "... because the wrath of the gods followed the wrong pronunciation of a single letter of the sacrificial formulas; ..."[2]

In Europe, we are used to thinking that precision and accuracy are characteristics of sciences, above all mathematics. In India, this need stemmed from religion. The *Rig Veda* speak of constructing an altar for the fire of the sacrifice, the *agni* [fire]: "Like experts a house, they have made it, measuring equally".[3] Unlike the relative *Samhita*, which are *Collections* of *mantras* [sacred formulas to be recited and sung], the *Brahmana* explain and comment on the execution of the rites. The *Śatapatha Brahmana [Brahmana of the hundred pathways]*, and the *Taittirya Brahmana [Brahmana of the Taittirya]* are among the most ancient commentaries (eighth century B.C.?) on the *Veda*.

In the *Śatapatha Brahmana*, the ground for the sacrifice, the *vedi*, must be of a trapezoidal shape because it is female, like the earth, "... it should be broader on the west side ...". In the ritual of creation, it is called the "womb". By means of the sacrifice, "they obtained (...) this entire earth, therefore it [the sacrificial ground] is called *vedi* (...). For this reason they say, 'As great as the altar is, so great is the earth'; for by it, they obtained this entire [earth] ...".[4] The *vedi* was assigned precise dimensions: 15 steps are taken to the South, 15 to the North, 36 to the East, and from there, 12 to the South, and finally 12 to the North; in this way, an isosceles trapezium is obtained (Fig. 4.1).

From the precise measurements of the trapezium, it can be seen how these could be used to proceed in the construction of the altar. The height of 36, together with the semi-base of 15 and the diagonal 39, form an exact right-angled triangle, $15^2 + 36^2 = 39^2$ (Fig. 4.1). This makes it practically certain that in order to mark off the ground of the sacrifice, the Indian priests knew the fundamental property of right-angled triangles. The presence already in the *Śatapatha Brahmana* of a figure like this means that the Indians' knowledge of what we call today the theorem of

[2]Thibaut 1875, p. 227; reprint Thibaut 1984, pp. 3 and 33. The rite even contemplated an officiant, called a *brahman*, whose task was only that of indicating the errors committed and the way to correct them; *Chandogya-Upaniṣad*, IV, XVI–XVII; ed. 1995, pp. 271–274. Malamoud 2005, p. 111.

[3]Seidenberg 1981, p. 271.

[4]*Śatapatha Brahmana* I, 2, 5, 15–16 and 7; Seidenberg 1960/62, p. 520.

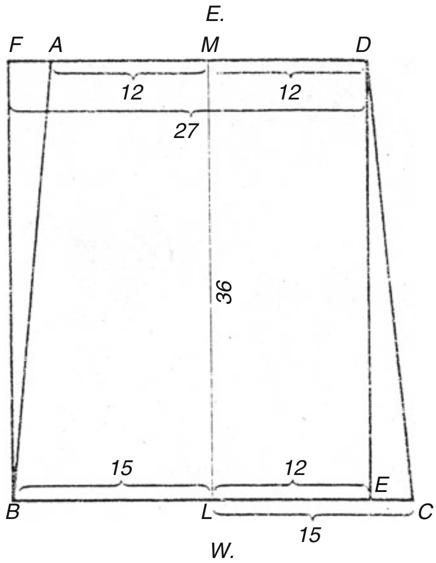

Fig. 4.1 Plan with the dimensions of the altar *vedi* (*Śatapatha Brahmana* III, 5, 1, 1–6; also *Taittirya Samhita [Taittirya Collection]* VI, 2, 4, 5; Seidenberg, 1960/62, p. 508; Springer V)

Pythagoras was particularly ancient. The trapezium-shaped *vedi* is also found in the *Taittirya Samhita*, which goes back at least to the sixth century B.C., though some scholars say even earlier.

The *agni*, the sacrificial fire-altar, was built with layers of bricks, and could be of various shapes, depending on the purpose. "... He should pile in hawk shape who desires the sky; the hawk is the best flier among the birds; verily becoming a hawk, he flies to the world of heaven. [...] He should pile in the form of a triangle who has foes; verily he repels his foes. He should pile in triangular shape on both sides who desires, 'May I repel the foes I have and those I shall have'. [...] He should pile in the form of a chariot wheel, who has foes; the chariot is a thunderbolt; verily he hurls the thunderbolt at his foes."[5]

At the beginning, the *rishi* [airs of life] created seven people in the shape of squares. "Let us make these seven people one Person!" otherwise they would not have been able to generate. For this purpose, they were united in an altar with the shape of a hawk (Fig. 4.2).
"Sevenfold, indeed, Prajapati was created in the beginning. He went on constructing his body, and stopped at the one hundred and one-fold one."[6] Also the layers of bricks followed a symbology, which fixed particular numbers. Five layers for the earth, ten for the atmosphere. Fifteen for heaven. Thus an altar with ten layers was dedicated to Indra, the guardian god, the god of the atmosphere, who fought the miasmas of the plague with his winds. "He who has an adversary should sacrifice

[5] *Taittirya Samhita* V, 4, 11; Seidenberg 1960/62, p. 507.

[6] *Śatapatha Brahmana* VI, 1, 1, 1–3; X, 2, 3, 18; Seidenberg 1960/62, p. 492.

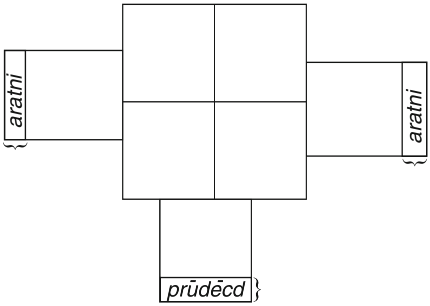

Fig. 4.2 Plan of the altar *agni* (Seidenberg, 1960/62, p. 495; Springer V)

with the sacrifice of Indra, the Good Guardian. Thus he smites this sinful, hostile adversary and appropriates his strength, his vigour."[7]

The need for precision is also satisfied for the *agni*. The altar was increased from 7 and a half square *puruṣa*[8] to 8 and a half, and then to 9 and a half. And so on, up to the required 101 and a half square *puruṣa*. It was written that, during this procedure: "He [the sacrificer] thus expands it [the wing] by as much as he contracts it; and thus, indeed, he neither exceeds nor does he make it too small."[9] "Those who deprive the *agni* of its true proportions will suffer the worse for sacrificing."[10]

The dimensions described were obtained as follows. "Now as to the forms of the fire-altar: Twenty-eight [from west to east, square] *puruṣa*, and twenty-eight [square] *puruṣa* is the body, fourteen [square] *puruṣa* the right, and fourteen the left wing, and fourteen the tail. Fourteen cubits [*aratni*] he covers [with bricks] on the right, and fourteen on the left wing, and fourteen spans [*vistasti*] on the tail. Such is the measure of ninety-eight [square] *puruṣa* with the additional space for wings and tail."[11] Thus the total arrived at 101, as prescribed.

Perhaps we can imagine the rite as a gigantic bird that comes down the faraway heaven, looking bigger and bigger, until it settles on the sacred ground of the sacrifice in order to make it fertile.

4.2 Rules and Proofs

In order to avoid incurring the wrath of the gods, with the relative baleful consequences, the brahmana established that the reciting and the singing of the *mantras* should follow the pronunciation, intonation and rules fixed with precision

[7] *Śatapatha Brahmana* XII, 7, 3, 4; Seidenberg 1960/62, p. 495.

[8] Unit of length corresponding to a man with his arms raised.

[9] *Śatapatha Brahmana* X, 2, 1, 1–8; X, 2, 2, 7–8; Seidenberg 1960/62, pp. 507–508.

[10] Quoted in van der Waerden 1983, p. 13.

[11] *Śatapatha Brahmana* X, 2, 3, 11; Seidenberg 1960/62, p. 508.

by the sacred language: Sanskrit. Between the fifth and the fourth centuries B.C., Pânini expounded the principles of phonetics, grammar and morphology for the language of the priests, in his *Aṣtadhyayi [Collection in eight sections]*. Even the rites of sacrifices had to follow precise procedures. We have seen how complex these could be. As a result, an ancient, centuries-old oral tradition gave rise, eventually, to written texts which gave the details necessary for the prescribed performance of liturgical rituals: the *Śulvasutra* [*Sutra*, or *Aphorisms of the chord*]. These, too, were written in Sanskrit, for the same group of people. This religious language was used for the deepest layers of Indian scientific culture. Various schools of the *Śulvasutra* existed: Baudhayana, Apastamba, Katyayana. The uncertain antiquity of these has generally been subject to different evaluations. George Thibaut, who first studied and translated these texts, assigned them to the fourth or third century B.C.[12]

In the relative extant pages, we may read the rules followed by the Indians for the required geometrical constructions. Among other things, "The chord drawn through the square [the diagonal] produces an area of twice the dimensions."[13] The sentence recalls Plato's *Meno*, which we have already seen previously.[14] Then a rectangle was constructed, as wide as the side of a square, and as long as its diagonal, concluding that "the diagonal equals the side of a square thrice as wide."[15] Thus the Indians used the fundamental property of right-angled triangles. It was even enunciated in its generality in the following terms: "What the length and the width [of the rectangle] taken separately construct [*kurutas*, that is to say, the squares], the same the diagonal of the rectangle constructs [*karoti*] both."[16]

We can interpret the sentences in two ways. In the first case, as in the *Meno*, we have an argument which implicitly aims to justify the fundamental property of right-angles triangles, but we have no figure, and in any case it only deals with the particular case of triangles with two sides equal. No justification is supplied for the general case, unless we imagine that the Indians intended to arrive at the general case step by step, from one diagonal to another by induction. In the second case, the rules to calculate the various squares constructed on the diagonals derive from the general rule, which is not demonstrated. Apastamba and Katyayana put the general

[12]Seidenberg 1960/62, p. 505. In his introduction to Thibaut 1984, pp. i–xxii, Debiprasad Chattopadhyaya tried to backdate the geometrical art of vedic altars to the brick constructions of the culture that developed in the valley of the Indus during the second millennium B.C., but he was unconvincing in various points. Here, we cannot wonder how much implicit geometry still emanated from the ruins of Mohenjo-daro, but rather how much precision brickmakers needed to have, as different subjects from the brahmana, and whether they could write in Sanskrit.

[13]For the *Śulvasutra*, we use the edition (with relative verse numbers) of S.N. Sen and A.K. Bag, who also offer us an English translation: *Śulvasutras* 1983. But various critical passages cited in our chapter have been translated directly from the Sanskrit *ex novo* by Giacomo Benedetti. B.[audhayana] 1.9; A.[pastamba] 1.5; K.[atyayana] 2.8; Ś.[*ulvasutras* 1983], pp. 78, 101, 121. Cf. Thibaut 1984 (1874–1877), p. 73. Cf. Seidenberg 1960/62, p. 524.

[14]Above, Sect. 2.3.

[15]B. 1.10; A. 2.2; K. 2.10. Ś. pp. 78, 102, 122. Thibaut 1984, p. 74. Cf. Seidenberg 1960/62, p. 524.

[16]B. 1.12; A. 1.4; K. 2.7. Ś. pp. 78, 101, 121. Thibaut 1984, p. 74. Cf. Seidenberg 1960/62, p. 524.

case before the particular ones in the text, and thus favour the second interpretation. Baudhayana favours the first one, because it starts by calculating the particular cases, No arguments are to be found, in these deeper layers, which can be compared to the Greek theorem of Pythagoras-Euclid or to the Chinese figure of the string.

However, proofs of another kind exist. How do you transform a square into a rectangle? "If it is desired to transform a square into a rectangle, the side is made as long as desired; what remains as an excess portion is to be placed where it fits."[17] But we are not told how "to fit" it. Various ancient and modern commentators have proposed solutions like the following one: given, for example, a square whose side is 5 and desiring a rectangle whose side is 3, cut out from the square a rectangle whose sides are 5 and 3. What remains is a rectangle whose sides are 5 and 2. Cut out from this a rectangle 3 by 2 and add it to the rectangle 5 by 3. What remains is a square 2 by 2, in the place of which we take a rectangle 3 by 1 and a third. We add this rectangle to the previous two, thus obtaining a rectangle 3 by (5 plus 2 plus 1 and a third), the area of which is equal to that of the square 5 by 5.

Here we have a proof which returns to the starting-point: obtaining a rectangle from a square. And in the end, numbers are used to solve the problem: $2 \times 2 = 3 \times (1 + \frac{1}{3})$. Are these commentators indifferent to vicious circles? Why not use numbers at once, then, to calculate $5 \times 5 = 3 \times (8 + \frac{1}{3})$? Did geometry enjoy a greater consideration than arithmetic at that time in India? Certain Indians appear not to have suffered the rigid distinction between numbers and geometry, typical of the classical Greek world.

And yet other commentators [Dvarakanatha, Sundararaja] offered a different argument for the same text, based on a figure (Fig. 4.3). "Having increased up to the desired length the two sides towards east, the diagonal-chord is stretched towards the north-east corner. The line cuts the breadth of the square lying inside the rectangle; the northern portion is cut off; the southern side becomes the width of the rectangle." Note that in Fig. 4.3, East is at the top, as in the plan of the *vedi*.

The square ABCD is transformed into the rectangle with the base HC. In order to obtain the height, prolong the diagonal GC till it meets the prolongation of AB at E. The desired rectangle is then the portion JHCF of EBCF, seeing that it contains, with JGFD, the missing area ABGH. But why are these the same? The comment does not explain this explicitly. And yet it would be sufficient to take away from the two equal triangles EBC and ECF the couple of equal triangles EAG, EGJ and HCG, GCD.[18] The proof would be somewhat similar to that of Euclid's *Elements*, Book 1, prop. 43.[19] Is it too similar? Did one draw its inspiration from the other one? Before answering, those who are interested should solve the complex,

[17] A. 3.1; B. 2.4. Ś. pp. 103, 79. Cf. Seidenberg 1960/62, pp. 517–518; Seidenberg 1977/78, pp. 334–335.

[18] Seidenberg 1960/62, pp. 517–518. Seidenberg 1977/78, pp. 334–335. *Śulvasutras* 1983, pp. 157–158.

[19] Euclid 1956, p. 340.

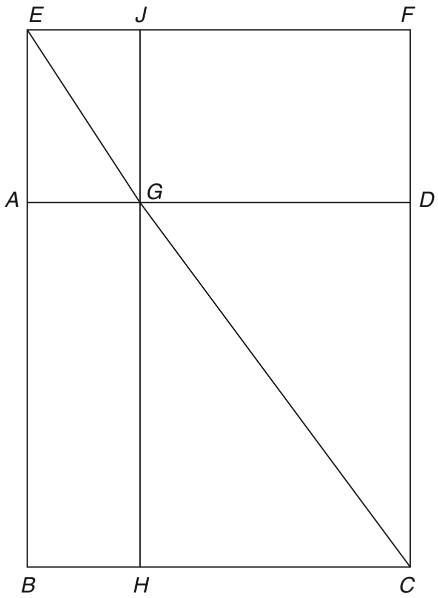

Fig. 4.3 Geometrical construction to transform a square into a rectangle with the same area (Seidenberg, 1960/62, p. 517; Springer V)

controversial problems of the dates, and of the relationship between text and commentary.

We also find a demonstration to transform a rectangle into a square. "If it is desired to transform a rectangle into a square, its *tiryanmani* [width] is taken as the side of a square. The remainder [left out, when cut off the square] is divided into two equal parts and placed on two sides. The empty space [in the corner] is filled up with a [square] piece. The removal of it has been stated [to get the required square]." That is, the rectangle is first transformed into the L-shaped figure, which Euclid called a gnomon, equivalent to the difference between the two squares. The fundamental property of right-angled triangles, already described above, then made it possible to transform the difference between the two squares into the desired square with the same area as the rectangle.[20]

The argument recalls the one to obtain, from one square, another larger one. "Now there follows a general rule. One adjoins the [two rectangles], both [... produced] with the increment in question [and with the side of the given square], to two sides [of the square ABCD, namely ADFJ on the eastern AD and ABEH on the northern AB]; and the square [GHAJ] which is produced by the increment [AJ] in question, to the north-eastern corner."[21] (Fig. 4.4)

[20]B. 2.5; A. 2.7; K. 3.2. Ś. pp. 79, 102–103, 122. Thibaut 1984, p. 77. Cf. Seidenberg 1960/62, p. 524; Seidenberg 1977/78, p. 318.

[21]A. 3.9; Ś. p. 103. Cf. Seidenberg 1975, pp. 290–291.

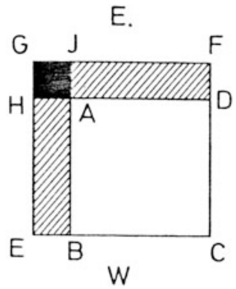

Fig. 4.4 Geometrical construction to transform a square into another larger one (Seidenberg, 1975, p. 290; Springer V)

The rule is the same that increases, in modern algebraic formulas, a square whose side is a into another whose side is $a+b$, in accordance with the formula $(a+b)^2 = a^2 + ab + ab + b^2$. It recalls Euclid's *Elements*, Book 2, prop. 4.[22]

In the following verse, of this page dedicated by the Indians to squares, we read: "With half the side of a square, a square one-fourth in area is produced, because four such squares to complete the area are produced with twice the half side. With one-third the side of a square is produced its ninth part."[23]

In the *Śulvasutra*, therefore, we find, at times, together with the results, also the procedures to follow to obtain them. These are their constructions, which are thus the arguments in favour. Of course, they are not deductive theorems. But I do not see any reason not to consider them as equally valid proofs as the Greek or Chinese ones. Also Indian culture, therefore, produced its own proofs, with the characteristics of the context in which they were generated and used.

We have other rules about how to transform a square into a circle, and a circle into a square. "To transform a circle into a square, the diameter is divided into eight parts; one [such] part, after being divided into twenty-nine parts, is reduced by twenty-eight of them, and further by the sixth [of the part left], less the eighth [of the sixth part]."[24] This means taking as the ratio between the side of the square and the diameter of the circle with the same area the value

$$(\frac{7}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}) : \frac{8}{8}.$$

Thus $\frac{l}{2r} = 0.8786817$, and as the ratio between the side of the square and the radius of the circle with the same area is equal to $\sqrt{\pi}$, that is equivalent to taking the value of 3.08832... for π, instead of 3,1415... Another value for this ratio was $\frac{13}{15}$, which, however, is more approximate, because it is equivalent to taking about 3.0044 as the ratio between the circumference and the diameter.[25]

[22]Euclid 1956, pp. 379–380.

[23]A. 3.10; Ś. p. 103. Cf. Seidenberg 1975, pp. 290–291.

[24]B. 2.10; Ś. p. 80. Thibaut 1984, p. 78.

[25]B. 2.11; A. 3.3; K. 3.12; Ś. pp. 80, 103, 123. Thibaut 1984, pp. 33–36.

As regards the ratio between the side of the square and the diagonal, it was written that: "The measure is to be increased by its third and this again by its own fourth, less the thirty-fourth part; this is the diagonal of a square; this is approximate."[26]

$$1 : (1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34}).$$

The ratio supplies a value of 1.414216... compared with 1.414213... for $\sqrt{2}$. No justification was given for this formula, either, in which the brahmana appear to be influenced by the fascination of numerical symmetries, as in the previous one. We note, however, that one possible interpretation of the text makes them aware of the "approximation" of this number.

Some have attributed the discovery of irrational numbers to the Indians, though others have provided valid reasons to deny this. Reading the *Śulvasutra*, we find that the authors are confident that it is possible to assign a numerical value to every magnitude, without any limitations. As for the Chinese, so also for the Indians, the Pythagorean distinction between whole numbers (or ratios between whole numbers) and the others does not seem to make sense.[27]

The *Śulvasutra* calculated the area of the *Mahavedi* [large trapezium-shaped *vedi*], for which the *Śatapatha Brahmana* had fixed the precise measurements, and explained the procedure.

"The *mahavedi* is 1000 minus 28 *padas*.[28] From the south-east *amsa* [corner D, a line] is dropped towards the *śroni* [south-west corner C] at 12 *padas* [from point L of the *prsthya*, backbone, E]. The portion cut off [DEC] is placed inverted on the other side. That makes a rectangle [FBED]. By this addition, [the area] is enumerated."[29] The number 1,000−28 = 972 is obtained by multiplying 36 by 27 (Fig. 4.1). Here, we have not only a rule, but also its proof. Even Thomas Heath, who otherwise denied the existence of "proofs" for the Indians, admitted here "the nearest approach to a proof."[30] Only his overly Eurocentric love for Euclid stopped him from admitting that this was a proof *tout court*.

The geometrical construction fixed by the *Śatapatha Brahmana* for the *agni*, in the shape of a hawk, was even more complex. Now the *Śulvasutra* suggested their solutions. "The excess to the original form [the area of 1 *puruṣa*, to be added to make the altar grow] should be divided into 15 parts, and 2 parts be added to each *vidha* [to each of the 7 *puruṣa*; the remaining 15th part is added to the

[26]K. 2.9; B. 2.12; A. 1.6; Ś. pp. 121, 80, 101. Sen and Bag interpret *saviśeṣa* as deriving from the word *saśeṣa*, which means "with the rest, incomplete". But if the sentence is divided differently, as *sa viśeṣa*, the text would simply say, "that is the diagonal", and the approximation would disappear. Cf. Thibaut 1984, pp. 18 and 79.

[27]Seidenberg 1960/62, p. 515; Euclid 1956, pp. 363–364. *Śulvasutras* 1983, pp. 168–169.

[28]Fraction of the *puruṣa*.

[29]A. 5.7; Ś. p. 105. Cf. Seidenberg 1960/62, p. 519.

[30]Euclid 1956, p. 361.

half *puruṣa*]. The [new] *agni* is to be laid with such [increased] 7 and one half *vidha*."[31] (Fig. 4.2)

A possible interpretation could be obtained by imagining that the brahmana constructed increasingly large bricks, maintaining the same shape, and increasing the unit of measurement. In modern formulas, if we desire to increase the first altar from 7 and a half to 8 and a half, by adding 1, we can calculate $(7 + \frac{1}{2}) + 1 = (7 + \frac{1}{2}) \times p^2$, where p stands for the new increased *puruṣa*. Thus $p^2 = 1 + \frac{1}{7+\frac{1}{2}} = 1 + \frac{2}{15}$, as in the previous quotation. Using the above-mentioned procedures, the rectangle with the area $1 \times (1 + \frac{2}{15})$ is transformed, finally, into the new desired larger square.

In the *Śulvasutra* of Katyayana a procedure was described that is slightly different. "To add one *puruṣa* to the original falcon-shaped *agni*, a square equal to the original *agni* [of 7 and a half *puruṣa*] with its wings and tail is to be constructed, and to it is added one *puruṣa*. The original *agni* is to be divided into fifteen equal parts. Two of these parts are to be transformed into a square. This will give the [new] *pramana* [unit] of the *puruṣa*."[32] Here a square with the area of 7 and a half square *puruṣa* was increased by one *puruṣa*, adding the relative square by means of the above-mentioned fundamental property of right-angled triangles (theorem of Pythagoras). The new square of 8 and a half *puruṣa* was divided into 15 equal rectangular parts, and lastly, 2 of these are transformed into the square whose side gave the new unit of measurement for the *agni* of 8 and a half. However this is interpreted, the brahmana must have used the above geometrical properties to carry out their religious rites with the required precision, even if they did not always leave us the proofs.

All this leads us to believe that the fundamental relationship between the sides of a right-angled triangle was already known to the brahmana in particularly ancient times. The *Śulvasutra* can be compared with the period of Euclid, but in any case the *Brahmana* are considered as prior to the fifth century B.C., to which Pythagoras is attributed. How else could the priests have followed the liturgy described with such precision, if they had not already possessed the relative geometrical rules? This has led some scholars to advance the hypothesis that the most ancient Greek mathematics might derive from India, or that Greece and India had drawn from a third common source. This could not have been the Babylonian scientific culture (generally more ancient than both of them), because the geometrical style present in the other two was totally lacking there. Consequently, it has been hypothesised that the common origin is to be sought elsewhere, without indicating, however, exactly where.[33] More recently, it was conjectured that the solution to the problem might lie in an area between Persia, the Caspian Sea and Central Asia, a region once known as Bactriana. The relative languages and the findings in the respective archaeological

[31] B. 5.6; Ś. p. 83. Thibaut 1984, pp. 62 and 87–88. Cf. Seidenberg 1960/62, p. 525.

[32] K. 5.4 e 5.5; Ś. p. 124. Cf. Hayashi 2001, p. 729.

[33] Seidenberg 1960/62; 1975; 1977/78.

sites are said to show links both with those of the valley of the Indus, and with others of Asia Minor.[34]

However, all these speculations critically depend on Eurocentric, anachronistic mathematical categories, such as geometry, algebra, arithmetic, geometrical algebra, algebraic geometry. The proposers of one thesis or another are too strongly influenced by their own university training as algebraists, formalists, and the like, in the search for some mythical origin of their discipline, which may celebrate the triumphs of the past or of the present. Thus, failing certain proofs based on documents or ascertained material passages, here we prefer to continue to think that the Indian brahmana and the Greek philosophers developed their mathematical cultures in a relative autonomy, maintaining their own characteristics. This was also wisely sustained by the pioneer, George Thibaut.[35]

We must, however, note that on passing the Himalayas on our return towards Europe, we begin to breathe a more familiar kind of air with respect to China. To understand better what this means, let us now examine the place of numbers in the Indian culture.

4.3 Numbers and Symbols

After expounding in a general form the fundamental property of right-angled triangles, as observed above, the *Śulvasutra* of Baudhayana continued: "This is observed in rectangles having sides 3 and 4, 12 and 5, 15 and 8, 7 and 24, 12 and 35, 15 and 36".[36] Note that 15 and 36 are the numbers chosen to measure the *Mahavedi*.

$$3^2 + 4^2 = 5^2\,;\, 12^2 + 5^2 = 13^2\,;\ldots\,;\, 15^2 + 36^2 = 39^2$$

In other words, the list presented couples of numbers which, squared and added together, give another perfect square. With the passing of time, in the West, these groups of three numbers were called, for obvious reasons, Pythagorean triplets. In the *Śulvasutra*, they are presented together with the squares and the rectangles for which they offer this significant property. Among numbers and magnitudes to be measured, did the brahmana create any hierarchy similar to the Western Pythagorean tradition, or not?

> The diagonal of a rectangle of sides 3 and 4 is 5. When these increased by three times themselves, the two eastern corners, and with these increased by four times themselves, the two western corners [are determined].[37]

[34] Staal 1999.

[35] Thibaut 1984, pp. 3–4. Cf. Seidenberg 1977/78, p. 306.

[36] B. 1.13; Ś. p. 78. Thibaut 1984, p. 74. Cf. van der Waerden 1983, p. 9.

[37] A. 5.3; Ś. pp. 105, 238.

$$(3 + 3.3)^2 + (4 + 4.3)^2 = (5 + 5.3)^2;\ 12^2 + 16^2 = 20^2$$

$$(3 + 3.4)^2 + (4 + 4.4)^2 = (5 + 5.4)^2;\ 15^2 + 20^2 = 25^2$$

> The diagonal of a rectangle of sides 12 and 5 is 13. With these, the two eastern corners, and with these increased by twice themselves, the two western corners [are determined].[38]

$$(12 + 12.2)^2 + (5 + 5.2)^2 = (13 + 13.2)^2;\ 36^2 + 15^2 = 39^2$$

> The diagonal of a rectangle of sides 15 and 8 is 17. With these the two western corners [are determined]. The diagonal of a rectangle of sides 12 and 35 is 37; with these [are fixed] the two eastern corners. The knowledge of these [squared numbers] makes possible the *vediviharanani* [construction of the geometry for the *vedi*]."[39]

Thus, in the *Śulvasutra*, (whole) numbers had the function of permitting the construction of altars of the exact shape prescribed by the ritual. But they do not seem to play exclusively this basic role typical of the Western Pythagorean tradition. The general emphasis of the texts lies in the construction of squares, rectangles, triangles trapezia and so on. Numbers served to measure them variously in the appropriate units.

For example, the *vedi* for the sacrifice of animals is a trapezium whose dimensions are 10 for the larger base, 12 for the height, and 8 for the lesser base, which contains the right-angled triangle whose sides are 5, 12, 13.[40] Certain circular holes for the sacrifice whose diameter is 1 are said to have a circumference of 3.[41]

Apastamba begins by declaring explicitly: "We shall explain the methods of constructing figures".[42] Baudhayana: "We shall explain the methods of measuring areas of their figures", in order to build the altars for the sacrifice.[43] Let us not forget that the geometrical figures had an anthropomorphic sense. In a certain kind of trapezoidal *agni*, the angles to the East are the shoulders, those to the West the hips: "It is like a wooden doll."[44]

Here in India, we find one thing considered beyond all measurements. "Those who desire heaven should construct [the *agni*] by increasing the height measure with *aparimitam* [innumerable] bricks; …".[45] The transcendent religious element brought with it a number that arised beyond all numbers.

In the *Śulvasutra*, altars of a precise shape were prescribed in order to obtain from the gods particular results. Against enemies, for example, the shapes established were the isosceles triangle, the rhombus, and the cartwheel; for food, the feeding

[38] A. 5.4; Ś. pp. 105, 238.

[39] A. 5.5 e 5.6; Ś. pp. 105, 237–238.

[40] B. 3.9; Ś. p. 81. Thibaut 1984, p. 81.

[41] B. 4.15; Ś. p. 82. Thibaut 1984, p. 86.

[42] A. 1.1; Ś. p. 101.

[43] B. 1.1–2; Ś. p. 77. Thibaut 1984, p. 69.

[44] A. 4.5; Ś. p. 104.

[45] A. 10.8; Ś. pp. 109–110.

trough; for heaven, the hawk. Sometimes the model could profit by the choice between two versions: either square or circular. The shape corresponded to that of the thing desired, or the means to obtain it. The basic brick was square, but if necessary, it was divided differently to obtain the shape desired. Of course, everything was accurately measured.

Regardless of the shape, the bricks were counted. In general, there had to be 200 of them in five layers, making a total of 1,000, or a multiple of it. Thus numbers now assumed a symbolic significance in the ritual.[46]

And yet the brahmana also realised that all the constructions, for all their fixedness and precision, were approximate, subject to change, and situated on the earth. "What is lost by burning [and drying] is to be made good by loose earth because of the flexibility of its quality."[47] "The decrease suffered by the bricks due to drying and burning is made good by further addition, so as to restore the original shape."[48] "When dried and burnt, bricks lose one-thirtieth."[49]

What, then, does the relationship appear to be between numbers and geometry in the most ancient Indian scientific culture? Baudhayana first gave the geometrical formulation of the fundamental property of right-angled triangles, and then the relative triplets of whole numbers.[50] In the *Brahmana* and in the *Śulvasutra*, a balance seemed to be maintained between the figures, their measurements and the relative numbers. The approximations seen above to measure the diagonal of a square and the circumference of a circle fit in well with this equilibrium.

The Sanskrit term for square is *varga* or *krti*. These are used to indicate both the geometrical figure and its numerical area, but also the product of a number multiplied by itself. "A square figure of four equal sides and the area are called *varga*. The product of two equal quantities is also *varga*."[51] It is so defined by Aryabhata I (475–550), even though it corresponds to the current use of the term 'square' in many languages. The square root *varga mula* or *pada* shares the same geometrical origin. Brahmagupta (598–c. 665) defined it as follows: "The *pada* [root] of a *kṛti* [square] is that of which it is the square".[52] In their translation of Brahmagupta, the Arabs will choose the term *jadhr*, the base of the square, for it. The same word was also to be chosen by Al-Khwarizmi (780–850) to indicate the root of an algebraic equation.[53] *Varga mula* and *jadhr* are thus to be compared with the Latin *radix* and the use made of it by Leonardo da Pisa.[54]

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[46]Ś. pp. 111–119. "The mystic number eighteen"; *Bhagavad Gita* 1996, at the beginning, page unnumbered. Malamoud 1994, p. 299.

[47]A. 9.8; Ś. p. 109.

[48]Manava, 9.1; Ś. p. 133.

[49]Manava, 13.17; Ś. p. 138.

[50]B. 1.12–13; Ś. p. 78. The geometrical style of the *Śulvasutra* was underlined by Thibaut, who corrected the arithmetic readings of subsequent Indian commentators, Thibaut 1984, pp. 60–64.

[51]Quoted in Datta & Singh 1935, I, p. 155. Thibaut 1984, pp. 64–66.

[52]Quoted in Datta & Singh 1935, I, p. 169.

[53]Datta & Singh 1935, I, p. 170. Joseph 2003, pp. 301 and 318.

[54]See Sect. 6.3.

In the *Śulvasutra*, the side of the relative square was called *karani*.[55] "In later times, however, the term is reserved for a surd, i.e. a square root which cannot be evaluated, but which may be represented by a line".[56]

In his *Bijaganita [Calculating with the element* or with the unknown *avyakta*] (1150?), Bhaskara II (1114–1185) made a distinction between this kind of mathematics and arithmetic. "Mathematicians have declared *Bijaganita* [algebra] to be computation with demonstrations: otherwise there would be no distinction between arithmetic. [...] The method of demonstration has been stated to be always of two kinds: one *kṣetragata* [geometrical] and the other *râśigata* [symbolical]." The geometrical kind was attributed to no better specified "ancient teachers". It is likely that Indian mathematicians thought of the *Śulvasutra*.[57] At this point, even Datta and Singh include three meagre pages on the *Śulvasutra*. It is a pity that they did not, unfortunately, write or publish the third part of their book, due to be dedicated to the history of Indian geometry.[58]

> What the *samkhyah* [philosophers of the Samkhyah school, or learned calculators] describe as the originators of intelligence, being directed by a *satpuruṣa* [wise being] and which alone is the *bija* [primal cause] of all *vyakta* [knowns], I venerate that Invisible God as well as that Science of Calculation with *avyakta* [Unknowns] ....[59]

All their counting and numbering bricks in the order established by the ritual, and blessing them with the *mantras* before fixing in position played their part. In the Indian scientific culture, we do not find that faith that is typical of the Chinese: a primordial *continuum* from which everything originated. Whole numbers also appear to be encumbered with particular religious symbolisms. With the appearance of the first texts explicitly reserved for the mathematical sciences, written by figures whose names are known, arithmetic prevailed. And yet something happened which would not have been expected. These books were not written using for numbers the Indo-Arabic symbols which today have spread all over the world. The *ganaka*, Indian astrologers-cum-astronomers-cum-mathematicians, preferred to use letters and words for them.

The famous grammarian Pânini (fourth century B.C.) used as numbers the vowels of the Sanskrit alphabet: a = 1, i = 2, u = 3, ... For the numbers of astronomy, Aryabhata I used the alphabet in the classification fixed by Pânini, with 5×5 *varga* [classified] letters for the odd positions, and the *avarga* [non-classified] letters for the even positions, in a positional notation. Other systems of notation with letters appeared to be variants of this one, like that of the *jaina* mathematicians.[60]

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[55] B. 1.10–11; Ś. p. 78. A. 1.5; Ś. p. 101. K. 2.3–4; Ś. p. 121. Thibaut 1984, pp. 73–74.

[56] Datta & Singh 1935, I, p. 170. But isn't this a bit too anachronistic, and too Greek an interpretation of the term? In K. 2.4, *rajjurdaśakarani* literally means "chord that constructs [a square whose area is] ten". Thibaut 1984, pp. 65–66.

[57] Datta & Singh 1935, II, pp. 1–8.

[58] Datta & Singh 1935, I, p. xi.

[59] Datta & Singh 1935, II, p. 4.

[60] Datta & Singh 1935, I, pp. 63–74.

The idea that the numerical value of a symbol may be made to depend on its position in the relative sequence goes back to a *jaina* text of the first century B.C. Here, the number of human beings is calculated to occupy 29 places.[61] "... from *place to place* each ten times the preceding." wrote Aryabhata I.[62] Thus we have here also a numeration with the base 10. Starting from the sixth century A.D., the dates in inscriptions also contain the *śunya* [void] to indicate an empty place in the sequence of numbers.[63] When studying the metrics of the *Chandahsutra [Aphorisms of metrics]*, Pingala (third century) used the symbol 0 to calculate the combinations with 2^n repetitions of two syllables, one short and one long, distributed over n positions.[64] In the *Panca-Siddhantika [Five final expositions]* of Varahamihira (sixth century), a *bindu* [dot] "•" was used to indicate the degrees or constellations, or to write very large numbers with " ... eight zeros". The 0, as an empty place in the sequence of numbers, was used by Bhaskara I (c. 600). Inscriptions with a 0 go back to the eighth century.[65] Thus we have found all the characteristics of our positional way of writing numbers with the base 10, and with a 0.

The sacred language of the priests, Sanskrit, played a role in our history which cannot be underestimated. Pânini's subsequent theory, which consecrated it as a model of rigour, is equally important. The *sutra*, the aphorisms, became fixed rhyming rules to be repeated and without difficulty learnt by heart. This form of writing thus had a property that was indispensable for the transmission of rituals, orally in the most ancient period, from master to pupil. Numbers, too, needed to be words that could be included in the sentences in verses. The *Veda* needed members in order to be operative; thus the *Vedanga* were born. These contemplated the *Kalpasutra* for the rituals, divided between the other rituals in the *Śrautasutra* for the sacrifices and our *Śulvasutra* to obtain the exact measurements of the altars. If we translate *sutra* as aphorisms, we are using a Greek term, *aphorismos*, which means "definition", which in turn indicates a limit. Thus we are moving away from a culture like that of China, which attempted to explain phenomena by means of links, in the direction of distinctions and separations. They begin to appear on the horizon, although we are still at a considerable distance from the definitions contained in Euclid's *Elements*. The word used by Euclid for definitions is rather $o\rho o\iota$ [ends, limits, borders].[66]

On the other hand, the Indian scientific culture made very little use of ratios or proportions. We only find a *trairaśika* [rule of three] in the following form. *pramana* [argument] : *phala* [fruit] = *iccha* [requisition] : [unknown].[67] Here, on the contrary, fractions, called *bhima* [broken], were currently used in calculations. In the most ancient texts, the following were used, as words, of course: *pada* for $\frac{1}{4}$; *tripada* for

[61] Datta & Singh 1935, I, p. 84.

[62] Datta & Singh 1935, I, p. 13.

[63] Datta & Singh 1935, I, p. 38.

[64] Datta & Singh 1935, I, p. 75.

[65] Datta & Singh 1935, I, pp. 75–82.

[66] Euclid 1969, pp. XLII-1. Cf. Euclid 1956, pp. 153–194.

[67] Datta & Singh 1935, I, p. 203.

$\frac{3}{4}$. Later, they were written as we write them today, though without any line between the numerator and the denominator.[68]

Following their religious convictions regarding time and space, the *ganaka* considered very large numbers. In the *Veda*, *tallakṣana* meant 10^{53}.[69] Both the *jaina* scholars and the Buddhists needed to describe enormous intervals of time. In this, the positional system revealed its advantages.[70] They even arrived at infinity, the prelude of which we have already found above in the *Śulvasutra*. In a work by *jaina* scholars, attributed to the beginning of the Christian era, we find the following passage: "Consider a vat whose diameter is that of the earth (100,000 *yojannas*, that is to say, about one million kilometres) and whose circumference is 316,227 *yojannas*. Fill it with white mustard seeds, counting them one by one. In the same way, fill other vats of the size of different lands and seas with white mustard seeds. We still have not arrived at the largest number that can be counted." Then the "uncountable" numbers arrived, divided into the "almost uncountable, absolutely uncountable and uncountably uncountable". Lastly, the "infinities" were reached, distinguished into "almost infinite, absolutely infinite and infinitely infinite." Compared with the modesty of the Greeks, with their horror of infinity, which was hidden and exorcised in every possible way by means of paradoxes and geometry, we are in a different world.[71]

In the *Bijaganita*, Bhaskara II described the *kha-hara*, that is, division by 0, in the following way. "In this quantity, consisting of that which has cipher for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God, at the period of the destruction or creation of worlds, though numerous orders of beings are absorbed or put forth."[72]

A manuscript on birch-tree bark, almost exclusively dedicated to arithmetic, was casually found at Bakhshali (north-western India), but was uncertainly and controversially dated some centuries A.D. Here, the sum of numbers was indicated by *yu*, from the word *yuta* [added], multiplication by *gu*, from *guna* [multiplied], and division by *bha*, from *bhaga* [divided]. It is curious that for subtraction, the sign + was used, as an abbreviation of *kṣaya* or of *kanita* [diminished]; seeing that the initial in these Brahmanic characters is a cross with a flourish at the base. The symbol for the square of a number was *va*, from *varga* [square], the root *mu*, from *mula* [root] or also *ka*, from *karani*. *ka* was placed in front of numbers, but in general these symbols were placed after them. Negative numbers, which were admitted like all other numbers, together with all kinds of roots, had a dot or a circle at the top.[73]

68 Datta & Singh 1935, I, p. 188.

69 Datta & Singh 1935, I, p. 11.

70 Datta & Singh 1935, I, p. 86.

71 Joseph 2003, p. 249. But we are not, as Joseph believes, in the European paradise of Georg Cantor (1845–1918); cf. Tonietti 1990.

72 Datta & Singh 1935, I, p. 243.

73 Datta & Singh 1935, II, pp. 12ff.

Some indicated the unknown as "0" in equations. Around the year 628, Brahmagupta used, for unknowns, the kaleidoscopic range of the *varna* [colours and letters of the alphabet]: *kalaka* [black], *nilaka* [blue], *pitaka* [yellow], *patalaka* [pink], *lohitaka* [red], *haritaka* [green], and *śvetaka* [white].[74] The same term, *varna* is used for the castes into which Indian society is divided, but also for the modes of Indian music in its most ancient theoretical text, the *Gitalamkara [Ornament of song]* of Bharata, which we shall encounter below in Sect. 4.5. Lastly, the 63 sounds of Hindi phonetics are also called *varna*.[75]

This habit of using letters in mathematical and astronomical texts has given some historians the idea that the course towards modern Western symbolic algebra started in India.[76] This hypothesis is not very convincing. Apart from the question, which will be discussed in due time, of exchanges that are documented between Indians and Arabs, the relationship between mathematical symbolism and language is not the same. In India, attempts were made to reinforce the link with the sacred language, a language raised to the level of a universal model of precision and rigour for the religious reasons already seen. For these reasons, Sanskrit acted as a paradigm (also in the literal sense), even for sciences like astronomy, geometry and arithmetic. On the contrary, Western mathematical symbolism, as represented and stabilised by Descartes and Leibniz, took the opposite direction. In the Europe of the seventeenth century, on the contrary, the purpose of symbols was to detach scientific reasoning as far as possible from the relative languages, which had by now become historical and national with the progressive decline of Latin.

Also in China, we found that scientific discourse was comfortably lain down in the literary language of the imperial functionaries in charge of the various branches of the administration. It had no desire to become detached with the invention of some specialised symbolism. But now, south of the Himalayas, we find a language quite different from the expression in characters of Chinese vocabulary. The alphabet of Sanskrit, which represents sounds in a linear manner, facilitates the idea of studying its combinations as a way of penetrating into a universe conceptualised, in turn, through the multiple combinations of things and essences. We have already encountered the combinations of syllables calculated by Pingala. Above all scholars of the *jaina* religion loved to group philosophical categories together, two by two, three by three, or the five senses, or men, women and eunuchs, or tastes, etc., exhausting all the possible combinations.[77]

Furthermore, unlike Chinese, where it is lacking, Sanskrit displays an abundant, and frequently implied, use of the verb *as, sat* [to be], which on all sides is capable of uttering the properties of any entity or phenomenon. With these linguistic properties, Indian scientific culture then started to draw closer to the West, which expressed itself in Greek and Latin, and at the same time to the well-known families of relative

[74] Datta & Singh 1935, II, pp. 17ff.

[75] Bharata 1959, p. 165. Cardona 2001, p. 741.

[76] Datta & Singh 1935; Joseph 2003.

[77] They calculated how to distribute among the seven hells one living creature, or two, three, ... ten, countable or uncountable beings; Hayashi 2001, p. 775. Joseph 2003, p. 252.

languages. Thus we have to prepare ourselves to see the appearance in Indian sciences of some of those aspects which made European sciences different from those of China.

In time, the original interests of the *Vedanga* in geometry were surpassed by arithmetic, combinations and equations. These were used to achieve famous results, for example entire complete solutions in the case of indeterminate equations of the first and the second degree.[78] In Europe, mathematicians ended up by calling these kinds of problems with whole numbers Diophantine equations, after the name of Diophantus of Alexandria (c. 250). But in India, famous scholars like Brahmagupta and Bhaskara II had dedicated studies to them, on the basis of their reasons of a religious nature.

For them, those whole numbers represented the number of complete cycles covered by the stars within their particular cosmogony. In this, the universe is born and dies periodically, at enormous intervals of time, known as *kalpa*, or "day of Brahma", equivalent to 4,320,000,000 terrestrial sidereal years. The *kalpa* is divided into 1,000 *mahayuga* [large yokes] of 4,320,000 years each. In turn, the *mahayuga* can be broken down into other, smaller *yuga*, which stand in the ratios 4:3:2:1. For this reason, we are today living in the last *yuga*, called *kaliyuga*, an iron age, with wars and violence (as is clear), of 432,000 years. It began with a planetary conjunction in 3,102 B.C., and will finish with the entire destruction of the universe. Then the cycle will begin again, with a new creation of the universe, and so on *ad infinitum*. In the most popular religion of the *Purana [Ancient Texts]* that followed the *Veda*, Brahma creates the universe that Viṣnu maintains, until Śiva destroys it at the end of the first cycle. The incessant alternation of the eras, the inexorable revolving of the stars in a circle, are represented also by means of the rhythmic movements of a cosmic dance: the *Nataraja* of Śiva, where creation and destruction blend together.

Cakra [wheel] is called the cycle of reincarnations. A similar idea is also found in the mathematical procedure, *cakravala*, followed by Bhaskara II to obtain complete solutions of a Diophantine equation of the second degree, today called Pell's (John, 1611–1685). If the stars were all in line and synchronised at the beginning of every era, one of the most important astrological problems for a similar cosmogony was how to calculate when their conjunctions would be repeated. One particularly clear case is represented by eclipses. Hence the emphasis on whole number solutions of certain equations with whole-number coefficients.[79] May it be legitimate, then, for us to suspect that the shift in emphasis from geometry to numerical equations took place under the influence of the parallel detachment from the Vedic sacrifices, towards a religion guided rather by cycles of cosmic recurrences?

And yet some aspects of ancient geometry must have remained. It was interesting to see the elegant, geometrical way in which Nilakantha (1445–1545) calculated the sum of arithmetical series, as shown in a figure. It is clear from this that the sum of

[78] Datta & Singh 1935; Joseph 2003.

[79] Van der Waerden 1983, pp. 113–154. But I am not convinced by Van der Waerden's hypothesis that the Indians had been subject to Hellenistic influence, because there is a lack of documentation about them.

an arithmetical series of n terms, which begins with a and ends with f, is equal to $\frac{1}{2}n(a + f)$.[80]

However, one of the greatest prides of Indian mathematicians remains the introduction of the *jya* [sine] in astronomy, with the relative calculation of highly precise tables. Starting from the *Surya Siddhanta [Final exposition of the sun]* (400) and from Aryabhata I, up to the *Siddhanta Śiromani [Crowning of the final exposition]* of Bhaskara II, the art was developed of calculating the length of chords with respect to the angle at the centre of the circle. While Hipparchus (150 B.C.), Menelaus (100) and Claudius Ptolemy (150) chose to operate with the entire chord of the circle, Indian astronomers preferred semi-chords, that is to say, the *sine*; in this way they took a decisive step forwards towards modern trigonometry.[81]

In the *Brahma Sphuta Siddhanta [Final corrected exposition of Brahma]*, Brahmagupta constructed a quadrilateral inscribed in a circle, combining some right-angled triangles with whole-number sides quoted in the *Śulvasutra*. The figure obtained allowed him to calculate the fundamental formulas of trigonometry for the *sine* and the *cosine*.[82]

The mathematician Mahavira, who lived in southern India during the ninth century, and was a follower of the *jaina* religion, attributed a universal importance to his studies. "In the science of love, [...], in music and drama, in the art of cooking, in medicine, in architecture, [...], in poetry, in logic and grammar, [...], the *ganita* [science of calculation] is held in high esteem." It allowed him to follow the movements of the sun, the moon, and eclipses. "I glean from the great ocean of the knowledge of numbers a little of its essence."[83] Also from the *Upaniṣad*, we can understand the place assigned to mathematical sciences in the Indian culture. The 'Seventh reading' from the "Chandogya Upaniṣad" recites: "O Lord, I know the *Rg-veda*, the *Yajur-veda*, the *Sama-veda* and lastly the *Atharvana* as fourth, the *itihasa* [traditional sayings] and the *purana* as fifth, the Veda of the *Veda* [grammar], the ritual of the Manes, calculation, divination, knowledge of the times, logic, rules of conduct, etymology, knowledge of the Gods [of the sacred texts], knowledge of the supreme spirit [philosophy], the science of arms, astronomy, the science of serpents, of spirits and of genies."[84]

4.4 Looking Down from on High

Whether scientific or not, whether religious or not, among the general characteristics of Indian culture, we often find two aspects which play a role, as we have seen, in the differences between the orthodox Chinese culture and the culture prevailing in

[80] Joseph 2003, p. 291.

[81] Joseph 2003, pp. 278–282.

[82] Zeuthen 1904a, pp. 107–111.

[83] Datta & Singh 1935, I, pp. 5–6.

[84] *Upaniṣad* 1995, p. 311.

Europe. These are the impulse to transcendence and dualism. Such characteristics are found with greater emphasis in the *Śastra* [science of language] earlier than in mathematical sciences. It is in Pânini's studies on grammar that we observe a higher level of abstraction and symbolic formalism. The absence, in certain cases, of the ending for the plural of a word was indicated by means of a special symbol assimilable to zero.[85]

The desire to indicate 'what is not there' expressed the need to represent the invisible. A symbol of this kind had to be expressly created, with its conventional meaning, because it cannot exist among the common words used for the most common events of life. It found its natural context in a religious language addresses to gods that transcended everyday experiences, invisible divinities which thus needed to be represented in some other way. As a result, the search for a special precision assumed the form of metalinguistic *sutra*, formalised and superimposed on expression. What Elements of transcendence do we find, then, in Indian culture, and what links do they display with relative scientific discourse?

Those who have faith in a single universal science (past, present or future), and in scientific progress, let themselves go to support a greater wealth of India compared with China: "in the field of abstract sciences, such as, for example, mathematics, logic and linguistics".[86] I do not intend to follow them along this road leading to the construction of value hierarchies, measured with the yardstick of the abstract sciences of the West, ignoring the characteristic Indian context. In order to maintain whatever characteristics may be present also in Indian sciences, I prefer to call this relative abstraction "Elements of transcendence", and show that these depended to a large extent on a particular choice of religious philosophies that contained them, and facilitated them, creating the general cultural environment.

Let us remember immediately, however, that even more than in the case of China, it is not possible to reduce India to a single culture or philosophy. Anyway it has never suffered any submission to a single empire, but on the contrary, it underwent various colonial invasions, the last of which saw the English as dominators. Nor has one single language ever been spoken in the vast and varied subcontinent that extends from the Himalayas to the coasts of Malabar. They do not even use the same alphabet here. The ancient classical culture is united only by Sanskrit, whereas the modern visitor is surprised to hear Indians talking together in English!

During their history, this has not impeded the formation of various orthodoxies, which may be compared with heretical minority positions. Hinduism is prevalent, but is divided into numerous sects that are devoted to a 100 million sundry divinities. Together with them, we find the authoritative variant of *jainism*, in turn divided between the *Digambara* [dressed in heaven, i.e. naked] and the *Śvetambara* [dressed in white]. The most radical heresy is Buddhism, which has become rare in India. Its texts have partly been lost, because burnt, and have survived at times only in Tibetan or Chinese translations.

[85] Staal 2001, p. 629.

[86] Staal 2001, p. 615.

Let us look at a few transcendent Elements that are significant for sciences. From the *Veda* to the *Śastra* and the *sutra*, the texts are generally presented as outside time. In that eternity without any history, the various commentaries thus become confused with the most ancient layers. For Kumarila (seventh century), one of the most famous representatives of the *darśana* [vision] *Purva Mimamsa* [Ancient exegesis], the *Veda* are totally impersonal, since they do not have any human or divine author. Here we have a transcendence that does not even need any prophets. In the *Upadeśasahasri [Thousand verses on teaching]*, S̀ankara (seventh century), the author of the *Uttara Mimamsa* [Further exegesis], related the apologue of the "tenth man". Ten young men were crossing a river on a raft. When the doubt arises that one of them may have fallen into the water, they start counting their number. But the answer is always nine. How terrible! Only a wayfarer on the bank counted ten of them. The young man had forgotten to count himself. Out of negligence? Because he was too personally involved? Out of fear? Revelation needed to be based on something external. S̀ankara was in a minority among these religious philosophers.[87]

Were these *darśana* too religious? But also that of the *Vaiśeṣika* [specifications] (first and second centuries), which deals with the systematic cataloguing of things and essences, presented them as without any history, synchronic and without any processes: they are supposed to be eternal and independent.[88] In the *Samkhya* [enumeration] (fourth and fifth centuries), knowledge in accordance with truth is obtained by detaching the soul from matter and from the world. In the *Samkhyakarika [Rooms of the Samkhya]*, the *mulaprakrti* [radical nature] was presented as not manifested and not created. Anyway, "as a dancer leaves the scene after presenting her spectacle to the public, so the *prakrti* withdraws after revealing itself to the soul . . .". Nature and the soul have to collaborate in order to find their way out of the wood, as a blind man gets a lame person to help him, but in the end they separate. Thus this appears to be a model of knowledge that discriminates, because it does not see the relationships between all the phenomena. It is a *darśana* that seems to be without revelation and without any divinities, and yet its roots lie in the *Katha-Upaniṣad* and in the great epic-religious poem of the *Mahabharata [Great Bharata]*, that is to say, the Great India.[89]

The *Mimamsa [Intense application to reflection]* is, as its name says, explicitly dedicated to religious exegesis. With its concentration on sacred texts, it developed the analysis of language, concluding that the relationship between meaning and word was eternal and unchangeable; the phonemes that compose them are eternal, in turn, and only phonation makes them contingent.[90] In a similar context, should we be surprised that the highest degree of transcendence and abstraction was reached above all in linguistic studies, and only as a consequence in the mathematical

[87]Torella 2001, pp. 642–643.

[88]Torella 2001, pp. 655ff.

[89]Torella 2001, pp. 661ff.

[90]Torella 2001, pp. 666ff. Cf. Balslev 1986.

sciences? In his *Aṣtadhyayi*, Pânini had sought the general essence of the text in meta-rules like "in the place of", specially constructing an artificial metalanguage.[91] His last *sutra* is written as "a a". After this, what else can come but the silence of a void?[92]

Along the road that had opened up towards symbolism, a school like *Navyanyaya* [New method] was born, which created technical terms to obtain "the highest degree of precision, in contrast with the ambiguities of natural language . . . fully exploiting the intrinsic ability of the noun phrase in Sanskrit to express every imaginable level of abstraction, thanks partly to the ease of coining abstract expressions by adding secondary suffixes . . . (e.g. *jneya/jneyatva* 'intelligible/intelligibility)".[93] This had to be subtracted from human beings, who would condemn it to an unstable, contradictory fate. Therefore, in the *Nyaya* [Method], the convention of the language was attributed to the supreme Creator, who thus fixed it at the beginning of all cosmic eras, just as he recreates the *Veda* every time. Among the various schools of thought, long discussions were held about the status to assign to the "word", to the "phrase" and their hierarchic relationships,[94] which is a subject studied in Europe mainly by Ferdinand de Saussure (1857–1913) and twentieth century formalistic structuralism.

Already at the time of Pânini, the *sutra*, the aphorisms, had become definitions and 4,000 rules of grammar, called *Śivasutra*, as they were dictated by Śiva. They were formulated by means of letters of the alphabet, which acted as signs, creating abbreviations in a complex system of symbols.[95] Although the differences between schools gave them various forms, the widespread, prevailing tension towards transcendence in Indian culture made itself felt above all in the linguistic sciences. But the attitude of justifying the rules by making them descend from on high, into an earthly reality swarming, on the contrary, with innumerable forms of life, appears to be generalised. It can even be found in treatises on architecture.[96]

One extreme form of transcendence appears in the *Uttara Mimamsa*, which harked back to the *Vedanta*, and in particular to the *Upaniṣad*. According to this, there exists a universal principle of a spiritual nature, *Brahman*, and a second individual spiritual principle, *atman*. Only these are real. The world of phenomena, on the other hand, suffers from the unreality of dreams: the flow of things that are born and die are likewise purely apparent. ". . . the concept of *maya* [illusion] . . . according to which the empirical world of differences, of the relative, is a kind of mirage that the Absolute produces, like a *lila* [game] . . .".[97]

[91] Staal 2001, pp. 637–638.

[92] Torella 2001, p. 644.

[93] Torella 2001, p. 653.

[94] Torella 2001, pp. 668–669.

[95] Cardona 2001, p. 745.

[96] Dagens 2001, p. 903.

[97] Malamoud 1994, p. 180.

For them, the entire universe is *ajata* [unborn], and as a result, every apparent multiplicity loses value, leading to the total negation in the One. Their favourite parable was that of the cord, which stands for the *brahman*, mistaken for a serpent, the world of the multiple.

At the opposite extreme, as radical antagonists of the general brahmanic culture, we find those who preached: "The world does not go beyond what enters into the field of the senses ... it is foolish to chase after the invisible after abandoning the visible ... The *dharma* [law, duty] is not to be placed before the *kama* [longing, satisfying one's desires]". Here, the possibilities of knowledge encountered serious limits, as the entire infinity of phenomena could not be contemplated. Known as *Lokayata* [the worldly, from *loka* world], no sufficient traces of their heresy would be extant, were it not for the interest shown in them by other less radical heretics like the *jaina*.

The latter, who came from the rival caste of the *kṣatriya* [warriors], contrasted the absolutes of the brahmana, and accepted the limitations of human beings. They depicted us as blind people around an elephant: whatever can this be, a leg, a tusk, the trunk, the wrinkled skin, the tail? They theorised with their *anekantavada* [non-absolutism] a plurality of points of view, all of which perceive a fragment of truth, trying not to fall into total scepticism. They preferred rather to admit several possibilities, gradually using up all the combinations, as we have already seen for numbers (countable, uncountable, arriving as far as the uncountably uncountable) and for other Elements.[98] But in this kind of universe, an object does not have only two choices: to be or not to be. The *jaina* used the optative form of the verb to be, *syat* [it might be] and concluded by combining as many as seven different possibilities: "[a thing or a property, in a certain sense] 1) is, 2) is not, 3) is and is not, 4) is inexpressible, 5) is and is inexpressible, 6) is not and is inexpressible, 7) is, is not and is inexpressible."[99]

This sophisticated dialectic refinement enabled them to perform the necessary acrobatics to live in history and in this world. Although they had been born in the caste of warriors and kings, the *jaina* had developed the most radical of the opposing principles: *ahimsa* [non-violence]. Naturally, they were vegetarians, but what if they killed an ant or a gnat by mistake, while lighting a fire or cooking? For this reason, they got others to do this work, and begged for food from house to house. But these other people worked for them, and were thus guilty of the violence committed on animals. But they answered in the following way: "No, because the ascetics visit houses without notice, and so the cooking has not been carried out deliberately for them". The final conclusion of the logical argument was that the *jaina* were the most virtuous of all.

Lastly, Buddhism dissolved the continual becoming of the external world of phenomena in the illusions of a subject placed at the centre of every problem. As

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[98] See Sect. 4.3 above.

[99] Influenced by India, in Europe, the mathematician L.E.J. Brouwer avoided the *tertium non datur* rule; Tonietti 1983a and 1990.

a similar philosophy of life was expelled from orthodox India, except for the brief period of the reign of Aśoka, and spread, on the contrary, in Tibet, China, Japan and the rest of the world, acquiring a greater visibility, we refer readers elsewhere on this subject.[100]

In order to cut what would otherwise be very long stories short, we, too, shall formulate few *sutra*. The brahmana say: the world of the phenomena is a dream, only the single universal spirit, the absolute *Brahman* exists. The buddhists say: the world of the phenomena is the painful illusion created by the subject, who needs to be freed from it into *nirvana*. The *jaina* say: in the world, we can, at most, grasp only a few fragments of truth, trying to classify them and recombine them. The *lokayata* say: the world does not go beyond its innumerable sensations, let us follow its course, guided by our desires. Among the brahmana, typical aspects of realistic Western Platonic philosophies could be found. In *jainism*, we rather see traces of certain contemporary reductionist philosophies.

The prevailing brahmanic orthodoxy let develop a conception of life and of human beings seen as a unitary whole. Among these, the most famous is undoubtedly the *ayurvedic* medicine, in which the *ayus* [long life] is intended as "union of body, organs, *sattva* [mind] and *atman* [self]". It is reached by means of forms of behaviour dictated above all by a pacific morality which respects every form of life.[101] Under the protective shield of a universal, all-engaging *Brahman*, but faced with a multifarious world, full of variety, the Indian culture took shelter in a rich series of dualistic and ternary classifications distributed everywhere.

We have already seen some of these above, and others are the same as in the European culture, and so there is no need to recall them. These, for example from the famous *Artha Śastra [Treatise on what is useful]* by Kautilya (fourth and third centuries B.C.), include: "right or wrong in the *Veda*, useful or useless in economic activities, correct or incorrect in government".[102] Or the *samanya* [universal] as opposed to the *viśeṣa* [particular].[103] Two antithetic principles lay at the root, one dynamic and the other static, one female and the other male.[104] They exerted their influence at every level, and finally arrived at the opposition between world and soul, and between *brahman* and *atman*.

Right from the *Śatapatha Brahmana*, we read that creation opened with the conflict between gods and *asura* [demons], both children of Prajapati. As they do not succeed in overcoming each other by means of arms, they "… try to defeat one another by speech, by *brahman* [sacred writ]! He who cannot follow up our uttered speech by [making up] a pair shall be defeated, and lose everything." Indra starts for the gods by saying: " '*Eka* [one, masculine] for me!' The others then said, '*Ekâ* [one, feminine] for us!' and thus found the desired pair." Thus they continued,

[100]Torella 2001, pp. 670–689.

[101]Comba 2001, p. 835.

[102]Torella 2001, p. 640.

[103]Torella 2001, p. 660.

[104]Torella 2001, p. 661.

whereupon Indra wins by saying: " '*Panca* [five]', because then the others found no pair, for then both [masculine and feminine] are *panca*".[105] From the myth, we can understand once again that also dualism finds its clearest representation in the structure of Sanskrit, with its masculine, feminine (and neuter) cases, which we see dominating the scene, as usual, in India.[106]

The opposite is easily expressed by means of the prefix *a*, as in *nrta/anrta* [order/disorder of the world], *loka/aloka* [world/non-world], *nirukta/anirukta* [explicit/implicit], *parimita/aparimita* [limited/unlimited].[107] In the *Śatapatha Brahmana*, the rite of *agnicayana* [stacking (bricks for) the fire] is interpreted with numerous dualistic symbols, such as the same/different, one/multiple, continuous/discontinuous.[108] Through words, or rather, with the sacred Word, the *mithuna* [couple] male/female represented in the masculine/feminine cases passed into the objects and symbols of ritual.[109]

Also the two most famous epic-religious poems narrate the struggle between good and evil, between *dharma* and *adharma*, that is to say, between Rama and Ravana in the *Ramayana*, or between the Pandava and the Kaurava in the *Mahabharata*. It is only in the alchemy of the *tantra* [weave, doctrine] that the male and female principles represented in sexual union are merged to produce the desired results. But here, extra-Indian influences are suspected; undoubtedly it recalls in part Taoism.[110]

In the *Brhadaranyaka Upaniṣad [Upaniṣad of the great forest]*, the *atman* was described as follows: "He does not grow as a result of good actions, nor does he decline as a result of bad ones. He is the sovereign of all that exists, the sovereign of beings, the lord of beings, the king of beings. He is the dyke that separates the worlds, stopping them from becoming confused. He is the one that the brahmana try to know by means of the study of the *Veda*, by means of sacrifice, by means of alms, ascesis, abstinence . . .".[111]

The orthodox brahmanic culture, which presented itself as a culture of knowledge, thus indicated its roots in the *atman*, seen in his function of separating the worlds. Likewise, as we shall see below, they tried to keep the gods distinct in a myth that involved music.

Unlike the Chinese culture, and coming closer in this respect to the European one, the orthodox Indian brahmanic culture appears to be dominated by the *dharma* [law]. The *dharma* was contrasted with the *artha* [material interest] and the *kama* [sexual desire]. In spite of some circular references to one another, we can see a

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[105] *Sacred Books of the East*, v. 12, p. 153; Seidenberg 1962b, p. 33.

[106] Sani 1991.

[107] Malamoud 2005, pp. 102, 193, 200, 212.

[108] Malamoud 2005, p. 116.

[109] Malamoud 2005, pp. 89–121.

[110] White 2001, pp. 868–869. Cf. Chap. 3 above. However, the *ardhanariśvara* [androgyne] created by the couple Śiva Parvati, comes close to this. Malamoud 1994, p. 180.

[111] *Upaniṣad*, "Brhadaranyaka", IV, 4, 22; cf. Malamoud 1994, pp. 95–96.

hierarchy parallel to that of the castes. The *dharma* is above everybody, and it is the task of the brahmana to study and maintain it, just as the *kṣatriya* had the task of defending it.[112] The law decided and made a division between truth and error. Avoiding errors meant respecting the *dharma*. Hence the great care dedicated to the geometry of sacrificial altars, which was our starting-point.

> This being rules over the worlds that are beyond the sun
> and human desires. Therefore those who sing
> accompanied by the *viṇa*, sing of him, and aided by him,
> obtain riches.
>
> Then he who knows this and sings of the *saman*,
> sings of both. By the former, he acquires the worlds that
> are beyond the sun, and what the gods desire; by the latter,
> he acquires the worlds on this side of the sun, and reaches
> what men desire.
>
> *Chandogya Upaniṣad*, I, 7, 6–8.

4.5 Did a Mathematical Theory of Music Exist in India, or Not?

Mathematics was considered among the *Vedanga [Members of the Veda]* in India, together with *Jyotisa [astronomy]*: "As the crest on the heads of peacocks, as the gems on the hoods of snakes, so is *ganita* [calculation] at the top of the sciences known as the *Vedanga*."[113] But then, like peacocks, mathematical sciences in this country seem to be birds that do not sing.

We find ourselves faced with a paradox that we must try to explain. We shall immediately see that Indian culture assigns a prominent, generalised position to music. It has already been narrated that the most ancient religious tradition presented the need for a particular precision, which gave rise to aphorisms and even mathematical rules. Thus we would expect to find similar considerations also for music. Instead, these appear to be lacking, thus raising an interesting historical problem. We have seen mathematical theories for music in Greece, and various others were invented on the other side of the world in China. We shall also find mathematical theories for music in the following chapter, presented in Arabic or Persian. But in India, they appear to be lacking. Why? If this were true, would this not give us valuable, significant information, among other things, about relationships between, or the independence of various cultures? For example, how could the legendary Pythagoras, the one who put musical ratios at the centre of his system, have taken his inspiration from India, or vice versa?

[112] Malamoud 1994, pp. 143–167.

[113] Datta & Singh 1935, I, p. 7.

Music does not appear explicitly among the six *Vedanga*, even though it might be considered as hidden between *śikṣa* [phonetics] and *chandas* [metrics]. Later (ninth and tenth centuries), as *gandharva*, it joined the institutional disciplines to be known, together with medicine and archery.[114] And yet one of the *Veda*, the *Sama Veda [Veda of melodies]*, was reserved to the singing of *mantras*. With these, a tradition was created in which the phrases and words, divided up into syllables,[115] lost their literal meaning, and the sense of ritual was assigned to the sounds and the notes handed down from the master to the pupil. Consequently, it was *gita* [singing, vocal music] that was most cultivated.[116]

The most famous, most commentated and most translated page of Indian culture is the *Bhagavad Gita [Song of the Blessed]*, from the *Mahabharata*. Here, Krṣna, the eighth reincarnation of Viṣnu, called the Blessed, explained to Arjuna, the archer of the Pandava, why he should kill his cousins, the Kaurava, in spite of all his moral doubts.[117]

The rite was considered to be effective, not only if the right words were pronounced at the right moment, but also if their sound was right.[118] When the oral tradition passed into writing, and the rules of grammar were fixed, 63 sounds, called *varna* [colours], were classified, distributed over three tones: *uDatta* [high], *anuDatta* [low] and *svarita* [high followed by low]. These were indicated by accents: <u>*a*</u> low, *á* high and low, *a* (without any accent) acute.[119]

In Europe, the theory of music was regularly included for centuries among the interests of mathematicians and natural philosophers. We have already seen this in Greece, and we shall see it again in the following chapters. In India, music is not among the activities to which a *ganaka* devoted himself. No books by *ganita* speak about it. It is absent from the Bakhshali manuscript, from the *Aryabhatya* and from the comment of Bhaskara I on it. Likewise it is missing from the *Brahma Sphuta Siddhanta* by Brahmagupta and from the *Ganita Sarasamgraha [Compendium on the essence of mathematics]* by Mahavira (ninth century), from

[114]Pingree 2001, p. 694

[115]Perinu 1981, p. 19, n. 21. "The hymns are broken down into *mantras*, and these mantras often attract attention only because they contain a certain word, a certain morpheme, a certain syllable, regardless of the general sense of the phrase. . . . Disjointed and dispersed, the *mantras* quoted by the *Brahmana* form a collection of divisible, mobile Elements, much more than a textual *continuum* that draws its meaning from the linking together of words and phrases: . . ."; Malamoud 1994, p. 298.

[116]Together with *atodya* [instrumental music]; Pingree 2001, p. 703. In the myth about the rite of the *soma*, the stringed instrument was used by the gods only in order to ensure the return of the Word, exchanged with the *gandharva* [see below] to obtain the sacrificial plant. But it appears to be a fraud; it was an instrument used only because the sacrifice with the offer of the *soma* is based on the Word of the *Veda*. Malamoud 1994, pp. 181–182.

[117]*Bhagavad Gita*, 1996.

[118]Minkowski 2001, p. 709.

[119]Cardona 2001, pp. 740–743.

the *Siddhanta Śiromani [Crowning of the Siddhanta]* and the *Lilavati [The playful one]* by Bhaskara II.[120]

And yet, even in India, certain ideas would seem to connect music with mathematics. The same notation *katapayadi* [that which begins with *k, t, p, y*] of letters of the alphabet was used for music as for numbers.[121] The combinatorial analysis so appreciated by the *jaina* was also used for music.[122] In the *Taittiryia Brahmana* (3.4.15), for the *puruṣamedha* [human sacrifice], the *ganaka* was offered to the divinity of song, together with the *viṇa* (stringed instrument) player. Was a connection thus established between the *ganita* and music?[123]

Greater precision was promised by cosmology. It has already been said that the Indian classification of cosmic eras arrived at the *mahayuga* of 4,320,000 years. This was separated into four periods, in the ratio 4:3:2:1. These are the same figures used by Pythagorean sects to divide the monochord, and generate the notes of their scale. In Europe the music of the spheres was being created, linking these notes to the stars of the sky. In India, as far as the contents of current literature indicate, we do not find anything similar, and those numbers remain confined to cosmology. Perhaps a suggestion in this direction is the event that the teaching of music to humanity is attributed to Śiva, who created the world with his dance.[124]

The last period of 432,000 years, called *kaliyuga* in India, is also found in Babylonian myths.[125] The game with the first four numbers, 1, 2, 3, 4, which add up to 10, may have struck the imagination both of the Babylonians, and of the Greeks and the Indians. But we should find out how different cultures arrived at different consequences.

We may suspect that the characteristics of the Indian historical context did not stimulate the development of previous ideas, to form explicit relationships between numbers and music. How is it possible to imagine numbers for notes, if the latter are generated by the voice? Numbers, in Greece, could be suggested by the length of strings, and in China by the length of pipes. Perhaps the music of the Vedic period was preceded by a relative geometry which did not lend itself so readily to numerical considerations? In this period, the *svara* [resounding sounds, notes] were called *prathama* [first], *dvitiya* [second], *trtiya* [third], *caturtha* [fourth], *pancama* [fifth], *mandra* [deep], and *atisvarya* [that resounds intensely] or *kruṣta* [acute]. Thus they were counted, and subsequently they were also to be grouped together in the *saptaka* [seven]. But this seems to be the only relationship between them, and no other mathematical form was imagined to classify their height.[126] When the

[120] Hayashi 2001, pp. 776ff. Unless these scholars, afflicted by blinkers and ear-muffs, neglected to quote it.

[121] Hayashi 2001, p. 783.

[122] Hayashi 2001, p. 775. Cf. Leibniz 1666 and Tonietti 1999a. See Part II, Chaps. 9 and 10.

[123] Hayashi 2001, p. 727.

[124] Bharata 1959, p. vi.

[125] Pingree 2001, p. 720.

[126] Perinu 1981, pp. 18 and 24.

famous *ganaka* later concentrated their studies on numbers, had the *Sama Veda* lost prestige and consideration in their eyes? The brahman caste had started to decline, substituted by the *kṣatriya* [warriors], while Vedic singing was being transformed into *gandharva*, music for instruments.[127]

In the *Mahabharata*, the universe started with *akaśa* [ether] from which spouted *apah* [water], *vayu* [air], *tejas* [fire] and *prthivi* [earth] in accordance with their qualities of space, movement, heat, . . ., or sound, taste, smell, . . . For the thought of the *Vaiśeṣika*, it was an infinite ether, unitary and without atoms, which made sound possible, as it acted as an environmental substance. According to this philosophy, qualities depend on the substance to which they belong, but they cannot belong to other qualities or to movements. Thus it was impossible to speak of numbers for sound, seeing that also numbers were a particular quality.[128] Consequently, it would have been difficult here to find a mathematical theory of music. But why did other philosophies not elevate the sounds of the *mantras* to the transcendent absolute, through that of numbers? Perhaps they felt no need for this, because they had placed it in the letters of the *Veda*?

Nowadays that music is no longer generally among the professional interests of mathematics scholars, these ancient Indian *ganaka*, likewise indifferent to it, might appear to us singularly modern in this respect. Not even in the most ancient texts, which deal with Indian musical art in general, do we find any pages about sounds, divided in accordance with numerical ratios.

The *Gitalamkara [Ornament of singing]* by Bharata (date uncertain, perhaps a few centuries B.C.) declared from the beginning (repeating it at the end) that it was "the hook to control drunken elephants, which are the adversaries."[129] The aim was to fix rules in a field subject to discussion. As the title, *Gita* [singing], says, it dealt with vocal music expressed in words. Maybe for this reason, the last chapter contains a list of the different languages in which it could be sung.[130]

Everything here is obtained by singing: "virtue, success, pleasure, freedom . . .". With a sweet song, even the ugliest singer would obtain whatever he wanted from a beautiful woman.[131] Then "the thief of songs commits a crime equivalent to that of killing a brahman, raping the wife of a master, killing a cow or an unborn baby."[132] A subject considered so important deserved to be studied with special precision. "There are seven *svara* [notes], three *grama* [villages, gamuts], twenty-one *murchana* [modal scales], forty-nine *tana* [bases of the modes], three *matra* [units of time], three *laya* [times]. There are three *sthana* [registers], three *yati* [rhythms], six *hasya* [ways of smiling], nine *rasa* [emotions], thirty-six *varna*

[127]Perinu 1981, p. 21.

[128]Torella 2001, pp. 656–659.

[129]Bharata 1959, pp. 3 and 213.

[130]Bharata 1959, Chap. XV.

[131]Bharata 1959, pp. 3–5.

[132]Bharata 1959, p. 35.

[colours, modes] and six times seven *bhaṣa* [styles of singing]."[133] "All that belongs to language in this world, whether it be the word or the artificial [sounds of instruments], has as its subject the seven notes ... The note *Ṣadja* [born from six] has the colour [red] of the lotus petal. *Rṣabha* [bull] is greenish-yellow like a parrot. *Gandhara* [fragrant] is golden. *Madhyama* [medium] has the colour of jasmine. *Pancama* [fifth] is black. *Dhaivata* [harmonious] is bright yellow and *Niṣada* [rest] is multicoloured."[134] They were described in accordance with the course in the human body that generates them from the belly button up to the head. They are related to the castes, to animals, to the divinities and to the *Veda*.[135]

The notes are grouped together in *murchana* [modal scales], described in accordance with the circumstances to which they are suited. The same criterion was followed for the *tana* [bases of the modes]. The *Sama Veda* was sung in the Brahma *tana*, whereas for love, the *Geha* [dwelling-place] *tana* was used.[136] Like syllables, notes had a *hrasva* [short], *dirgha* [long] or *pluta* [prolonged] duration, measured in *matra* [the time of a blink]. The relative emotions were defined, from *śrngara* [love] to *śanta* [peace]. The six different kinds of *hasya* [laughter] will appear strange to us, but they served to attract the listener.[137] Lastly, there were the *varna* [modes] classified as masculine, feminine and generated, with a list of their effects: *sauri* [of the sun], the eighth feminine mode makes listeners happy; *dravidi* [of the Southern people], the fourth generated mode, brings good luck.[138]

Among all these particulars, we do not find even a mention of the issue of how to fix the pitch of notes with precision. "Syllables can be [pronounced] in three *sthana*: from the chest, from the throat, from the head. People learned in the Scriptures say they are named after the three libations [the drink-offerings to the Gods, the *soma*, in the morning, at noon, and in the evening]."[139] Unfortunately, at this vitally interesting point for us, the text appears to be mutilated, and is completed by commentators, following treatises on phonetics, or referring to much later books on music.

In the *Samgita Ratnakara [Mine of gems for music]* (1.3.7), we read: "In practice, however, they are in three areas: in the heart [the octave] considered *mandra* [deep], in the throat, the *madhya* [medium] one, in the head the *tara* [acute] one, [each] double in ascending order."[140] But the treatise continues as late as Śarngadeva (between the ninth and the thirteenth centuries) and it is better conserved in the commentary of Simhabhupala (thirteenth century). Otherwise, at last, we would have found here the "double" ratio, which would allow us to identify, as scholars usually do, the *sthana* as the octave, had it not been for the fact that a dozen centuries

[133] Bharata 1959, p. 47.

[134] Bharata 1959, pp. 51ff.

[135] Bharata 1959, pp. 51–73.

[136] Bharata 1959, pp. 121 and 125.

[137] Bharata 1959, pp. 147–149.

[138] Bharata 1959, pp. 185 and 189.

[139] Bharata 1959, p. 139.

[140] Bharata 1959, pp. 138–139.

Fig. 4.5 Sarasvati, goddess of knowledge, plays the *viṇa* (Sarasvati, anonymous popular art)

passed in the meantime. At that point, Indian culture would have had all kinds of opportunities (unfortunately?) to be influenced by the Arabs, who, as we shall see in the following chapter, brought Greek theory with them.

Furthermore, the musical parts of the *Natya Śastra [Science of drama]* (a few centuries A.D.) already sounded different from the *Gitalamkara* because they were particularly for instrumental music.[141] In the twelfth century, Jayadeva composed the famous *Gita Govindam [Song of Govinda]*, in which he sang of the *madhura bhakti* [pure, heavenly love] of Radha for Krṣna, representing the love of the soul for God. However, its *raga* [delights], as the ancient *varna* modes of the *Gitalamkara* had been called in the meantime, are no longer extant. Nowadays, every Indian musician plays them in his own style.

Śarngadeva was given the epithet of *Niḥśanka*: "One who does not have any doubts".[142] How, then, may he have solved his doubts about tuning and intonation? Perhaps the brahmana, who wrote treatises for everything else, had obtained the precision required by the ritual by following a method different from the mathematics used for the altars, in the case of singing. Did they trust above all their ears in the oral transmission of that tradition from master to pupil?

In placing the numerous frets on the neck of the *viṇa* (Fig. 4.5), may craftsmen and musicians have used systems which did not pass into books, because they generally belonged to castes that were different from the first, that reserved to the

[141]*Natya Śastraa* 1996, Chaps. 28–30, 33. But, in spite of the opinion of a scholar like Alain Daniélou, even here it is not possible to find any relationship with the predominant Greek musical concepts, as we shall soon see. Bharata 1959, p. viii.

[142]Satyanarayana 2005, p. 184. The adjective also means "fearless".

priests? The profession of the musician was developing, less connected to religious rite and more often used in scenic representations.[143] It is true, dividing a length into a number of equal parts might be relatively easy, but would the ear have been satisfied?

In the *Natya Śastra*, the following words were dedicated to the tuning of stringed instruments:[144] "These notes become low or high, according to the adjustment of the strings, and the diversity of the *danda* [neck, measuring 120 cm] of the *viṇa*, and of the sense organs [...]. The difference that occurs in *pa* [*pancama*, the fifth note], when it is raised or lowered by one *śruti* [hearable], and when the consequential *marda* [slackness] or *ayatatva* [tension] of strings occurs, will indicate a typical *śruti*. We shall explain the system of these *śruti*. Two *viṇa*s should be made ready, with the *danda*s, strings, of a similar measure, and with a similar adjustment of the latter in the *ṣadja grama* [gamut on the *ṣadja*].[145] Then one of these should be tuned in the *madhyama grama* [gamut on *madhyama*],[146] by lowering *pa* by one *śruti*. The same *viṇa*, by adding one to *pa*, will be tuned in the *ṣadja grama*. This is the meaning of decreasing a *śruti*." "The notes of the *vamśa* [flute] should be perfected, and accomplished with the help of the *viṇa* and of the human throat ... A unison of the human throat, the *viṇa*, and the flute is specially praised." "Sound is airy [i.e. air-dependent], and it is considered to be of two kinds, one equipped with *svara*s [notes], and the other with *abhidhana* [name-words with meaning] ... Seven *svara*s have been proclaimed in the *viṇa* as well as in human vocal cords. The same are produced in *atodya*s [instruments] as well. Notes coming out of the human body are transmitted to the wooden *viṇa*, then to the *puṣkara* [drums], and ultimately to the *ghana* [solid instruments] ....".[147]

Also in the *Natya Śastra*, the three *sthana* [registers in height] were situated in the human body: "There are three *sthana* [locations, registers]: 1) the *uras* [chest], 2) the *kaṇtha* [throat], and 3) the *śiras* [head]."[148] Thus, to tune a stringed instrument like the *viṇa*, their length was taken into consideration, while modifying the tension. Guided by his "sense organs", the musician added or took away *śruti*, which are the "audible" intervals. The human voice remained the fundamental origin, while everything was traced back to Brahma, Śiva and to the *Veda*.[149] Numerical ratios are completely missing.

Starting from the *Natya Śastra* to the *Samgita Ratnakara*, however, the art gradually developed of breaking the *sthana* down into 22 *śruti*. But only if this had been identified as the Greek-Western octave could the *śruti* have become quarters of a tone. However, we did not find any space for them in the orthodox Pythagorean

[143]Perinu 1981, p. 21.

[144]*Natya Śastraa*, Chaps. 28, 23 and 27–28; ed. 1996, pp. 388–389.

[145]*ṣadja* [born from six] is the first note.

[146]*madhyama* [middle] is the fourth note.

[147]*Natya Śastraa*, Chaps. 33, 29–35; ed. 1996, p. 486.

[148]*Natya Śastraa*, Chaps. 19, 38–40; ed. 1996, p. 268.

[149]*Natya Śastraa*, Chap. 1.

Greek culture. Only the rival school of Aristoxenus theorised quarter tones, and did so, as we have seen,[150] using the ear, instead of numerical ratios, in the practice of music. Thus even if we hypothetically admitted, though not conceding, either in the merits of the case, or in our historical style, "une parenté certaine avec les théories musicales de la Grèce, ..." [an undoubted relationship with the musical theories of Greece], as sustained by Alain Daniélou,[151] this could have existed only with heretics like Aristoxenus, and clearly not with the Pythagoreans. To arrive at quarter tones by means of numbers, European theoreticians of music would have to learn to count beyond four, indeed, as far as seven. But they would have to wait, with Euler, until the eighteenth century.[152]

Thus the study of the texts has helped us to understand how far the music of India was, in its origins, from the mathematical theories of Greece.[153] In particular, neither the original *sthana*, nor the three relative *saptaka* (low, medium, high) could be identified as the Western octaves without some serious distortion.[154] The *murchana* organise the *svara* along lines that could be likened to modal scales. Inside these, the *jati* are born, that is to say, the melodies which arouse the various feelings in the listener (better known subsequently by the term *raga*). And yet, the whole system was constructed on sounds whose relative intonation was not fixed,[155] which were arranged in two *grama*, the form of which may best be rendered, perhaps, by a circle, rather than a Western scale, in accordance with the model of a village, which is the real meaning of the word.[156] Around the circle, the 22 *śruti* would fit in comfortably.

Lastly, it should not be overlooked that we find the idea of a relationship between the *svara* and the sounds made by animals not particularly good at singing. Thus the first *ṣadja* corresponds to the intonation of the peacock, ... , the sixth *dhaivata* to sounds like those of the horse and the lizard, the seventh *nisada* to the elephant. It was only the fifth *pancama* that was undoubtedly connected with songbirds, such as the nightingale and the blackbird. The medium note *madyana* was connected also with the natural sounds of water falling and the rustle of the forest.[157]

Even the choice of the new name for music, *gandharva*, different from *samgita* [song with], revealed significant attitudes. The *gandharva* were semi-divine beings, aerial spirits that inspired music and love. They were accompanied by the *apsaras*, celestial nymphs that were a symbol of the senses, which distracted the conscience from the *atman* [the soul, self]. They particularly tormented ascetics. In one myth,

[150]See above, Sect. 2.5.

[151]Bharata 1959, p. v.

[152]See Part II, Sect. 11.1. Euler 1739. Tonietti 2002b.

[153]Cf. Perinu 1981, pp. 11 and 141.

[154]A.N. Sanyal: "... the concept of the octave ... cannot be accepted for classical Indian music ... Neither the ancient traditions nor modern practice show signs of it ..."; quoted by Perinu 1981, p. 40, n. 16. Otherwise they would have written *aṣtaka* [the eight].

[155]Perinu 1981, p. 24.

[156]Perinu 1981, pp. 25, 29–30 and 44, n. 42.

[157]Perinu 1981, p. 42, n. 30.

the *gandharva* did not take any position in the struggle between gods and *asura* to gain control of the heavenly and terrestrial worlds. Consequently, when the gods won, they did not want to concede any part to them. Above all because they wanted heaven and earth, which had once been united, to be separated. But the *gandharva* discovered *saman* [harmony, melody], which allowed them to inhabit those worlds, while the gods continued to connect them with singing.[158]

The story is a good representation of the position of music in this country's culture. On the one hand, with sacred singing, it separates and elevates towards heaven; on the other, it remains attached to the earth, like the *gandharva*. In another myth about the rite of the *soma*, the *gandharva*, "who love women", yield the *soma* to the *Vac* [Word, feminine in gender]. And they liked her so much that they wanted to keep her with them. But how then could any rite take place, as they were performed by reciting *mantras*? In order to win her back, the gods seduced her by singing to the sound of the *viṇa*, which was specially created.[159] In the recurring dualism, music was at times pulled upwards, as in the singing of the *Veda*, at times downwards, as an instrument of seduction. The well-hidden, or probably absent, mathematical theory of music in India suffered from this ambiguity. How could it have survived in that world split into two parts? Did not taking the one or the other side mean being condemned to invisibility or absence? Today, however, we see it and hear it only on the earth.

A similar concept of music continues to stop us from landing directly on the beaches of Magna Graecia, seen from afar, to return to Plato's schools. It appears to follow ears, and not numbers, and even escapes from those counted by the brahmana at the moment of the birth of names. It appears to be poised between heaven and earth, but it prefers to come down here among living creatures. In India we have found the prevalence of theories of music, elaborated by musicians and not by the *ganaka*, which are similar to that of Aristoxenus. In Europe, we have to wait until Jean Le Rond d'Alembert in the eighteenth century, to find a mathematician who, for this purpose, preferred to trust his own ears.[160]

In Indian music, the indispensable role assigned to the many types of drums finds no equal in Europe. The myth of their invention has been narrated in the *Natya Śastra* in the following terms: "Thanks to the velocity of the gust of wind, clear sounds were made over the leaves of the lotus clusters by the falling columns of water. On hearing the sound arising from the water columns falling down, [Svati] considered it a mysterious occurrence, and hence observed it carefully. He returned to the hermitage with the full knowledge of the high, medium and low sounds made on the lotus leaves as majestically deep, sweet, and delightful."[161] The Greek myth,

[158] Malamoud 2005, pp. 204–205.

[159] Malamoud 1994, pp. 181–182. *Gandharva* was also a kind of marriage based on love, thus defined: "Voluntary union of a girl and her lover, with desire as the cause and sexual pleasure as the end." Malamoud 1994, p. 172.

[160] Part II, Sect. 11.2; Tonietti 2002b.

[161] *Natya Śastra*, Chaps. 33, 4–13; ed. 1996, p. 484.

instead, told of Pythagoras, who classified sounds in accordance with the size of the hammers that he heard in the workshop of a blacksmith.[162]

Śarngadeva recorded the division of the Indian tradition for singing between the 'hindustani' music of the North and the 'carnatic' kind in the South.[163] In that period, foreigners had arrived from the Middle East, with their cultural differences.

4.6 Between Indians and Arabs

In the seventh century, the Syrian bishop Severus Sebokt confronted the arrogance displayed by those who only considered the Greek culture valid. On the contrary, he called attention to the astronomy of the Indians, and to "their precious methods of calculation, which surpass all description … I would simply tell you that these calculations are performed by means of nine signs".[164] Thus what had been achieved in India began to become known also moving westwards. In 773, induced by the caliph al-Mansur, Ibn al-Fazari translated into Arabic the *Brahma Sphuta Siddhanta* by Brahmagupta. Three centuries later, also the *Aryabhatya* by Aryabhata I was translated into Arabic. Arabic and Persian astronomic tables were calculated, purportedly using Indian methods. Also the famous Muhammad ibn Musa al-Khwarizmi (c. 780–c. 850) compiled digests based on the *Siddhanta*, and dedicated another entire book to the Indian mathematical sciences. No longer extant in its original version in Arabic, it has survived in the Latin version, under the title *Algorithmi de numero indorum [Calculations regarding the numbers of the Indians].*[165] Since then the term algorithm has meant, as it does today, a procedure for calculation. This was the perfect case which ought to help us understand both the value of the various scientific cultures, and how much better it is to maintain, above all, the differences.

In the eleventh century, a famous Persian encyclopaedic scholar who came from the Arab world, Abu Rayhan al-Biruni (973–1048) remained for several years in the north-west of India. Here, he learnt Sanskrit and visited the scholars of the area, compiling translations and books, including the *Kitab al-Hind [Book of India].*[166] But in this case, the direct contact between the two cultures encountered resistance, misunderstandings and abuse. Our Persian's comparison with the mathematical and astronomical sciences of his hosts, who were just starting in that period to be invaded and conquered in the name of Mohammed, was described by the *Kitab al-Hind* as follows: "I can only compare their mathematical and astronomical literature … to a mixture of pearl shells and sour dates, or of pearl and dung, or of costly crystals and

[162] Above, Sect. 2.1.

[163] Satyanarayana 2005, p. 202; Perinu 1981, p. 9.

[164] Boyer 1990, p. 250.

[165] Joseph 2003, p. 301.

[166] Subbarayappa 2001, p. 796.

common pebbles. Both kinds of things are equal in their eyes, since they cannot raise themselves to the methods of a strictly scientific deduction".[167] Al-Biruni overlooked the outstanding calculations of Brahmagupta for eclipses, and rather criticised his Vedic religion which said that it was a dragon that swallowed the moon or the sun. Also Muslims prayed at certain hours of the day. But "the sun has nothing to do with our worship".[168] And in that book, Indian alchemy did not meet with a better fate, compared with Arab alchemy.[169]

On the contrary, Amir Khusraw, another Islamic scholar who moved to India, but who wrote in Persian, expressed the opposite judgements in his poem *Nuh siphir [Nine spheres]* (1318). He loved India and its culture, which he considered to be superior, because it was capable of producing great results in mathematical sciences, in chess and in literature.[170] A similar variety of behaviour, distributed over time and in space, puts us on our guard against simplistic generalisations. And yet they are all a part of the historical background of the penetration of the Arabic culture into the North of India, amid flashes of scimitars and galloping horses.

Inspired by a religious idea like Mohammed's Islam, the nomadic populations of the Arabian peninsula conquered half of the Iberian peninsula, half of the Mediterranean, North Africa, Persia, central Asia and the North of India in a few centuries starting from the VII. Arabic, the language of their sacred book, the *Koran*, became the predominant language of this vast empire: *Dar al-Islam* [Dwelling-place of Islam]. To the east and to the south, the Arabs, meaning by this word populations that were heterogeneous in their history, geographic areas and culture, but united by their holy book and its language, came into contact with the Indians. To the north and to the west, their neighbours were the Greeks and the other populations of Europe. Thus the cultural exchanges that took place through them became more and more intense in many directions, with alternating fortunes.

The closure of the school in Athens, ordered by Justinian in 529, was a sign that a cultural epoch was coming to an end. Scholars were also moving eastwards, in search of new spaces, thus creating novel aggregations, where they found patrons prepared to provide them with the necessary means. In this way, Greeks, Persians and Indians began to meet one another. As he was interested in the medicine of India, the caliph Harun al-Rashid (766–813) invited some Indian doctors to Baghdad, so that they could translate their books from Sanskrit into Arabic and Persian.[171] However, the opposing current proved to be stronger, seeing that it was the Arabs, through their various components, that took possession of North India, and not vice versa.

Apart from a sporadic attempt in the eighth century, the Turkish sultan Mahmud moved his steps from Afghanistan in the eleventh century, with al-Biruni in his

[167] Kline 1972, p. 190. Casari 2001, p. 909.

[168] Casari 2001, p. 915.

[169] Speziale 2001, p. 927.

[170] Casari 2001, p. 919.

[171] Speziale 2001, p. 921.

retinue. The Turkish-Afghan phase lasted until the invasion of Tamburlaine in 1398. On this basis, his Turkish descendant, Babur, set up the long *mughal* dynasty in 1526, which was destined to last until the mid-nineteenth century, leaving several famous monuments. The *mughal* initially favoured mass emigration from Persia. Beyond all these variants, the Arabs took the trouble, from the beginning, to set up public schools, where, among other things, they taught the geocentric astronomy of Ptolemy, arithmetic, algebra, geometry and music. According to their style, they also built hospitals in the North of India, where the Arabic medicine called *yunani* [Ionic, that is to say, Greek] was practised. As the word suggests, this had been derived from the medicine of Galen. Starting from al-Biruni, as usual, translations of text-books of medicine and pharmacological catalogues were prepared in the sundry languages used, such as Greek, Arabic, Persian or pahlavi, and Sanskrit, without forgetting Urdu, where necessary.[172]

With all these difficulties in understanding the various languages, and interpreting cultures based on religious systems that were so distant, but equally pervasive, with the difficulties of supporting the burden of a military occupation and foreigners' policies, we may wonder whether any exchanges actually took place or not in certain scientific disciplines. No contact of a theoretical nature was made between *ayurvedic* medicine and the Greek-Arabic variety.[173] The former came from the *Veda* and from the predominant Indian philosophies. Connecting the psychic self, therefore, with the other organs of the body, it preferred words, ointments, massages, preventive diets and spiritual peace.[174] The latter searched rather for a balance between humours and Elements, and cured by administering the opposite qualities.[175]

Were relations better between the different mathematical sciences? Al-Biruni should declare that he had translated into Sanskrit both Euclid's *Elements* and Ptolemy's *Almagest*. But these versions are not extant today, and other confirmations of their existence are lacking.[176] Was this at least a sign of their scarce success? Only in the eighteenth century did the sovereign of Jaipur, Sawai Jayasimha, a vassal of the mughal king Muhammad Sah, have the Arabic version of Ptolemy's *Almagest* and Euclid's *Elements* translated into Sanskrit, giving them, respectively, the new titles *Samratsiddhanta [Final exposition of the universal sovereign]* and *Rekhaganita [Calculation of straight lines]*. Euclid thus would have stayed in India only one century after China, and for very different reasons: Jayasimha was particularly interested in astronomy, both seen from an Indian point of view and from an Arabic or European one.[177]

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[172] Speziale 2001.

[173] Speziale 2001, p. 926.

[174] Comba 2001.

[175] Speziale 2001.

[176] Joseph 2003, p. 410.

[177] Subbarayappa 2001, pp. 795–796.

Following the other road, during the *mughal* period, both the *Lilavati [Playful]* and the *Vijaganita [Calculation of the seeds]* by Bhaskara II were translated into Persian. Apart from the Indian capacity for calculation, even astronomical sciences seemed to be irreconcilable in the two cultures. In 1370, *Yantraraja [King of the instruments]* an Arabic treatise on the astrolabe, was translated into Sanskrit. But in the translation into Persian of the *Brhatsamhita [Great collection]* by Varahamihira (died in 587), eight chapters were censured, because they were considered to be idolatrous.[178] Furthermore, Indian astronomy is thought by some to have been influenced both by that of the Assyrio-Babylonians and by the Greeks.[179]

Within this complex picture, what had happened to music? Taking it from the four main mathematical sciences of Greek-Pythagorean derivation, music was taught in the Arab schooling system, transplanted into the North of India, from the beginning of the conquest.[180] And we should not forget the role assigned to music, even in medicine. We trust that some other patient and interested scholar will sift through the enormous quantity of material that remains in the Muslim libraries in India, for a total of at least 1,671 works written in Persian and 1,219 in Arabic.[181] In the meantime, we feel justified in suspecting that the origin may be similar in the case of the scanty allusion to a "double" ratio for the octave found in the *Samgita Ratnakara* by Śarngadeva.[182]

In the opposite direction, towards the West, the Arabs brought two mathematical notions which were destined to meet with great success, and to remain in the manuals of the whole world. After being used by the Indians in astronomy, the semi-chord of the arc, *jya*, became *gib* in Arabic, and then *gayb* [sine, pocket], eventually entering in time into modern trigonometry as *sin*, the *sine* of an angle.[183] Even more interesting is the course of Indian numerical symbols. We have already seen that in this culture based on the sacred Sanskrit words, the letters of the alphabet were preferred to numbers.[184] It was equally surprising that also the Arabs preferred to 'Arabic' numbers the 28 letters of their alphabet *abgad* [ABJD], which associates a number to each of them, as in the succession A = 1, B = 2, J = 3, …, 9, 10, 20, 30, …, 90, 100, 200, …, 900, 1,000. Was this system, too, inspired by Greece? In any case, it was the same as the Jewish *kabbalah* and *gematria*. That symbolism was used for calculations, algebra, astronomy and the musical notes. "Paradoxically, the Indian positional system transmitted to the Arabs in Europe was rarely used in Muslim scientific literature, only for large numbers, as, for example, the tangents of angles close to 90 degree".[185]

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178 Casari 2001, pp. 915–917.

179 Pingree 2001.

180 Casari 2001, p. 913.

181 Casari 2001, p. 914.

182 See above, Sect. 4.5.

183 Casari 2001, p. 908.

184 See above, Sect. 4.3.

185 Casari 2001, pp. 913–914.

Before the symbols of numbers entered into our current use, much more was to take place between the Arabs and Europeans like Leonardo da Pisa, called Fibonacci. The most (Muslim) and least (Hindu) transcendent religions, more or less revealed, were to make way for the calculations of merchants.

Chapter 5
Not Only in Arabic

Search for knowledge, also in China.

Muhammad

Music is a written work that men conceive
and women carefully draw up.

Ishaq al-Mawsili

Writing is spiritual geometry that
appears by means of a physical instrument.

[Saying attributed by the Arabs to Euclid]

Ah, but my calculations, people say,
have squared the year with a human compass, eh?,
so to speak, cancelling from the calendar
tomorrow not yet born and yesterday already dead.

Umar al-Khayyam, *Robaiyyat [Quatrains]*

5.1 Between the West and the East

The roots of the Greek, Chinese and Indian mathematical sciences lie so deep as to appear fathomless. We can, anyway, imagine the first written texts preceded by oral traditions, by archaeological artefacts of all kinds, and by anthropological customs which in time have become myths, and then legends.

In the case of Arabic scientific culture, on the contrary, the first written documents, which are not so ancient, appear from the start to refer mainly to other books. In view of its geographical position, it could easily have rested on what was already at hand, and what had remained from the ancient Egypt of the Pharaohs and

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the very ancient Babylonian civilisation.[1] Yet the effects of these relative influences seem to be limited or negligible, compared with what was absorbed from the Greek-Hellenistic and Indian areas. We dwelt briefly on the exchanges with the latter at the end of the preceding chapter. Let us now see the relationships of the Arabs with the mathematical sciences and the culture of the Greeks, as well as, obviously, what Arab scholars succeeded in inventing with respect to both of these.

Continuing to emphasise the importance of the language used, this time Arabic, above all, we shall use the adjective 'Arab/Arabic' in general to refer to this particular culture and its capacity for dominating in its period and area of influence, which was considerable. However, we shall see the emergence also of scholars with heterogeneous cultural roots, and with some heretical positions, who expressed themselves in particular in the language of Persia, *Palhavi*. All of them were forced to abandon their mother tongues in favour of the dominant language imposed in the administration of political power. Only if expressed in that language did every argument gain in prestige, diffusion and readability. Arabic was the language used for the revelation and the writing of the sacred book which was the main characteristic of this culture: the *Koran [Recitation]*. Thus, in writing of Arabic science, we shall allow the religious lymph transported by the language to flow and surface, where necessary, without causing it to dry up. In so doing, we can conserve one of the main characteristics of *Islam* [Submission (to Allah)], though not explicitly using the word. Their books, including those about science, were written "in the Name of God, the Forgiving one, the Merciful . . . We invoke God as our help: we invoke His clemency so that He will allow us to complete our task."[2]

5.2 The Theory of Music in Ibn Sina

Faithful to our style, let us begin by narrating the history through music. Unlike India, here in the *Dar al-Islam* [Dwelling-place of Islam] a mathematical theory of music was displayed with confidence. Among the most important scholars, we find it immediately in Avicenne (975–1037),[3] as he was called in the West. His full name was actually Abu Ali al-Husayn ibn Abd-Allah Ibn Sina, which places our figure in the genealogy between fathers and sons.

[1]If any of the calculations performed by the Babylonians in cuneiform characters survived, the Arabs would have received it from the Alexandrian Heron (second century); Boyer 1990, pp. 201–202 and 270. The classic reference-point for the mathematical sciences of the Egyptians and the Babylonians remains Neugebauer 1970, or van der Waerden 1983. However, these authors present the achievements of these peoples, emphasising the historiographic hypothesis of mutual transmission, instead of their cultural differences.

[2]Ibn Sina 1935, pp. 105–106. Nasr 1977, p. 17.

[3]Ibn Sina 1986, I p. 261. Leaving aside the usual uncertainties and controversies about dates, the Arab world adopted a lunar calendar, whose years are counted from the Hegira, the departure of Muhammad from the Mecca for Medinah. Consequently, the dates for Avicenna became 365–428.

His most complete text on music is contained in the *Kitab al-Šifa [Book of Healing]*. But although his greatest fame comes from the art and studies in medicine contained in the renowned work *Kitab al-Qanun fi 'l-tibb [Book of the Law for Medicine]* destined to leave its mark even in the West in the translation *Liber Canonis* of Gerardo from Cremona (1114–1187), now he wanted to heal the spirit from its errors, and not the body from its diseases. In the encyclopaedic philosophy of the book, among numerous other disciplines such as logic, physics and metaphysics, Ibn Sina also contemplated geometry, arithmetic, astronomy and music, in which we recognise the Greek *quadrivium*.

However, from it he did not accept the idea of music uncritically. "We shall not make any attempt to establish a relationship between the states of heaven, the characters of the soul and musical intervals. This would mean acting like those who do not recognise the typical characteristic of each science; they are heirs of an out-dated, nerveless philosophy, and confuse the essential attributes of things with accidental ones. Some have imitated them with digests. But those who understand the correct philosophy, those who have grasped the right distinctions, corrected the mistakes deriving from imitation, and cancelled the errors that hide the beauty of ancient thought, these people deserve a favourable reception; because too many habits have been wrongly applauded, and too much praise has been attributed without reflection."[4]

Thus our Arab (who was actually a Persian) intended to give us a theory without the relative music of the spheres devised by the Pythagorean sects and taken up by the Platonic schools: "out-dated, nerveless philosophy". For him, the "correct philosophy", on the contrary, was that of Aristotle. Sound was presented in relation to hearing, the judge of what is, or is not, pleasing. In this sense, it was not so different from the other objects of sense, such as smells. The sounds of the voice serve for communication between human beings: they express their ideas. But as these are "without any limit", language is not capable of giving expression to them all. A convention, based on human arts, makes an idea correspond to every inflection of the voice. "Sounds combined in a specific way and in a harmonious order act more deeply on the soul." If two notes are to create a pleasant impression, the second must be "a return of the first in another form: the two will together create a relationship that is pleasing to our ear."[5]

In discussing consonances and dissonances in music, "the sound principles" are those of physics. The material data are clothed with forms, from which derive the "arithmetic principles ... Effectively, these forms dispose musical data to receive numerical ratios thanks to which they appear to be harmonious or discordant."

[4] Ibn Sina 1935, p. 106. For the musical parts, we use the French translation of Chap. XII contained in the *Kitab al-Šifa*, prepared by Rodolphe d'Erlanger for the second volume of his *La Musique Arabe*. At times, however, Michele Barontini controlled the Arabic originals, translating them directly into Italian.

[5] Ibn Sina 1935, pp. 109–110. Erlanger identifies the relative passages of Aristotle taken from books on the soul, on politics, etc.; Ibn Sina 1935, pp. 259–260.

Sounds can be distinguished as "acute or deep". The causes of acuteness are identified by Ibn Sina as lying in the "cohesion of molecules in the body generating it; the strength that animates this body; the smoothness of its surface, and an intense compression of the layers of air due to the wavelike movement that transports the sound. The causes of deepness are the opposite of those that cause acuteness." Among these, our Persian-Arab listed "the hardness of the body struck, its smoothness, its shortness and its tension." In the case of wind instruments, "the smallness of the opening and its closeness to the mouthpiece." The opposite is true for deep sounds: "softness of the body, ... roughness of the surface, ... its large dimensions, ... its relaxation." For wind instruments, "the extended diameter of the openings and the distance from the mouthpiece".[6]

"The causes of acuteness and deepness increase or decrease in direct proportion to the intensity of each of these conditions. Consequently, a string for which the tension remains unchanged will produce sounds with a different degree of deepness or acuteness, depending on its length ... the same holds for all the conditions mentioned." Lastly, Ibn Sina noted that "the length of the strings, the diameter of the openings in wind instruments, the distance that separates these from the origin of the blowing are measurable quantities, the only ones that allow us to evaluate the degrees of sounds".[7]

In these pages, we see the reappearance of ancient Greek ideas about direct or inverse proportions of acuteness/deepness to the dimensions of the resounding bodies. And their merits were absorbed, together with their limitations. Proportionality was direct, also between acuteness and the tension of the string. The idea of measuring the tension by comparing weights attached to the string did not occur to them. Thus, only in the sixteenth century, and above all with the European scientific revolution of the seventeenth century, was direct proportionality reduced to the square root of the tension.[8]

And yet, leaving aside the eccentricities like the smoothness of the resounding body, this page also presents some interesting novelties, which are unknown to the prevalent Greek theories. Sound was described as a "wavelike movement" of the air, that is to say, continuous, and not as the discrete succession of impacts, as sustained in Pythagorean texts.

Besides strings, also wind instruments were taken into consideration, measuring not only their length, but also the dimensions of their openings. The effect of this on the deepness of sounds was recorded, but without succeeding in establishing a precise relationship for it.[9] Ibn Sina must have been attached to his idea of sound as a wave in the air; we find it again also in the treatise *Asbab huduth al-huruf [Causes that produce the sounds of a language]*. Here, the direct cause of sound is the *tamawwuj al-hawa* [air waves]. When they strike the nerve of the ear, they

[6] Ibn Sina 1935, p. 111.

[7] Ibn Sina 1935, pp. 111–112.

[8] See Sect. 6.7 and Part II, Chaps. 8 and 9.

[9] Compare with the results of the Chinese, in Sect. 3.2, and with Vincenzio Galilei in Sect. 6.7.

create sound. "The wavelike movement itself creates sound. On the contrary, the high and low tone of the sound is created by the *hal* [mode] of the undulation, in the sense of a greater or lesser closeness of its single parts [the waves?]: the acute sound is produced by a greater degree of closeness, and the deep one by the opposite".[10] This would appear to be a good approximation of future European wave theories, like that of Marin Mersenne (1588–1648).[11]

When he considered the intervals between notes, dividing them into consonant and dissonant sounds, Ibn Sina initially appears to recall Aristoxenus and Aristotle. In order to obtain a consonance between two notes, he required that there should be a certain relationship and similarity between them. As in Aristotle, this could be in *esse* or in *posse*. But by similarity in *esse*, our doctor and philosopher intended two notes, "one ciphered by 8 and the other by 4", that is to say, in the ratio of double or half. And for the similarity in *posse*, 6 and 4 or 6 and 2, which meant, respectively, superpartial and multiple ratios. Above all, he concluded: "The two degrees of a dissonant interval may be in a numerical ratio, or not." A non-numerical dissonance will be one generated "by the portion of a string whose tension is indefinite, and the other by the string in all its length, if these two lengths of the string stand in the ratio of the side of a square to its diagonal".[12]

Thus, although it was placed in an Aristotelian framework, the theory of music proposed by Ibn Sina remained fundamentally Pythagorean. Such was the central nucleus of numerical ratios certainly, seeing that this excluded irrationals as non-numbers, and maintained their concept of consonances, perhaps with one exception which will be seen below. The ear initially invoked had not arrived thus far. In justifying the ratio 8 to 3 as consonant, on the same plane as the ratio 4 to 3, because it is obtained by adding the octave, our Persian-Arab ended up by explicitly defending the "old Pythagorean law", and accused those who wanted to reject it. "They do the same as those who dive into the water to avoid the raindrops".[13] He maintained the division of the octave according to the arithmetic and harmonic means, respectively 2, 3, 4 and 3, 4, 6.[14] He admitted the geometric mean to divide an interval in half, only when the numbers were perfect squares. Otherwise, "it would not be possible to find an exact geometric mean for them". Thus, between 8 and 9, the interval of one tone, he considered only the arithmetic mean, which does not divide it into two equal parts.[15]

And yet, on the Pythagorean basis, Ibn Sina-Avicenne constructed different spaces. We can find variants in the terminology, whereby the octave, the Greek *diapason*, "is qualified as 'complete' in the double ratio".[16] He christened as

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[10]Bausani 1978, pp. 196–197.

[11]See Part II, Sect. 9.1.

[12]Ibn Sina 1935, p. 116.

[13]Ibn Sina 1935, p. 118.

[14]Ibn Sina 1935, p. 124.

[15]Ibn Sina 1935, p. 136.

[16]Ibn Sina 1935, p. 120. Translated literally, *diapason* means "through all".

"intricate, tangled" small [melodic] intervals such as 5 to 4, which were considered consonant.[17] The notes were "ciphered" by numbers such as 8, 4, 5, 3, 2, 3, 6, . . ., but the ear was often invoked to decide a consonance. 8 to 3? "The ratio of this interval is neither multiple, nor superpartial, and yet the ear recognises it as consonant".[18] The intervals of one tone were not to be divided into more than four (unequal) parts, because "A smaller interval would have an unpleasant sonority".[19] Attention was paid to the limitations placed by material conditions in the execution: "our throat cannot produce similar notes", "the voice gets tired when passing often over their notes". Above all, he thought of vocal performances, considering every other kind of music an imitation.[20]

He "ciphered" the notes, but he did not intend to reduce them to numbers. He made use of them to calculate additions, subtractions and divisions of intervals between notes. Was he really satisfied with the traditional pattern of ratios? He wrote successions of numbers: 6, 8, 9 and 8, 9, 12. To combine the fourth 3 to 4 with the tone 8 to 9 in the fifth, he multiplied 4×9 and 3×8, as if they were fractions, but then he did not simplify to obtain 2 to 3. Rather, multiplying also 4×8, he arrived at the succession 24, 32, 36.[21] At one point in the text, he seems to make an explicit use of fractions. Ratios were often given in the form $1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}$ . . . reminiscent of the Graeco-Latin terminology *sesquialtera, sesquitertia, sesquiquarta*. Discussion of the genres of melodies was carried out by means of successions like the following[22]:

$$64^{\frac{8}{7}}\,56^{\frac{8}{7}}\,49^{\frac{49}{48}}\,48$$

What there was of contrasting and incompatible in Greek scientific culture, the Pythagoreanism of Plato and Euclid compared with the Aristotelian Aristoxenus,[23] Ibn Sina now tried to reconcile in a personal synthesis. Thus we discover one of the orthodox characteristics of Arabic culture: novelties were to be fitted into tradition, without which value and perspective were lost. What was the tradition for music? As rarely happens in our histories, the Persian-Arab spelt it out clearly for us: "He who is interested in studying these problems can complete our exposition by referring to

[17]Ibn Sina 1935, p. 120.

[18]Ibn Sina 1935, p. 125.

[19]Ibn Sina 1935, pp. 137–138.

[20]Ibn Sina 1935, pp. 121, 138 and 140.

[21]Ibn Sina 1935, pp. 130–131.

[22]Ibn Sina 1935, pp. 120 and 146. The subject is delicate. We are unable to decide at the moment whether the translator Rodolphe d'Erlanger was sufficiently accurate in maintaining in French the highly significant distinction between "ratio" and "fraction". Note 25 on p. 254 arouses doubts. But see also pp. 263–265. *Nisbatu* is the Arabic term for ratio. I would like to be able to verify in the most ancient Arabic editions whether the fractions interpolated in the successions are in the text or were added by d'Erlanger.

[23]See above, Chap. 2.

what Euclid teaches in his book entitled *The Canon*".[24] Was its influence too heavy, stopping any developments different from Greek-Pythagorean orthodoxy? Was this the reason why the Arabs' deep innovations in the field of numbers and equations had little or no influence on the theories of music?

Ibn Sina seemed more at his ease with Greek philosophy than with the mathematics of the Arabs. However, his theory of music allowed for a variety of cases and genres. And he even dedicated attention to how musicians could place their hands and fingers in order to play their instruments.[25] He divided up melodies into three genres: *mulawwanah* [coloured], *ta' li fiyyah* [enharmonic], and *rasimah* [sketched]. These terms were taken from the Greek distinctions between the diatonic, enharmonic and chromatic genres, but evidently, he preferred to 'colour' the first of these. In any case, the effects on the Arab soul were similar to those on the Greeks, because the first one remained strong and hard, the second weak and soft, and the third intermediate. In order to obtain them and explain them, our encyclopaedic doctor divided the interval of a fourth, 4 to 3, in various ways. In this, he followed variants of the classical tetrachord, because its ratios were not limited to the tone, 9 to 8, but also considered 8 to 7 and 10 to 9, with the relative combinations and the consequent remainders needed to complete the fourth.[26]

The principal source of his theory was *ΑΡΜΟΝΙΚΩΝ* by Claudius Ptolemy, where we find numbers and considerations that are similar.[27] Like Euclid, he, too, was given the honour of explicit recognition, and thus we find the other Hellenistic basis of Arabic sciences. "Divided in the same way, the interval of a fourth generates the intervals $1 + \frac{1}{7}$ and $1 + \frac{1}{6}$ ... Ptolemy prefers the genre at which we have arrived to all the others."[28]

Ibn Sina must have arrived at Aristoxenus through Ptolemy, who often criticised him, though this Greek musician does not seem to have played a significant role. However, a small step was taken in his direction when, in the Pythagorean scale, the *limma* 256 to 243 (the remainder after taking away two tones 9 to 8 from the fourth) was "wrongly called a semitone", because it is smaller than that. Our Persian-Arab even attempted to calculate a better approximation of the semitone, with 256 to 240 or 256 to 241, without daring to extract a square root, though.[29]

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[24] Ibn Sina 1935, p. 129. Cf. above, Sect. 2.4.

[25] Ibn Sina 1935, p. 142.

[26] Ibn Sina 1935, pp. 146–155.

[27] Ptolemy 1682. See Sect. 2.6.

[28] Ibn Sina 1935, p. 148. Ptolemy 1682, pp. 80–81, says: "Furthermore, as regards the division of the whole tetrachord into two ratios, in this genre [diatonic], it is obtained from those ratios which come closest to equality, and are closest together: without doubt, the sesquisixth [$1 + \frac{1}{6}$] and the sesquiseventh [$1 + \frac{1}{7}$], which divide the whole distance between the extremes into two [equal] parts. On the basis of what has been said above, this appears to be the most pleasant for the hearing." See Sect. 2.6. Cf. d'Erlanger in Ibn Sina 1935, p. 282.

[29] Ibn Sina 1935, p. 149.

In choosing from the variety of genres and melodies, on the one hand, and the innumerable possible divisions of the fourth on the other, Ibn Sina wavered between the effects on the ear and numbers. He even criticised the musicians of his age because they confused certain intervals among them, tuning their instruments without distinguishing a tone increased by $1 + \frac{1}{12}$ or by $1 + \frac{1}{13}$. "They play one for the other without perceiving the differences involved ... Their ear does not realise these differences. ... It is not a rare thing, however, to meet artists whose ear is sufficiently trained to distinguish these differences".[30]

For our doctor-cum-philosopher, music was "the art of combining notes and establishing a rhythm". In his pages, he listed combinations of two or three notes; going further, "the number of combinations is infinite."[31] He dwelt above all on rhythm, following the Aristotelian principle that art imitates nature: "... everything natural is measured; but what is measured is not always natural ... Be aware that custom does a lot to give a melody, a rhythm or a poetic metre natural qualities."[32] At the end of the section dedicated to music, Ibn Sina explained how to tune the four strings of the basic instrument, the *ud* [wood, lute], distributing the frets on the neck. They were given the names of the fingers that were used to play them. *Khinsir* [little finger] presses the first one, which is placed at one quarter of the length of the string, thus allowing it to vibrate for the remaining $\frac{3}{4}$ and attuning the fourth. *Sabbada* [index finger] lengthens the string to $\frac{8}{9}$ and generates the tone. *Binsir* [ring finger] adds another tone, while *wusta* [middle finger] presses the fret fixed at the length $\frac{3}{4} \times \frac{9}{8}$ to take away one tone from the fourth. It was also called "ancient or Persian".[33]

The other strings were tuned by fourths. Sometimes the *ud* had a fifth string and other frets. "The lute can be tuned in many other ways; the majority cannot be distinguished from the method explained above, that for the tuning of a single string". As "theory and practice do not coincide", Ibn Sina even gave instructions about how to make up for variations in the length or the width of strings. "In order to arrange the ligatures so as to reconcile theory with practice, we need a trained ear to guide us."[34]

It seems that Ibn Sina included a chapter on music in his *Danesh Nama [Book of science]*. Written in Persian, his native language, it contains a compendium of his scientific thinking. He organised it in accordance with Aristotelian philosophy: logic, metaphysics, physics or natural sciences, followed by the *quadrivium* of geometry, astronomy, arithmetic and music. But the Persian version of the final parts of mathematics, including music, were not from the pen of our doctor-cum-philosopher. His direct pupil, Gowzgani, declared at the beginning of the part on geometry, that those pages had been lost, and that he had translated them into Persian

[30] Ibn Sina 1935, pp. 149–150.

[31] Ibn Sina 1935, pp. 167 and 163.

[32] Ibn Sina 1935, p. 179.

[33] Also the Greeks gave one note the name of a finger, *lichanos* [index finger].

[34] Ibn Sina 1935, pp. 234–238. Shiloah 2002, p. 530.

from other works by his master. His pupil also possessed a pamphlet on music, but nothing on arithmetic, which may therefore have been derived from the *Šifa*, above all choosing the problems that were necessary for music.[35] Further difficulties were added to these, which forced modern editors to collate other Arabic manuscripts.

All this may perhaps justify the striking mistake at the beginning of the last chapter on music. “Let … be the note that derives from the whole string and the one that derives from half of it: this second string is half as deep as the first one. Likewise, when the tension is reduced by half, the note generated is twice as acute as the first one.”[36] As in the *Šifa*, Ibn Sina Avicenne (or Gowzgani) should, on the contrary, have written: “… is half as acute as the first one” because if the string is less taut, the note becomes less acute. Also here, however, we can see the limitations caused by the Pythagorean idea of a direct ratio also between tension and acuteness.

For the rest, there was a repetition of what was written in the longer treatise regarding ratios. It may, however, be of interest to point out that the divisions of the fourth into three intervals were justified in the following terms: “… by a free choice of the best, and not by constriction.”[37] Lastly, our Persian, who wrote mainly in Arabic, must have possessed a certain competence in mathematics. He explained that the differences between the numerical terms of a progression of ratios (that is to say, geometrical) were not constant (as in an arithmetic progression).[38] We shall see that, on the contrary, the most distinguished theoretician of music in the West, widely read for many centuries, Severinus Boethius (c. 480–526), committed the mistake of considering them always the same, when carrying out a long calculation on Pythagorean *commas*.[39]

Ibn Sina had a particularly adventurous, tragic life; compared with him, the ups and downs that affected Galileo Galilei, Johannes Kepler or René Descartes, tossed here and there in search of protection and tranquillity, are like tourist trips with minor set-backs. He himself narrated his training in the autobiography that he dictated. Like everybody, he started from the *Koran*. But his father was rather an Ishmaelitish heretic, who discussed about religion with his brother. “They used to talk about philosophy, geometry and Indian calculation, as well. Then my father decided to send me to a pulse merchant who knew this form of calculation, to learn it from him.” He studied jurisprudence. “Under the guidance of [Abu Abdallah] Nateli, I read Euclid’s *Geometry* from the beginning to the fifth or sixth figure; as regards the rest of the book, I succeeded in solving all the difficulties by myself. Then I went on to the *Almagest* [by Ptolemy].” He studied the books of al-Farabi, and learnt Aristotelian medicine and logic. He approached problems by means of syllogisms. If he did not succeed, he adopted the following procedure. “I went to the mosque, prayed, and begged the absolute Creator of the universe to reveal to

[35]Ibn Sina 1986, II pp. 91–92.

[36]Ibn Sina 1986, II p. 222.

[37]Ibn Sina 1986, II p. 229.

[38]Ibn Sina 1986, II p. 223.

[39]Tonietti 2006b, p. 154; see Sect. 6.2.

me what was barred to me ... I returned home ... I drank a goblet of wine with moderation ... when I gave way a little to sleep, I saw in a dream exactly the same question, and so it happened that the solution of many problems appeared to me during my sleep."[40] We do not know to what extent Aristotle would have appreciated this singular use of his logic, but undoubtedly his behaviour will have run the risk of appearing heretical to a believer in Allah.

He entered into the good graces of the emir at Bukara, as he was already a good doctor while he was still young, but then for the rest of his life, he continually fled from one place to another to keep himself alive. Before the powerful sultan Mahmud, he was accused of being an infidel. As we saw in the previous chapter, Abu Reyhan al-Biruni submitted, and was enrolled in the army for the conquest of India. Ibn Sina, on the contrary, escaped across the desert, facing the same dangers as his companion in his flight, who died of exhaustion and thirst. It is not difficult to imagine that an important component of his continual persecution was the religious differences between the Sunnites, the Ishmaelites, the mystic Sufi and above all the Shiites of Persia. The sultan Mahmud distinguished for his bloodthirsty persecution of all the heretics among the ranks of Islam.[41]

At Hamadan, our doctor was appointed vizier, thanks to his effective cures, but then he was dismissed, due to certain cases of incompetence with soldiers. When those in power changed, he would have preferred to escape; he was forced to go into hiding, and when he was discovered, he ended up in prison for months. In the end, disguised as a Sufi, he fled to Isfahan, where was to remain until the end of his life. Unfortunately, religious violence and desire for conquest pushed Mahmud not only to India, but also to subject all the kingdoms of Persia, cruelly purging them from heretics and purifying the minds with the fire of their books. The soldiers of his son arrived also at Isfahan, which was sacked, including the house of Ibn Sina. And so on, amid fires, hangings and ferocious throat-cuttings, of which thousands of people were victims.

Also his death took place amid dramatic circumstances, though these have remained obscure in part. During a military campaign, our encyclopaedic Persian philosopher fell ill with dysentery. But the variety of treatments, taken at excessive doses, triggered a bout of epilepsy. Some sustained that he had been poisoned with opium by his servants, who had robbed him; others, that the prince had ordered him to be killed. In any case, in spite of all his attempts to cure himself, our doctor gradually grew weaker, albeit amid limited improvements. Finally, when he arrived at Hamadan, he lost all strength, gave up all other cures and rendered his soul to God.[42]

Ibn Sina dedicated the last three chapters of the *Kitab al Išarat wa'l Tanbihat [Book of instructions and warnings]* to the Muslim mysticism of the Sufi. Whether he was influenced by it personally or only interested in it because it was widespread

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[40] Ibn Sina 1986, pp. 12–15.

[41] Ibn Sina 1986, pp. 18–21.

[42] Ibn Sina 1986, pp. 22–30.

in Persia, he was harshly criticised by the defenders of orthodoxy, like Abu Hamid Muhammad al-Ghazzali. In order to make “the vanity of their doctrines” clearly understood, he translated the *Danesh Nama* into Arabic. Among the heresies of philosophers like Ibn Sina, he listed the hypothesis of the eternity of the universe. Another accused his protector, the prince of Isfahan of being an infidel: “This is the reason why in his kingdom Ibn Sina/Avicenne had the audacity to write books tainted with heresy, contrary to the divine laws.” In the end, not only the Sunnite rulers, but also the Persian Shiites quoted him as an example of a heretic, together with al-Farabi.[43]

Even if with different tones, in both the *Šifa* and the *Danesh Nama*, our Aristotelian doctor had attacked Plato and his theory of the ideas. In the latter, he mocked the consequences through the idea of a humanity that is “single and real, which accepted science when it was Plato, but likewise remained in ignorance when it was someone else.”[44] This emerged also from the criticism of the music of the spheres mentioned above. For this, we have a closer and more direct target in the theory of music sustained by the Ikhwan al-Safa [Brothers of purity] (tenth century).

5.3 Other Theories of Music

Among the 51 *Rasa’il Ikhwan al-Safa [Epistles of the Brothers of Purity]*, the fifth, about music, leaves no doubts about the fact that they largely took their inspiration from Pythagoras and Plato. To show this, the Brothers used the term borrowed from the Greek *al-musiqi*. This was presented as a blend of spirit and body. Sound was a stroke produced in the air by the collision of two bodies. Sounds were distinguished into large or small, fast or slow, *hadd* [acute] or *ghaliz* [deep], loud or soft, *muttasil* [continuous] or *munfasil* [discrete],[45] and are composed in accordance with certain ratios. Some are pleasant, others unpleasant. Men appreciate different combinations depending on the climate and nature.

Music has laws that are parallel to those of metrics in poetry. They derive from the correspondences generated by the One, like 1, 2, 3, 4 in arithmetic, or like point, line, surface, solid in geometry, or like the sun with its positions in astronomy. The most noble and perfect instrument is the *ud* [lute], which needs to be proportioned in length in accordance with $1 + \frac{1}{2}$ of the width, in depth with $\frac{1}{2}$ of the width, and in the neck with $\frac{1}{4}$ of the length.

[43]Ibn Sina 1986, pp. 37–46. “We must . . . consider as infidels both these philosophers [Aristotle], and their followers among the Islamic philosophers, like Ibn Sina, al-Farabi . . .”; quoted in Nasr 1977, p. 252. The mystic Shihab al-Din al-Suhrawardi (1153–1191) received even worse treatment: for his criticism of Islamic jurists, he was imprisoned and put to death. Nasr 1977, p. 267.

[44]Ibn Sina 1986, pp. 42 and 159–160.

[45]In epistle 31, a whistle is a continuous sound, whereas the vibrations of a string are discrete. Bausani 1978, p. 189.

The four strings will have a different thickness: the *bamm* [low string] 64 strands of silk, the *muthallath* [third] 48, la *muthanna* [second] 36, the *zir* [high string] 27, because the best proportion between them is that of $1+\frac{1}{3}$ [4 to 3]. This is the celestial ratio which gives balance and happiness to the soul of the listener, transporting him on high.

The heavens resound when they rub against one another; they are moved by the angels that listen to the harmony of the spheres. A correspondence exists between the heavenly melodies and those of the earth. Thanks to the purity of his soul, we are told, Pythagoras was able to hear even the music of the spheres, and thus dictated the rules of music. In Islam, the chanted singing of the *Koran* is similar to this, as well as poetry that rouses to the battle on the way of God. If music is prohibited according to certain religious laws, it is because it excites passions, rather than being used in this way.

The four strings of the lute correspond to the four elements and to the four humours of the human body; thus they are capable of changing their balance, re-establishing harmony among them and curing the sick. Also the heavenly spheres would stand to each other in the ratio $1 + \frac{1}{3}$. The favourite number of the Brothers was 8. Thus, if 8 was the radius of the earth, 9 would be the measure of the sphere of the air, 12 that of the Moon, 13 Mercury, 16 Venus, 18 the Sun, 21 and $\frac{1}{2}$ Mars, 24 Jupiter, 27 and $\frac{4}{7}$ Saturn and 32 the fixed Stars. The planets that are not in a ratio with 8, Mercury, Mars and Saturn, are inauspicious. The Brothers felt that they were universal, whereas the followers of the number 7 were attracted by the particular, and were different, in turn, from the dualists, from the followers of the three, like the Christians, of the four, like the naturalists and of the six, like the Hindus.[46] Even writing was an art guided by geometry and by proportions. Letters were thought to be derived from segments and arcs of a circle, in the proportion 1, $1 + \frac{1}{2}$, $1 + \frac{1}{3}$, $1 + \frac{1}{4}$, $1 + \frac{1}{8}$. The *hukama* [philosophers] confirmed the superiority of the sense of hearing over sight.[47]

In epistle 31, among those who had followed Pythagoras, the Brothers of purity quoted Ptolemy and Euclid.[48] Voices and sounds are diffused in the air in spherical surfaces, like glass blown by a glassmaker. As the spherical surface gradually increases, the strength of the sound diminishes, like the ripples of water produced by a stone thrown into a pond.[49] For Adam, God condensed everything into nine symbols, like the nine figures. These are the *huruf* [letters, sounds of a language], which the Indians expressed as 1, 2, 3, 4, 5, 6, 7, 8, 9.[50] In time, writing arrived, with the 28 letters of Arabic, the last, definitive letters, like the *šharia* [holy law]

[46]In epistle 32, on the contrary, the number 9 is attributed to the Hindus. Bausani 1978, p. 211. Cf. Shiloah 1995, pp. 50–52.

[47]Bausani 1978, pp. 56–65.

[48]Bausani 1978, p. 185.

[49]Bausani 1978, p. 186.

[50]I would like to know whether the Arabic original had the Indian figures or letters as the symbols of the numbers.

of Muhammad. In the macrocosm, they correspond to the 28 houses of the Moon, in the microcosm to the 28 members of man. The letters are formed by the *aql* [intellect] and by the *nafs* [soul]. The former is as straight as a straight line, the latter is curved, like a circle. The first letter [a] is straight "|", the second one [b] is curved, like a circle. The first is masculine, the second feminine. In the heavens, the inhabitants are erect, on earth they are curved. Number 1 corresponds to the straight line, 2 to the curve; all the others derive from them. In calligraphy, the ratios to follow are inspired by music. According to these Brothers, the Indians continued to speak using only nine letters.[51] When the symbols of the Indian numbers had arrived in the Arabic and Persian countries, does this indicate that they would have assumed the form of letters here, as well?

In epistle 32, the Pythagoreanism of the Brothers became explicit. Pythagoras, turned into a monotheistic philosopher of Harran, was said to be the first to have treated numbers as capable of giving to all things their nature. God is One. But in order to distinguish them, things must be ordered in couples, triples, sets of four, or five. The universal emanates from God, then the three souls, the four kinds of matter, the five natures, the six directions, the seven planets, ... The Brothers also particularly loved 5, like the five Platonic solids.[52]

In connection both with the training of Ibn Sina Avicenne in general, and in particular as regards music, we have already met al-Farabi (c.872–950), another encyclopaedic philosopher considered by Arabs as second only to Aristotle. Also in his case, the overall Aristotelian approach seems to be reconciled with a clearly Pythagorean theory of music, subject in turn to the poetry and the prayers of the *Koran*.[53] With him, we find another striking case, characteristic of Arab force aiming to conquer, devour, assimilate and transform for its own purposes heterogeneous elements taken from other cultures.

Among his various treatises, what stands out is the *Kitab al-Musiqi al-Kabir [Great Book of Music]*. Here he contained his melodies in the double octave, divided in accordance with the numerical ratios of the Greek tradition, that of Ptolemy in particular, where the notes were represented by letters of the alphabet.[54] That of rhythms, too, was "a classification inspired by Geometry and Arithmetic ... As all numbers can be traced back to one primitive number which serves as a unit for them, to 1 for example, and as all surfaces limited by a regular polygon can be traced back by subdivision to another kind of surface, the triangle for example, in the same way all rhythms will be traced back by us to only one as well ...".[55]

Almost everything was analysed through the relationship between numbers: from the separation of notes into acute and deep, to counting them in order to assign them to words in a song. "When the number of notes and that of the phonemes are

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[51] Bausani 1978, pp. 189–192.

[52] Bausani 1978, pp. 211–212.

[53] al-Farabi 1935, p. 65. It is quoted from the French translation of d'Erlanger.

[54] al-Farabi 1935, pp. 4–20.

[55] al-Farabi 1935, pp. 26–27.

not equal, we establish first of all their relationship; this will be unity plus a part, unity plus many other parts, double or double plus one or many parts, a multiple or a multiple plus one or many parts."[56] However, the universal extension of the Pythagorean model for the pitch of the notes appeared to find its limit in al-Farabi. For the human voice, "acuteness and deepness" were not enough, that is to say, it was not sufficient to know "... the quantity of the bodies struck, or in general of the bodies in which and through which a sound is produced; the other [characteristics] do not depend on the quantity, but follow, in largeness or smallness, the qualities of the bodies where the sound is produced." "After all, the same thing happens in optics, where we are not satisfied with what geometry teaches us."

Our encyclopaedic *hukama* [philosopher] then described the larynx from which our voice and our singing derives. His move away from an abstract Pythagoreanism to come closer to a healthy Aristotelian empiricism led him to consider "the passage of air through the larynx; it [the production of notes in man] is the consequence of the impacts on the concave walls of the larynx ... The larynx is actually a sort of natural flute, and the flute a sort of artificial larynx." From this, al-Farabi deduced that also in the larynx, the height of the voice emitted depended on the impacts on the walls and the measurements of the cavity, such as the width, "the wider it is, the deeper the sound", and the length, "... close to the place of power that gives the air its impulse, the note generated is more acute."[57] If, on the contrary, he had used the model of the four-stringed *ud* in his analogy, the four folds of the larynx would have become the four vocal cords. They are also missing in the description of the larynx given by Ibn Sina in his above-mentioned pamphlet on phonetics: where various parts and functions are distinguished there, but where the vocal cords are not found.[58] Here, historians of the sciences should see with their own eyes that no description, empirical though it may be, can claim to be independent of the theoretical models present in the head of the observer. But as we know, also science historians, in turn, have their philosophies and their models, from which, as often as not, this book takes its distance.

al-Farabi criticised other ideas of sounds. "There are, however, many people who, based on the facts that we have set out, attribute this or that quality of notes to the head, or to the chest, or to some other organ still lower."[59] Based on what we saw in the previous chapter, we are justified in suspecting that our Arabic philosopher was picking on the Indians.

According to this character from central Asia who wrote in Arabic, the aim of melodies was to arouse emotions in the soul. Some gave "tranquillity and peace". "They are useful also when inciting people to acquire spiritual virtues, such as wisdom and sciences. The ancient melodies attributed to the Pythagoreans are supposed to have had this property." Thus, while the main aim had to remain that of reaching the supreme good, he did not deny that also a little amusement could

[56]al-Farabi 1935, p. 68.

[57]al-Farabi 1935, pp. 54–56.

[58]Bausani 1978, p. 198.

[59]al-Farabi 1935, p. 57.

play a function; like the deserved rest after the effort made to perform important actions. But leisure was to be dosed, "... Aristotle says, like salt in food." To his mind, however, the public abused of it, transforming fun into "an obstacle to what really procures good."[60] These were probably sentences with which they tried to justify music, at least partly, in the eyes of those orthodox Islamic believers who were ready to condemn it because it was capable of arousing immoral passions.

As a musician and a player of musical pieces, al-Farabi conceived of music as the science of melodies and rhythms. The intervals between notes were calculated by arithmetic. In the octave, he set the fourth 4 to 3. This was divided not only into the Pythagorean tone 9 to 8, the relative ditone 81 to 64 and the remaining limma 256 to 243, but also into a variety of other intervals taken even from Persia. He used a terminology based on the fingers used to play the strings of the Arabic lute. Starting from the *mutlaq* [free string], he then put the *mugannab* [next to the index] 256 to 243 [the Pythagorean limma], another *mugannab* [next to the index which divides the tone almost in half] 18 to 17, and then the *mugannab furs* [next to the Persian index] 162 to 149, the *mugannab Zalzal* [next to the index of Zalzal] 54 to 49, the *sabbàda* [index] 9 to 8 [the Pythagorean tone], the *wusta* [middle finger] 32 to 27 [Pythagorean tone and limma], the *wusta furs* [Persian middle finger] 81 to 68, the *wusta Zalzal* [middle finger of Zalzal] 54 to 44, the *binsir* [ring finger] 81 to 64 [Pythagorean ditone] and lastly the *khinsir* [little finger] 4 to 3 [Pythagorean fourth]. The octave contained another tetrachord and one tone, so thus al-Farabi had divided it into 24°. He had gone beyond the first four numbers, arriving as far as 6, 7, 11, 17, 149, and going beyond the cage of the Pythagoreans, though without permitting himself to arrive at the extraction of roots.[61]

Many other Arabic books dealt with the science of music; together with these, there were the translations from Greek. All of them had to justify themselves in the face of opposition from the *ulama* [doctors of the Koranic law]. One of these went so far as to write a *Damm al-malahi [Condemnation of Leisure]* which targeted above all music. The pertinacity against music above all represented also the condemnation expressed by Koranic orthodoxy for sects like that of the dervishes. For such mystics, the *trance*, the loss of consciousness induced by music and the typical dancing in a circle, was the means to gain a more intimate direct contact with the divinity, leaving out all ecclesiastical hierarchies.

The Abbassides caliphs founded the Bayt al-Hikma [House of wisdom] at Baghdad, where they succeeded in concentrating the *élite* of the Arabic cultural world, including the scientific experts. One of their main tasks was to translate books from other cultures into Arabic. We have already seen, in the previous chapter, the arrival there of books in Sanskrit. Many books were translated from Greek, either directly or through the labyrinth of other intermediate languages, spoken, written

[60] al-Farabi 1935, pp. 95 and 98.

[61] al-Farabi 1930, p. 172. Shiloah 1995, p. 112; Shiloah 2002, p. 535; but here the middle finger of Zalzal has the ratio 54 to 49.

or dreamt by the various translators. The Bayt al-Hikma reached its highest point under the caliph al-Mamun, who reigned from 813 to 833.[62]

The *ilm al-musiqi* [science of music] took the name *musiqi* and the general theory from the writings of the Greeks; *ghina* was the Arabic word for the practice of the musical art. As in Greek culture, these learned Arabs debated whether it was art that imitated nature, in agreement with the writings of Aristotle, or whether, on the contrary, art, as a fruit of the intellect, was superior, in line with the opinion of Plato and the NeoPlatonics. In this case, nature would be imitating the abstract perfection of the world of the ideas. At Baghdad, at the court of the Abbassids, there worked al-Kindi (???–post870) who followed in the wake of the Pythagorean schools with his *Risala* [Epistles] dedicated to music. Another variant of the predominant Pythagorean Greek tradition, destined to count also in the Arab tradition, was subsequently elaborated by Safi al-Din (1216–1294). Now the Pythagorean tone 9 to 8 was divided into two *limmas* and the remaining *comma*.[63] His octave thus contained 18 unequal degrees, indicated by him both by the letters of the Arabic alphabet *abgad*, and by those of Hebrew, from *alef* to *yud heit*. Here again, we find a symbolism that united numbers with the letters of the alphabet and the notes of music. By means of their combinations, this musician and Arabic theoretician obtained 84 octave scales.[64]

The characters considered so far were above all doctors, philosophers, or musicians. Among the Arabic theoreticians of music, do we also find scholars mainly of mathematical sciences and astronomy, as we do in Greek culture? Tabit ibn Qurra (826–901) worked as a mathematician and astronomer at court. He translated from Greek and Syriac into Arabic; he also wrote about music in Arabic and in Syriac. In Arabic, we have his *Masala fi'l-musiqi [A Musical Problem]* which is reminiscent of al-Farabi and Avicenne in its approach.[65]

We have seen that in Arabic culture, not only among neo-Pythagoreans and declared Platonisers, but even among followers of Aristotle, some variants of Greek orthodoxy were prevalent which hark back to Pythagoras. The one whose absence is noticeable is Aristoxenus, because we only have memory of a Persian-Arabic scale which divided the octave on the basis of $\sqrt[17]{2}$, formed of the following notes: *alif, be, gim, dal, he, wan, zäin, alif*. We do not know where it came from, or how widespread it was, but it cannot have been well known if the major theoreticians completely ignored it. This was hypothesised and studied by Raphael Kiesewetter (1773–1850).[66]

[62]Farmer 1975, passim. Benoit & Micheau 1989, pp. 156–161. Shiloah 2002, pp. 525–529. Ihsanoglu 2002, pp. 113–115.

[63]If we take away the *limma* 256:243 from the tone 9:8, we obtain the *apotome* 2187:2048; taking away the *limma* from this, we obtain the *comma* 531441:524288.

[64]Shiloah 2002, pp. 536–537.

[65]Shiloah 1995, p. 49; Shiloah 2002, p. 534.

[66]These are the first letters of the Arabic alphabet. Kiesewetter 1842; Righini 1963; Bohlman 1986.

Among the various things that were to pass from the Arabs into the Europe of the scientific revolution, there was even a little music. While he was working on the *Harmonie Universelle* in 1610, Marin Mersenne tried to examine what the Arabs had left on the subject. He arrived at the *Kitab al-Inam fi marifat al-angam [Book of the Benefit of the Knowledge of Melodies]* by al-Dahabi al-Saydawi (fourteenth century).[67]

5.4 Beyond the Greek Tradition

The Abbassides caliphs implemented their scientific policy by means of the Bayt al-Hikma. Consequently, at the beginning of the ninth century, al-Haggag ibn Matar (active 786–833) translated the *Elements* by Euclid under the title *al-Ustuqusat* [sounds of *Stoicheia, Letters, Elements*] and the *Magiste Syntaxis Mathematica [Great Mathematical Composition]* by Claudius Ptolemy as *al-Magisti*.[68] These two books were leave a general impression on the mathematical sciences of the Arabs. What the Pythagorean style was for music, the style of Euclid was for the reasoning in books about mathematics. It was against this backdrop that Arabic scholars invented their main contributions, at times working in an independent manner. Tabit ibn Qurra even demonstrated the theorem of Pythagoras in a new way, different from Euclid's classic method. He extended the proof contained in the *Meno*, already seen above, to the case of a right-angled triangle with unequal sides.

In the figure on the left, the proof is obtained by moving the two white triangles with the angles at E (3 and 4) to the two white triangles with the angles at C (2 and 1). In this way, the square constructed on the hypotenuse EA is broken down and recomposed into the squares constructed on the catheti AA'BB' and DD'FB'. In the figure on the right, the large irregular pentagon is broken down in two different ways, by taking away equal areas. Therefore, if we subtract the three darkened triangles from the whole figure, what remains is the white squares constructed on the catheti. If we take away from the whole figure the three triangles with an equal area at the top, on the right and on the left, what remains is the square constructed on the hypotenuse. They are thus equal, because they are obtained by taking away equal areas.[69]

The figures are similar to the Figure of the String (Fig. 3.4), found in the Chinese *Zhoubi [The Gnomon of the Zhou]* discussed in the third chapter. About 1150, also Bhaskara II provided a similar proof, based on another figure.[70] There is no explicit, documentary evidence of any connection between these three proofs. The most ancient one, by several centuries, seems to be the Chinese figure. The probability

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[67] Shiloah 1995, p. 56; Shiloah 2002, p. 538. See Part II, Sect. 9.1.

[68] Rashed 2002, pp. 459, 492 and 481.

[69] Sayili 1960, pp. 35–36.

[70] Cantor 1922, pp. 656 and 680.

that Ibn Qurra had seen it, however, is extremely remote. A passage between Ibn Qurra and Bhaskara II might be more possible, because material channels existed for this contact, and we know of other well-documented cases, just as we know of exchanges between Indians and Arabs. Here, however, failing reliable elements to the contrary, we prefer to continue to believe that each of them had elaborated his own argument independently of the others.

What is certain, on the contrary, is the references by Ibn Qurra both to Plato's text and to Euclid. A friend of his had explicitly asked him to extend the "Socratic proof" to the general case. Having found it, he included it in a comment on the al-Haggag's translation of the *Elements*.[71]

Even if he came from central Asia, also Abu Jafar Muhammad ibn Musa al-Khwarizmi [Muhammad, father of Jafar, son of Musa, who came from Khorezm] (c. 780–850), became an authoritative member of the Bayt al-Hikma at Baghdad. Here, during the reign of the caliph al-Mamun, he diffused the *Kitab al-Gabr wa-l-muqabala [Book of Reduction and Restoration]*. But, first among the Arabs, and later in Europe and the whole world, this book was to become *Algebra*, and the scholars of the new mathematical discipline were called algebraists. This was the origin of a new technical term, because originally, *al-gabr* meant, among other things, also "to reduce" bone fractures, "to rectify" by force. Similarly, *al-muqabala* meant "to restore". And yet our man from Khorezm, who wrote in Arabic, dealt with second degree equations in his book. He divided them into six standard types, defined a priori. However, he did not expect every practical problem that he examined to assume precisely one of those forms. Thus he elaborated rules, operations or procedures, to lead the question back to them. By *al-gabr*, he meant the addition of the same term to both the sides of the equation; by *al-muqabala*, he intended transporting a term from one side of the equation to the other one.

In an attempt solely to make myself understood immediately by all modern-day students (who have not forgotten the subject), we may, albeit anachronistically, misusing post-Cartesian symbolic notations, describe the operations as follows:

$$x^2 + c - bx = d; c > d$$

al-gabr consisted of adding bx to both sides, obtaining

$$x^2 + c = bx + d$$

whereas the other operation consisted of the transformation

$$x^2 + (c - d) = bx$$

The majority of the ideas proposed by al-Khwarizmi had more or less illustrious precedents. Some were already in Euclid's *Elements*, others in the *Arithmetic* by

[71] Sayili 1960, p. 35.

Diophantus; particular calculation procedures for single cases go back as far as the Babylonians of the preceding millennia. However, none of them appear to be so capable of creating a general system suitable for every type of second-degree equation. Our Arabic algebraist gave automatic rules, easy to learn and simple to apply, which did not require a substantial baggage of other mathematical knowledge. They had been devised thinking of those people who needed to solve the practical problems of their culture. al-Khwarizmi's book was highly successful with them. But what were the requirements that our algebraist succeeded in satisfying with such surprising effectiveness? The historical context of the epoch explains this to us.

Starting from the seventh century, the Arabic culture was created around the Sacred book. The result was a culture of the word, particularly attentive to language, which prepared dictionaries, combined phonemes, and dictated rules for prosody. In the *Koran*, laws are specified for wills and hereditary succession, which establish complex procedures to be calculated with precision. Law schools flourished in this period, together with legal studies which produced books like the *Hisab al-wasaya [Computation of Wills].* Almost half of the book composed by al-Khwarizmi was dedicated to such computation. He openly declared in the introduction that he was thinking of how to survey land, dig canals, perform geometrical calculations; he had in mind commercial exchanges, the problems of inheritance, and "... all the rest of human business".[72]

Nowadays, algebra and mathematics are full of particular symbols with a technical meaning, but those who assigned these names were simply using words taken from the natural everyday language of his period and his city. And yet the Arabic language lent itself to abstract, formal interpretations, as well. In fact, an anonymous jurist wrote, for the purpose of producing innumerable combinations of legal sentences: "We designate the [juridical] examples by means of letters without any meaning, so that these may be productive in themselves, and not as a result of some real matter. Furthermore, they are short, and analogous to the things and the squares that the reckoner uses to determine unknowns."[73]

The algebra of our man from Khorezm was initially presented as an arithmetic of unknowns: *al-say* [the thing], the *res* of Latin,[74] the "thing" of the medieval abacus schools, destined to become the Latin letter x in the Europe of the scientific revolution of the seventeenth century. The term represented an infinity of things and cases. It was detached from real, specific things, and referred to whole classes of objects, freeing itself from their contingent particularities, and abstracting itself from them. But was this not also the deepest meaning of the words of the *Koran*? What meaning could be attributed to those words, believed to be the message of Allah, transported by the archangel Gabriel to the prophet Muhammad, and written

[72]Boyer 1990, p. 268. Joseph 2003, pp. 314–315.

[73]Rashed 2002, pp. 457–460.

[74]Rashed 2002, p. 492.

down by him in the sacred book, other than a sense that completely transcended all earthly realities?

al-Khwarizmi also included in his book the proofs of the algebraic formulas used. But for this purpose he made use of Euclid's geometry, recently translated into Arabic. He obtained the *illa* [cause] of his solutions by means of the areas of specially constructed figures. Thus, the solution of the equation $x^2 + bx = c$ will be obtained by identifying $AB = AC = x$, $BE = CD = \frac{b}{2}$. The sum of the areas *ABCM*, *BEMN*, *CDPM* will be equal to c, and the big square *ADOE* will become equal to $c + (\frac{b}{2})^2$. Therefore, extracting the square root, we obtain

$$x + \frac{b}{2} = \sqrt{c + (\frac{b}{2})^2}$$

from which we obtain the solution by extraction of the square root:

$$x = \sqrt{c + (\frac{b}{2})^2} - \frac{b}{2}$$

Two generations later, Ibn Qurra showed the general correspondence between the theorems of Euclid and the formulas of al-Khwarizmi, because algebra benefits from a geometrical representation, and geometry may have an algebraic reformulation. His purpose, however, was to distinguish the two procedures and give a more solid basis to the novelties introduced by the algebraists.[75] Apart from their success, the equations of al-Khwarizmi appear to have needed their justifications, and he found these again in the geometry of the Greeks. Even the Persian Umar al-Khayyam (1048–c.1131), made use of geometry in elaborating third-degree equations, though this time it was mainly Archimedes'. In his *Maqala fi l-gabr wa l-muqabala [Treatise on Algebra]* this scholar, who is more famous in the Western world as a poet than as a mathematician, used the intersection of the conical sections of spheres and cylinders.[76]

These mathematics scholars wrote in Arabic, and read the Arabic translations of the books in Greek left by Euclid, Archimedes, Diophantus, Apollonius; and yet they spoke Persian, Syriac, and other languages. They were born in various regions of Asia, but they often ended up in the centres of Islamic power. They moved ambivalently among the various cultures that the governing Arabs had imposed to keep in touch: on the one hand, they knew how to calculate in a new way the solution to their practical problems, like those deriving from hereditary divisions, on the other, they continued to make reference to the Greek tradition. But in their confrontation with algebra, two of their characteristic elements were modified: the quantities that the Greeks had considered *alogos*, without discourse, which Euclid

[75] Rashed 2002, pp. 459–460; Joseph 2003, pp. 319–323.

[76] Youschkevitch & Rosenfeld 1970, pp. 328–329. Rashed & Vahabzadeh 1999. Rashed 2002, pp. 464–467. Joseph 2003, pp. 323–327.

had treated by means of the geometry of the incommensurables in Book X of the *Elements*, now started to assume the same legitimate status as other numbers.

The tendency to reinterpret the *Elements* or *Arithmetic* by Diophantus in the light of algebra, where this latter book was translated with the title *Sinaat al-gabr [The Art of Algebra]*, culminated in the eleventh century with al-Karagi (???–c.1030) and his successors. He explicitly wondered: "... how can multiplication, division, addition, subtraction and the extraction of square roots be used for irrational quantities?" Thus, behind the word *gidr* [root] or "thing" of an equation, incommensurables, for which the Greeks had forbidden numerical representation, changed their meaning and could be used and calculated; the same rules for the other numbers were extended to them.[77]

Even though it is unfortunately got lost, we know that also al-Karagi dedicated a book to Indian calculation.[78] Thus it maintained its position in Arabic culture, offering an element of constant comparison for mathematical procedures. This can be seen from the continuous consideration received in books written by the most famous scholars: we may remember, for example, Ibn Sina or al-Khwarizmi. But, when they conquered Northern India, how much of what the Arabs acquired was integrated with the rest of their scientific culture, and in which way? It was more simple that this happened with algebra than with the Greek tradition. It is legitimate now to suspect that the novelties introduced by the Arabs had nourished themselves also with Indian calculation; that must have favoured above all the admission of *asamm* [surd, irrational] numbers, like $\sqrt{2}$, among all the others, because the distinction with *muntaq* [espressible, rational] ones did not exist in India.

Besides, the scholar who took a big step towards a non-Pythagorean idea of irrationals was also the one who was closest, linguistically and geographically, to India: the Persian Umar al-Khayyam. In his *Treatise on Algebra*, he quoted two Arabic books of Indian calculation, and stated that he had written a book (not available today) to demonstrate the validity of Indian methods to extract roots. Our renowned poet introduced a length taken as a unit in order to perform calculations with geometrical magnitudes. Thus he represented them as numbers, approximated if necessary. He was not satisfied with the way Euclid had dealt with ratios, because the latter considered only rational numbers. On the contrary, al-Khayyam considered all (positive) irrationals, including $\sqrt{2}$, or the ratio between the circumference and the diameter of the circle (π). In his *Sharh ma ashala min musadarat Kitab Uqlidis [Comment on the Difficulties of Certain Postulates in the Work of Euclid]*, he therefore greatly expanded the current notion of number. "We do not consider the magnitude g as a line, a surface, a body or time; but we consider it as a magnitude abstracted by the reason from all this, and belonging to the world of numbers. However, not among absolute, true numbers, because the ratio of a to b

[77] Rashed 2002, pp. 461–463. Qusta ibn Luqa (ninth to tenth centuries; Greek and Christian, with a knowledge of Greek, Syriac and Arabic) translated the term *pleura* [side], in the *Arithmetic* by Diophantus, using the Arabic word *gidr* [root, of the square]; Rashed 2002, p. 43.

[78] Crozet 2002, p. 498.

may frequently not be numerical, that is, often it may be impossible to find two numbers whose ratio is equal to this one."[79] In the movement towards irrationals here, they sought approximate numerical solutions.

Saraf al-Din al-Tusi (11??-c. 1213) moved in the same direction with his book *Al-Muadalat [Equations]*, where numerical approximations of the intersections between conics (to obtain the roots of third-degree equations) were calculated by means of *al-adad al-azam* [the largest number, a maximum]. For this reason, then, he should be included in the history of calculation by maximums and minimums, and consequently also infinitesimal calculus.[80]

These scholars had therefore started to fill in the too many gaps left among numbers by the Greeks and the Pythagoreans. In contemporary histories, may it be possible that those who have led us to remember only the jurist and mathematician Pierre de Fermat (1601–1665) and René Descartes (1596–1650) did so because they were too Eurocentric and too badly informed? Or was it because Umar al-Khayyam seemed to take too much interest in wine and poetry, rather than in mathematics?

Totally oblivious, or else variously glancing at the translated books of the Arabs, the protagonists of the European revolution were to develop, in any case, starting from the seventeenth century, the algebraic symbolism to which centuries of schooling have inured us. The characters of the Middle East mentioned here did not seem to have any inclination for this. May it not be true that to these people, who spoke Persian, Syriac, Aramaic and other languages, Arabic should already have seemed to be a sufficiently abstract, symbolic language? It was already undoubtedly transcendent to a great extent, as regards their own life.

In any case, these Arabs did not often use the numbers which we call Arabic today, to distinguish them from Roman numerals, but their letters in the above-mentioned alphabetic order *abgad*. Using these, they had even developed a particular kind of calculation called *hawai* [airy], considered alternative to Indian calculation, where the rules depended on the characteristics of the Arabic language, although the foundation and justification, as always, came from Euclid's *Elements*.[81]

In his *Kitab al-Manazil al-sab [Book of Seven Degrees]*, Abu 'l-Wafa (940–998) even included calculations with fractions, for which he offered the first known complete presentation. But he still made reference to the Euclid's notion of ratios, and chose the fractions among those that were *muntaq* [expressible] in the Arabic language; the fraction $\frac{2}{3}$ had a particular role, while the others were either combinations of these, or *asamm* [surds]. The line, as is used today to separate the numerator from the denominator, is thought to have been introduced

[79]Struik 1958, pp. 280–285. Youschkevitch & Rosenfeld 1970, pp. 325–327; these Russian scholars wrote that the methods of extraction were not exactly the Indian ones, and came closer rather to the Chinese methods of the *Art of calculating in nine chapters*. Rashed & Vahabzadeh 1999, pp. 271–390. Vahabzadeh 2002. Ben Miled 2002. Rashed 2002, p. 467. Joseph 2003, p. 305. Barontini & Tonietti 2010. Wymeersch 2008.

[80]Rashed 2002, pp. 467–471.

[81]Crozet 2002, pp. 499, e 503–504.

by the Western Arabs. This mathematician, who came from Persia, also wrote the *Kitab fi ma yahtagu ilay-hi al-kuttab wa-l-ummal min ilm al-hisab [On the Need of Arithmetic Science for Administrative Functionaries and Tax Collectors]*. The Abbasside empire effectively needed instruments for its administration based on functionaries distributed through the provinces, capable not only of communicating in Arabic, but also keeping precise accounts. However, in this respect, the Greek-Alexandrian tradition, even in the version in Arabic letters, was to prove limited and unwieldy.

Subsequently, a copier of Euclid's texts, actually known as al-Uqlidisi (tenth century) but whose full name was Abu 'l-Hasan Ahmad ibn Ibrahim (Ahmad father of Hasan son of Ibrahim), probably out of impatience with his job, wrote, at Damascus in the year 341 after the hegira, (i.e. between 952 and 953 A.D.), *al-Fusul fi 'l-hisab al-hindi [Chapters on Indian Calculation]*. It was an amusing paradox, worth recollecting, because the 'Euclidean' Ahmad opened up a new way: which was to move mathematical sciences increasingly further away, in time, from the then dominant Greek tradition. "Most arithmetic experts benefit from making use of the practice of it [Indian calculation]: thanks to its ease of use and speed, the few things that need to be remembered, the short time involved in the answers, the limited reflection on the subject that it deals with [...] this is a science and a practice that require an instrument that is useful for the scribe, the manufacturer and the knight in their professions [...] understanding this is not difficult or impossible, and there is no need of any preparation." Our Syrian al-Uqlidisi must have had direct personal experience of how difficult it could be in certain cases to calculate respecting those Euclidean procedures which were rigorously founded on the *Elements*.

As early calculations were made on tablets covered with sand or dust *gubar*, this remained the Arabic name of Indian figures. Among the various chapters, there were presentations of numbers written depending on their position with nine figures and zero, fractions, and the extraction of roots. The *sunya* [void] of the Indians, represented by '.', was called *sifr* [empty space] and was indicated by the Greek letter omicron with a bar over it $\bar{o}$. Subsequently, via the Latin *cipherum* and *zephirum*, the modern words cipher and zero were coined.[82]

With the use of the new language, the *asamm* [surds, irrationals] could be represented like the other numbers, and the calculation of roots assumed particular significance. Above all, decimal fractions appeared as the most convenient way of representing the result of divisions.[83] For fractions, therefore, another element of mathematics that has become common for us, the impulse of the Arabs was particularly important. And thus they were gradually to substitute ratios, which dominated the world of the Greeks. Also in order to approximate the roots of equations, the Arabs started to write numbers as *al-kusur al-asariyya* [decimal fractions]. In these, the whole part was separated from the decimal part by means of a dash over the number. One century later, the school of the Persian al-Karagi

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[82]Crozet 2002, p. 501. Hogendijk 1994, p. 73. Joseph 2003, p. 257.

[83]Crozet 2002, pp. 500–503.

(eleventh century) was to provide a systematic description of decimal fractions. The Indians also re-appeared in calculations in astronomy, which were more sophisticated and based on the method of interpolations: above all Brahmagupta, as used by al-Biruni.[84]

5.5 Did the Arabs Use Their Fractions and Roots for the Theory of Music, or Not?

We have seen Greek-Alexandrian mathematical sciences come into contact with those of India. Both of them were modelled by the demands that were characteristic of the new historical context expressed in the Dwelling-place of Islam: the Arabic language, the *Koran*, the calculation of wills,[85] jurisprudence, the administration of an empire that had become vast. Were these two heterogeneous traditions easily integrated together? Or did they clash, remaining separate in different fields of application? The most important and representative encyclopaedic philosophers help us to answer these questions.

Ibn Sina Avicenne also dealt with arithmetic in his books. Not only he himself, but also his subsequent biographers testified to his knowledge of Indian calculation and algebra. And yet, in his *Risala fi aqsam al-ulum al-aqliyya [Epistle on the Divisions of Rational Sciences]*, they appear only as secondary parts of arithmetic. "The secondary parts of mathematical sciences, branches of the [science] of numbers: the science of addition and of separation of Indian calculation; the science of algebra and of *al-muqabala* …". At the end of the list there were also "… the branches of music: the use of wonderful and curious instruments like the organ and similar things."[86]

Thus Ibn Sina classified the sciences according to a hierarchy within which even algebra and Indian calculation could find a place, but which had to leave the classical Greek disciplines in the foreground. How, then could it have contaminated that theory of music, a part of the *quadrivium*, ennobled by the attentions of Euclid himself, together with procedures like fractions and roots, derived from algebra and from India? The pattern of the *Kitab al-Šifa* was actually that of the Greek *quadrivium*, reserving a book each to arithmetic, geometry, astronomy and music. Also in the *Danesh Nama*, mathematics was divided in the same way. Here,

[84] Rashed 2002, pp. 476–483. Joseph 2003, pp. 312–314.

[85] "In Islam, as in no other religion in the history of mankind, scientific procedures have been applied to assist the organization of various aspects of religious ritual. These are: 1) a calendar whose periods are based on the Moon. 2) five daily prayers whose times are based on the Sun, and 3) a sacred direction [*qibla*] whose goal is a specific location [Mecca]. … 4) the distribution of inheritances, and 5) the geometry of Islamic decorative art." "The Prophet Muhammad is reported to have said that the laws of inheritance comprise one-half of all useful knowledge." King 1994, pp. 80 and 83. But compare with the geometry for Vedic sacrifices in Chap. 3.

[86] Rashed 2002, p. 490.

geometry was taken from the *Elements*: the fundamental property of right-angles triangles was demonstrated in accordance with proposition 47 of Book I.[87] The new proof of Ibn Qurra was ignored, as were his generalisations, and other cases. Probably these did not seem to be suitable for a compendium, even if it seems likely that Ibn Sina, or rather his direct pupil Gowzgani, knew the author.[88]

The choices of our Persian doctor cannot, therefore, be attributed to ignorance. They rather reveal the dominant role played by Euclid in that historical and cultural context, in particular as regards music. Do they perhaps also reveal a separation between the sciences considered to be higher, and others more common, because not equally metaphysical and heavenly?

For the neo-Pythagorean tradition, the new word *al-aritmatiqi* was introduced; this was the phonetic transposition from the Greek,[89] as in the case of *al-musiqi*. Also in the arithmetic part of the *Šifa*, as we have already seen in the musical part,[90] Ibn Sina attacked the Pythagoreans. He criticised them because they made reference "... to developments extraneous to this art [of numbers] and even more foreign to the use of those who proceed by means of demonstrations; these developments are more similar to the purposes of rhetoricians and poets." And yet the Persian doctor however did not come out of natural numbers with his arithmetic (just as he did not completely renounce Pythagoras). For the rest, he again referred to the *al-hisab* [calculation] "... what remains, in practice follows the example of algebra and of *al-muqabala* and of the Indian science of addition and separation. But it is better to mention the latter among the secondary parts." In any case, however, irrational numbers remained outside his considerations.[91] Also negative numbers were excluded systematically by all scholars, and this gave rise to the need to distinguish different cases of equations, where today we would see only one. The principal Greek tradition continued to dominate, even if it was cleaned up a bit from the music of the spheres.

In dealing with the complicated question of how the divine One could give rise to the multiple aspects of the real everyday world, Ibn Sina introduced the letters of the alphabet a, b, c, ... as formal symbols to indicate the various elements in the various passages of the process. Then he proceeded by means of successive combinations, which, according to the late comment of Nasir al-Din al-Tusi (1201–1274) were calculated two by two, three by three, ... and generated increasingly large numbers such as 4,095, 65,520, ...[92] We have already discovered that our Persian also conceived of music as the art of combining sounds and rhythms. Indicating the notes by letters, he combined them two by two, AA B J, A BB J, A B JJ, ... three by three,

[87] Ibn Sina 1986, II, p. 105.

[88] Rashed 2002, pp. 489, e 491.

[89] Rashed 2002, p. 491.

[90] See above, Sect. 5.2.

[91] Rashed 2002, pp. 491–492.

[92] Rashed 2002, pp. 493–496.

AAA BB JJ, AA BBB JJ, AA BB JJJ, ...[93] Here, it is difficult not to think of the *Dissertatio de arte combinatoria* of Leibniz.[94]

If we take the ideas of Ibn Sina as particularly characteristic, would other characters in our history confirm them? What scholar in disagreement with him do we have memory of? The philosophy of al-Kindi (ninth century), where he mixed Aristotle, neo-Platonism and the *quadrivium*, was based on both the Koranic Revelation, and the equally eternal *Elements* of Euclid. Through it, he tried, among other things, to refute the infinity of the world and of time. For these ideas, Ibn Sina had to suffer the charge of heresy and various persecutions. The same direction as al-Kindi, was also followed by Maimonides (1135–1204), in his *Dalalat al-khairin [Guide of the Perplexed]*, using arguments again taken from Greek geometry.[95] Few doubts can exist that the *Elements* by Euclid was "... at that time the main model of mathematical explanation",[96] and not only for the mathematical sciences, perhaps.

al-Farabi, instead, had been much more interested in irrational numbers, as can be seen from the event that in his *Ikhsa al-ulum [Enumeration of Sciences]*, algebra and *al-muqabala* did not exactly represent a secondary part of arithmetic. Our theoretician of music placed it, rather, under the heading "science of ingenious procedures", and connected it with physical bodies, because it was precisely the means by which the notions of geometry and numbers could succeed in being applied to matter and to the world of the senses. For him, these "ingenious procedures" made it possible to find both rational and irrational numbers "... of which Euclid has provided the principles in Book X of his work *al-Ustuqusat [Elements]* ..." "... every number is homologous to a certain rational or irrational magnitude. Thus, if the numbers homologous to ratios between magnitudes are determined, these magnitudes are determined specifically." For this philosopher from central Asia, the unknown *al-say* [the thing] of algebra indicated both numbers and geometrical magnitudes, and thus even numbers that approximate to the irrationals. al-Biruni was to write: "The circumference of the circle has a given relationship with its diameter. The number of the one with respect to the number of the other is also a ratio, even if it is irrational."[97]

But then, in his theory of Music, did al-Farabi admit the division of the tone into two equal parts, seeing that the geometric mean between 8 and 9 was an *asamm* [surd, irrational]? Or did even he remain inside that orthodox Greek tradition which denied it? Let us examine his formulation more closely. In the *Kitab al-Musiqi*, he continued to use only whole number ratios, excluding the other alternative. First he divided the octave into the main consonant intervals (fifth and fourth) and the *awdah* [return, tone]. This latter contained the *baqiyyah* [residue], that is, the Greek *limma*, the excess of the fourth compared with the ditone.

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[93]Ibn Sina 1935, pp. 163–165; see above, Sect. 5.2.

[94]Tonietti 1999a, pp. 196–198; see Part II, Sect. 10.1.

[95]Shiloah 1995, pp. 49–50. Rashed 2002, pp. 485–489.

[96]Rashed 2002, p. 496.

[97]Rashed 2002, p. 492. We need to remember his stay in India.

Then, "Trusting our ear . . .. with the help of our ear", he developed an argument, concluding that two 'residues' were ". . . equivalent to one tone; each of them will therefore have the value of a semitone, Q.E.D." But he also went on, "In this way, some have been induced to believe that the interval of the 'residue' is really equivalent to the semitone, even if things stand otherwise; let us admit it for the moment. Then the tone would include an exact number of 'residue' intervals, and as a result the 'residue' interval would be the measurement common to all the other intervals. The tone would include it twice; the fourth would include two and a half tones and the fifth three and a half tones. If we took the 'residue' interval as the unit of measurement, the octave would contain twelve, the fifth seven, the fourth five, and the tone two."[98]

This passage has a certain flavour that is reminiscent of Aristoxenus.[99] However, the conditional used with the hypothetical construction should put us on our guard. And actually, later on, al-Farabi wrote that the 'residue' was "smaller than the semitone". Placed one after the other and added to the fourth, the fifth would not be recognized. The ear does not perceive the difference with respect to exactly half of the tone when there are few 'residues', but if they are accumulated to reach the octave, then also the differences are accumulated, and the result will be different. Zeno had said that a mass of grains thrown to the ground made a noise, whereas a single grain by itself did not make any perceptible sound. Our musician and theoretician who had moved to Baghdad preferred to compare the phenomenon to a ship propelled by many oarsmen, though only one of them was unable to move it significantly.[100]

"As we cannot limit ourselves in this subject only to the principles that derive from the senses, we shall have recourse to theoretical principles; these will be axioms, or propositions demonstrated by other sciences." Having introduced ratios between numbers, in the end he wrote, "We have shown above that it is necessary to imagine notes and intervals to determine their value and to represent them with the help of whole numbers; and also from what point of view they are to be imagined in order to represent them with the help of fractional numbers. Clearly, these are two very different methods: one is that of the Aristoxenians, the other that of the Pythagoreans. It is easy to conclude which of these methods is to be followed in music."[101]

And yet al-Farabi was so familiar with musical instruments that he could not help feeling, even in his orthodox Pythagorean theory, a certain tension towards another way of dividing the octave. In every case, the way that the Pythagoreans

98 al-Farabi 1930, pp. 54–55.

99 Aristoxenus 1954, pp. 79–80. See above, Sect. 2.5

100 al-Farabi 1930, pp. 61–62.

101 al-Farabi 1930, pp. 64 and 76. In d'Erlanger's translation from French into English, we have changed the position of the "Pythagoreans" with that of the "Aristoxenians", because al-Farabi attributed "whole numbers" to Aristoxenus and "fractional numbers" to the Pythagoreans, as we see immediately afterwards.

explained the effects of music on the soul seemed wrong to him. "The idea of the Pythagoreans, that in their course, the planets and the stars give birth to sounds which combine in harmony, is wrong. It is demonstrated in physics that their hypothesis is impossible, and that the movements of planets and stars cannot generate any sound". Furthermore, "The fundamental principles of musical theory . . . can only be known with the help of sensation and experience . . . musical practice comes well before theory."[102]

Our musician and theoretician contrasted "Skilful musicians who have practised this art for a long time" with "masters who are incapable of bringing tangible proof of what they sustain in their books, since they have not trained their ears by listening to music." He even praised other Arabs, unfortunately without giving their names, as "skilful operators, addicted to music, . . . who only trust their ears . . . Thus they are closer to the truth than those theoreticians of our age who have preferred to follow the way of the Greek mathematicians of antiquity."[103]

The first numbers that al-Farabi used undoubtedly served to measure the intervals of Arabic musical genres, but the unit of measurement chosen at the beginning of his introduction was not that of the Pythagoreans. Rather, it is exactly the quarter of a tone, considered by Aristoxenus to be the minimum interval. Taking the value of the octave as 144, as a result the semitone was equal to 12 for him, and the quarter tone was 6; the other intervals were multiples or fractions of these, 8, 9, 18, 20, 24, 30, 36, 42, 44, 48, 60, 84. Here, then, he divided the octave into 12 equal semitones, in order to describe the genres to be played on the *ud*, which included notes that were foreign to the orthodox Greek Pythagorean tradition.[104] These are: the "Zulzul [or Zalzal] middle finger", played with the middle finger about a "quarter of a tone" above that of the ring finger; "close to the middle finger . . . placed at about a quarter of the distance from the index finger fret to that of the ring finger"; the "Persian middle finger . . . placed half-way between the frets of the index finger and the ring finger."[105] He used the exact semitone and the quarter tone as units, but in these pages, he did not measure them in turn in terms of numerical proportions. In this work, irrational numbers remain hidden. The ear that guided him in the distribution of frets on the neck of the lute clashed with the Pythagorean tradition. Was it imposing its prohibitions on him, as well?

al-Farabi described as "strong" and "masculine" the genres based on the division of the fourth into tone intervals = 24 or slightly less. By means of larger intervals, up to 48, and much smaller ones 6, 8, 9, he generated "gentle", "coloured" and "feminine" genres. In general, his classification was reminiscent of the Greek one, in its diatonic, chromatic and enharmonic genres. "The excessively gentle genres

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[102]al-Farabi 1930, pp. 14, 28 and 33.

[103]al-Farabi 1930, pp. 51–52.

[104]al-Farabi 1930, p. 59.

[105]al-Farabi 1930, pp. 56–57.

[*nadhim* that it constructs, *rasim* that it outlines] produce a weak, superficial impression similar to that of a drawing that is only sketched".[106]

It is clear that our musician and theoretician was pulled in different directions, that of Aristoxenus and that of the Pythagoreans. As a musician, he should have preferred the former, and he often actually referred to the ear and to practice. The Pythagoreans "reject the 'residue' [*limma*] interval, whereas the practical operators admit it. The 'residue' interval is effectively continually present in almost all the melodies [of Arabic music]. However, we cannot say that the practical operators accept the 'residue' interval because of an error of their senses . . . it is rare to meet a musician who makes a mistake about the sonority of intervals and modes, or, if a mistake occurs, it is made by the theoreticians who are not sure of practical aspects."[107] As a theoretician, on the other hand, he accepted the essential principles of the Pythagoreans. "Here we consider the notes of different degrees as produced by the lengths of commensurable strings compared together."[108] And at this point he turned to geometry and arithmetic. It should be noted that he chose strings of a commensurable length. "Our ear by itself does not allow us to define all the states of a note; for its part, theory does not supply us with the means to recognize whether a note is natural or not." According to him, then, it was necessary to make use of both of them: ". . . musical theory and practice complement each other, and together form musical science."[109]

In his *Kitab al-Musiqi* . . ., did al-Farabi succeed in his intent? In his synthesis, that geometry and that arithmetic (if these were of Greek origin) were a help or a hindrance? He even gave a methodological and philosophical form to the dilemma. The senses were to provide the initial elements to start from, in order then to rise again at the theoretical principles; another, different way would be to start from the principles and work down to the practical consequences.[110] Unfortunately, we have no knowledge of how this musician born in Persia played the *ud*. On the contrary, he has left us many pages about which principles of physics and mathematics he used, and how he tuned his instruments. The former were largely Pythagorean in their derivation, with some interesting variations. We find the most striking novelties in his distribution of the frets on the neck of the Arabian lute and in his description of flutes. The fundamental relationships of the octave (the Greek *diapason*, translated as the "interval of everything"), the fourth, the fifth and the tone (called the interval of 'return') were sought by starting explicitly from the lengths of strings, and applying them immediately to those of the lute.[111] This procedure thus appears to be more practical and concrete, for example, than that of Euclid's *Sectio Canonis*.[112]

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[106] al-Farabi 1930, p. 60.

[107] al-Farabi 1930, p. 68.

[108] al-Farabi 1930, p. 65.

[109] al-Farabi 1930, p. 66.

[110] al Farabi 1930, pp. 69–70.

[111] al-Farabi 1930, pp. 86–93.

[112] Cf. Sect. 2.4.

In any case, al-Farabi always argued using ratios and proportions. Thus, when he gave the arithmetic rules for adding, subtracting and dividing musical intervals, he proceeded in a more complicated way than if he had used the fractions introduced by other Arabic mathematicians, as we have seen above. In particular, when he wanted to divide the tone 9 to 8 into two parts, he obtained 18 to 17 and 17 to 16; then, for the quarter of a tone, he wrote 36 to 35. Avoiding the extraction of square roots, like all orthodox Pythagoreans, he was thus bound to arrive at fractional approximations of the exact values.[113]

He searched for general laws: "... we have ... established the laws that govern the production of notes and the formation of intervals." Those "... generally used, and also others which could be used in music, but which we musicians have not used so far." However, this musician with Persian roots did not want to regulate only his own musical style. If "imagination" had pushed him to search for other notes, it would be "... easy to do so, by observing the laws that have been expounded and defined ...".[114] al-Farabi reasoned using 15 notes distributed over two octaves. He indicated them by means of the letters of the alphabet A, J, D, H, Z, Ḥ, Ṭ, Y, K, L, M, N, S, 'A, F. He also indicated then using words like 'median', 'principal', 'low-pitched of the principal', ... Some of these were clearly translations of the Greek equivalents, such as *mese* or *hypate*, and he also supplied separately a list of all the Greek names, in order to facilitate the reading of books more ancient than his: from the *proslambanomenos* [added string] to the *nete* [acute].[115]

5.6 An Experimental Model Between Mathematical Theory and Practice

Many elements mentioned by al-Farabi had their roots in the Greek tradition. But two of them, on the contrary, were characteristic of his treatise, and are also particularly interesting for our histories of the sciences.

In order to represent consonant or dissonant relationships, he plotted the notes of the acute octave on the horizontal axis of a figure, and those of the deep octave on a vertical axis "perpendicular" to the former, where he placed the relative numbers (Fig. 5.1). "We shall give to each note its numerical value: Y will be indicated as 60, F as 30 and A as 120. The notes included between these will be represented by fractions." [...] "From the figure organised in this way, it is easy to determine the relative consonances and dissonances among the notes. It is sufficient to compare the notes together. We have two ways of proceeding: comparing each note with

[113]al-Farabi 1930, pp. 94–100.

[114]al-Farabi 1930, pp. 115–116.

[115]al-Farabi 1930, pp. 120–123. For the Greek names of the notes, see below Sect. 6.5 and also Maurolico 2010.

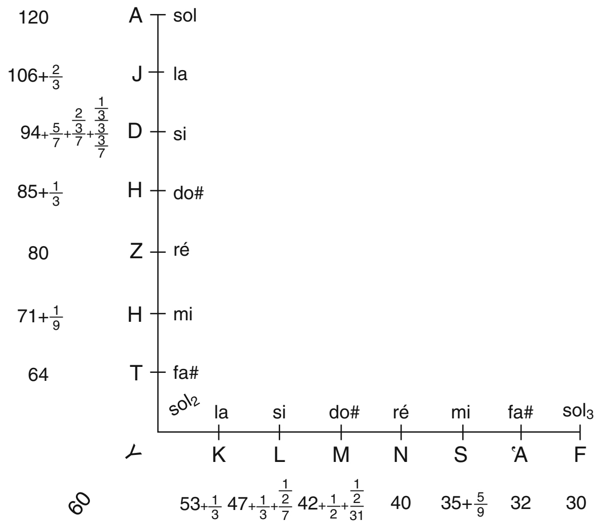

Fig. 5.1 How al-Farabi represented consonances and dissonances. The names of the notes used today in Italy and France were added by d'Erlanger (Al-Farabi, 1930, p. 148)

those fixed on the same line, or considering it with respect to each of those on the other axis." In the double or multiple ratio, the notes are consonant; likewise if the ratio is the whole plus a part.

Through the graph, our musician in search of theoretical foundations gave a better idea of the various melodies. He said that a melody was "direct" when it passed from one note to another along the line traced, "continuous" if it contained them all, "discontinuous" if it missed out some of them. A melody was defined as "sinuous" if it returned to notes already played, and even "circular" if at the end it returned to the initial note. In order to pass from one note to another that is dissonant with it, it was necessary to make use of a third note that is consonant with both of them. Hence the utility of "sinuous" and "discontinuous" melodies.[116]

Looking at these pages, it is difficult to avoid thinking of those graphs which were widely used in post-Cartesian Europe for sundry purposes. But even if we avoid the traps of alleged, improbable anticipations, and on the contrary, we continue to maintain our interest in differences between scientific cultures, we are forced to point out a second important page for the evolution of sciences. Did it remain unknown, or was it ignored in science history books, because it was found in a treatise on music?

al-Farabi gave detailed instructions for the construction of a theoretical musical instrument, different from the *ud*. He sought "truth" in "objective reality". He had based a theory regarding musical matters on numbers, but ". . . their existence must then be capable of being confirmed by sensation." ". . . we have judged it necessary

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[116]al-Farabi 1930, pp. 147–149. The manuscript in the Ambrosiana Library in Milan actually presents a drawing with two perpendicular strokes accompanied by letters and numbers: al-Farabi 1347, folio 76 recto. The French translator obviously added names that are anachronistic.

to describe an instrument imagined by one of the ancients [Ptolemy?] and conceived in a general manner. After constructing it and endowing it with bodies that are capable of producing notes . . . it will be made to emit notes that conform to what has been explained." The instrument described in the text had 15 strings, one for each note. At the beginning, they were all tuned to the same note, but subsequently, they were modified so as to produce the notes contemplated by the theory: by distributing some *ponticelli* along the strings, they were allowed to vibrate at the appropriate lengths, which were measured with precision, using a ruler.[117]

Our musician did not forget that notes serve for melody, and therefore they are not "chosen at random"; on the contrary, they have a purpose. ". . . when we desire to compose music, we have to agree, at the beginning, on a particular purpose that can be obtained with the help of musical means." [. . .] "It will be objected that for mathematicians, music is a part of mathematics [the Greek *quadrivium*]. Will it thus be necessary to search for the purpose of the existence of everything that belongs to music? In accordance with general opinion, mathematical philosophy does not deal with the reason for the existence of those things that belong to it. In its definitions, it gives us only one of the four kinds of causes; it tells us what the thing defined is . . . the question of why the thing exists does not enter into mathematics." Anyone who had learnt those "Elements of the musical art", for which al-Farabi had established the principles, ". . . will be able to work back, as he has succeeded in doing so far, towards the causes of the principles that music has derived from experience and sensations. He will also be capable of distinguishing what is right and what is false in the discourses of the various theoreticians . . .".[118]

How can we interpret these sentences today? Is it not true that they present the deliberate construction of an *organum* [instrument], as subsequent European natural philosophers were to write in Latin, in order to verify in the real world of the senses a theory that had previously been guided, in the world of the ideas, only by numerical rules? By simplifying and distorting to their own advantage the behaviour of the usual four or five famous characters, the best-known philosophers of scientific methodology have generally accustomed us for centuries to those stereotyped conventions called scientific experiments. To these, certain neo-positivistic thinkers went so far as to claim to dictate rules, formalising them in pompous expressions, such as 'observational protocols'. Current histories of the sciences abound with telescopes, pendulums, inclined planes, prisms, clocks, octants, sextants and so on. From the various discourses on methodology, their use and abuse was theorised, in the blind search for absolute eternal truths. However, on the bed of constraint where, sadistically, Francis Bacon desired to *dissecare naturam*, at the end of the task, only the pieces of a bleeding corpse were to be found. Nature was treated by the renowned philosopher of empiricism as an enemy to be defeated with the sword, in order to subdue it by force.

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117 al-Farabi 1930, pp. 158–160; Ptolemy 1682, p. 159. See above, Sect. 2.6, Fig. 2.5.

118 al-Farabi 1930, pp. 160–162.

With less cruelty, and greater pleasure, in the *Great Book of Music* written by al-Farabi, nature was seduced and its secrets were penetrated with the help of a musical instrument. He invented this experimental apparatus in an attempt to reconcile ideal neo-Platonic arguments based on numbers with the sensible reality perceived by the ears, preferred in Aristotelian schools. Note that it was the latter who suggested to him the construction of the experimental means. Curious, isn't it? Without doubt, the Arabic historical context of the tenth century was different from the Europe of the seventeenth century. Shouldn't we bear this in mind?

In the same spirit of verifying the principles and drawing the necessary consequences, our musician discussed the instruments used in the musical practice of his culture. It was necessary to know how to play these in order to produce notes. "He [the reader] will realise that theory is not simply a question of words, but finds confirmation in the testimony of the auditory sensations that are familiar to him."[119]

It is clear that al-Farabi refused the music of the spheres because he preferred to make music with instruments, and to listen to it with his ears. From them, he drew the notes capable of creating the melodies of his culture. He was convinced that in order to do this, introducing variations and extensions if necessary, it was sufficient to use the ratios between whole numbers derived from the orthodox Greek tradition. But did he not feel any tension between his requirements as a musician and that classical heritage? Did he love the variety of music without any doctrinaire exclusions, or did he think, like Plato, that only a part of it was to be practised, rejecting the rest as immoral, and thus following an interpretation of the Islamic law? To what extent may his sensibility as a musician have led him to stretch the meshes of Greek tradition? How close did he come to breaking point?

Our theoretician who had moved to Baghdad revealed an inclination for precision in measuring the lengths of strings for instruments. In the table for the tuning of the *ud*, then, he reduced them all to those whole numbers that expressed "their ratios [those of the notes] exactly".[120] However, he also knew that musicians used "... notes whose position on the strings of the lute is not determinated ... These notes are used to enrich melodies." On the other hand, not everybody used the same notes, or accepted the same consonances. He, too, knew perfectly well that the Pythagoreans, whether ancient or modern, did not admit among these the fourth united to the octave, which was "... weak, and difficult to hear."[121] For his part, among the various possible tunings of the lute, he stubbornly included one that was capable of producing 29 notes, which became 51 notes with the addition of the fifth string.[122]

But how can we judge consonances if our ear sometimes deceived us? With his customary, somewhat scholastic analysis, al-Farabi listed cases in which dissonances could be exchanged for consonances, and vice versa. It might happen that

[119] al-Farabi 1930, p. 165.

[120] For the 'free string', he took 20736, for the others in proportion. al-Farabi 1930, pp. 171–172.

[121] al-Farabi 1930, pp. 179 and 182.

[122] al-Farabi 1930, pp. 185–200 and 204–205.

the division of a string was not rigorously as desired. Thus it would be difficult to consider the *limma* dissonant, and the quarter of a tone as consonant, if the relative ratios were not respected exactly. Here, he seems to be addressing other musicians who are supporters of tuning by quarters of a tone. Does this prove that some of them openly followed Aristoxenus, if not Arabic theoreticians?

Another cause of error might arise from unintentional vibrations of the instrument. "With their vibrating movement, the strings stir the air surrounding them, thus imparting an undulatory movement. This air then penetrates through openings in the body of the instrument, and being compressed there, produces a humming noise." In judging consonances, the ear might have been confused by this. Between the temptations of quarter tones, again resisted, and ideal numerical ratios, then, did our musician in search of reconciliation regularly fall back on the latter? Not exactly, because his conclusion was: "In order to determine consonant and dissonant intervals, among those that can be produced by one instrument or another, it is not sufficient, therefore, to refer to theories founded on the division of strings; it is also necessary to verify consonances by means of the ear. We consider as dissonant those intervals which our ear judges as such ...".[123]

As regards the *tunbur* of Baghdad, "in favour in the country where we are writing this work",[124] al-Farabi proposed a distribution of frets and tunings, following the theoretical principles thus far established, but different from the one currently used at the time. The instrument must have been derived from a pre-Islamic tradition, because its frets were called *djahiliyyah* [infidels]. In general, they were placed at the same distances on the neck. Our theoretician-cum-musician judged it "... an incomplete instrument ... and we have shown how it can be completed and perfected."[125]

The *tunbur* was deemed suitable for "sweet, relaxed" melodies, and its system of tuning was said to be "feminine".[126] In the traditional arrangement of the frets, it was capable of playing micro-intervals. Taking a string of the length 40, $\frac{7}{8}$ of this was cut off, obtaining 35. The other frets were distributed at equal distances, 36, 37, 38, 39. The intervals thus generated were particularly distant from the classic Pythagorean ones: 40 to 39 (as a fraction 1.0256), which was quite close to the temperate quarter of a tone (1.0293); 40 to 38 (1.0526), close to the semitone (1.0594); 40 to 36 = 10 to 9 (1.1111), less similar to the temperate tone (1.1224), whereas the Pythagorean tone is 9 to 8 (1.125).

The *tunbur* of Khorasan, on the contrary, was more suitable for 'masculine', "strong" melodies. Its tuning continued to follow the Pythagorean manner, and its notes corresponded to those of the lute.[127]

[123]al-Farabi 1930, pp. 201–204.

[124]al-Farabi 1930, pp. 218–219. This is an instrument with two strings and a long neck, similar to the trichord, which belongs to the same family as the Indian *tamburi*; cf. Sachs 1981, p. 190.

[125]al-Farabi 1930, p. 241. Cf. Shiloah 1995, pp. 2–3.

[126]al-Farabi 1930, pp. 229–230.

[127]al-Farabi 1930, pp. 242–262.

Likewise, we obtain interesting information from the flutes, because we learn how he dealt with the diameter. The length of the pipe contributed in a linear proportional way to the depth of the note. "Acuteness or depth also depend on the diameter of the tubular pipe that the air passes through: actually the narrower this tube, the closer packed the molecules of air that pass through it, and the more violent the clashed produced between them and the walls. In this case, the sound produced is more acute. On the contrary, the wider the pipe . . . the deeper the sound produced."[128] Initially, al-Farabi considered the two factors that here determine the height of a sound to be independent. Pipes with the same diameter produce notes in proportion to their length. Furthermore, for him, pipes of the same length produce notes in proportion to their diameter. But later he also wrote: "All these causes can be combined together, concurring in the determination of the degree of the notes." Thus, in the end, he prudently concluded with the following sentence: "It is not possible to fix the place of notes on instruments of this kind by following the scale of other [stringed] instruments."[129]

This is, sure, the only part of the *Kitab al-Musiqi* . . . in which the ratios between relative magnitudes did not fix the values of pitch for notes with numerical precision. For a musician who used his ears, the orthodox Greek theory based on the monochord could not be directly extended to the pipes of wind instruments, or, at least, al-Farabi did not do so. As a result, to the best of my knowledge, the only ancient mathematical theory of sounds generated by pipes that took into adequate account the effect of the diameter on the pitch remains that of the Chinese.[130]

The numbers of Pythagorean tradition, with their usual variations, returned for the *rabab* and for harps.[131] At the end of the part dedicated to the tuning of instruments, the treatise even assumed an Euclidean approach. The paragraphs became *sakl* [propositions]: "In order to demonstrate this proposition, it is sufficient . . ." And they ended with the well-known "*Quod erat demonstrandum*".[132] al-Farabi had discussed examples in order to ". . . demonstrate the method" that could be applied to ". . . what the ancient authors and their successors talk about, and also to all the genres that can be imagined, but which are not described in any work."[133]

He also described another procedure based only on the ear, but concluding as follows: "However, [it] does not offer any guarantee of precision; it does not provide any certainty as regards the exactness of what we obtain by trusting our

[128] al-Farabi 1930, p. 263.

[129] al-Farabi 1930, pp. 263–268. d'Erlanger imprudently added a note here, which, instead of clarifying the problem of the so-called 'end effect', complicated the scenario of confused imprecision, as if he had never seen a modern organ.

[130] See above, Sect. 3.2.

[131] al-Farabi 1930, pp. 227–302. The *rabab* mounts two strings, which are played with a bow, without any frets. The rebec became the Western version of this instrument, from which the violin subsequently derived.

[132] al-Farabi 1930, pp. 299 and 297–302. Cf. Crozet 2002, pp. 326–341.

[133] al-Farabi 1930, p, 302.

ears. Only the first procedure described, and its variants, make it possible to accept the results obtained with the ear as manifest, and in line with the indications of the measurements of the strings." [. . .] "Everything that we have expounded in this treatise by means of theory is thus confirmed by sensations." This is why our musician and theoretician wrote not only for theoreticians, but also for practical musicians.[134] And he wondered in the end which instruments used for the executions were most suitable to verify the theoretical principles that he gave: perhaps the two-stringed *tunbur* of Khorasan, even if it had been recognized that the most appropriate was an instrument with one string (i.e. the Greek monochord)? But with this, demonstrations were difficult. Thus he excluded wind instruments, and also considered the lute to be imperfect. Consequently, only the theoretical instrument invented by him, described at the beginning of the section, would provide perfect demonstrations.[135]

His attempt to reconcile Pythagorean numbers with the ear was very similar to that of Ptolemy. Compared with the Alexandrian, however, our Turkish-Persian-Arab musician-cum-theoretician seemed to be more inclined to seek a balance than accept an imbalance in favour of the former.[136]

> Whoever says he does not find pleasure in sounds,
> in melodies and in music, is either a liar and a hypocrite,
> or else he is not in possession of his faculties, and
> is alien to the category of men and animals.
>
> al-Hudjwiri, *Kashf al-mahdjub*

> Anyone who speaks of things that do not regard him
> will risk hearing things that he does not like.
>
> *Alf laila wa laila [A thousand and one nights]*

> For since it scarcely in us lies to know the certain and the true,
> We cannot sit our whole life through conjecturing upon surmise.
> Best is it, that we do not set our precious cup of wine aside,
> But drink until beatified, not sober, not inebriate.
>
> Umar al-Khayyam, *Robaiyyat [Quatrains]*

5.7 Some Reasons Why

al-Khwarizmi also collected astronomical tables of Indian origin, together with elements of Ptolemaic astronomy, but his *Zig al-Sindhind [Tables of Indian Astronomy]* is extant only in a later Latin translation.[137] The Arabs proved to be great compilers

[134] al-Farabi 1930, p. 304.

[135] al-Farabi 1930, pp. 305–306.

[136] See Sect. 2.6.

[137] Benoit & Micheau 1989, p. 162.

of tables, with the measurements in degrees for the height of stars and planets, and also skilful builders of observatories: at Baghdad, Damascus, Rayy (near Teheran), and Samarcanda. Besides the names given to certain luminous stars, like Aldebaran or Altair, astronomical terms like zenith *samt ar-ras* [path overhead], nadir from *nazir* [opposite], azimuth *as-sumut* [direction] are of Arabic origin. These Arabic astronomers-cum-astrologers succeeded in constructing the valuable astrolabes and heavenly spheres to be found in museums today, which were used at that time both to calculate the planetary ascendants at the birth of heirs, and in open sea navigation.

In the case of chemists-cum-alchemists like the renowned Gabir ibn Hayyan (eighth to ninth centuries), this interest in observation also became a taste for active manipulation. This figure, subsequently known as Geber, controlled his combinations of metals, compounds and various preparations, measuring their quantities out with a balance.[138] Our word chemistry derives from the Arabic *al-kimia*, which is a deformation of the Greek *khemia* [art of transforming metals].

All these various activities provided results that went beyond the Hellenistic Greek heritage, and in some cases even criticised it. The Arabic scholars, that is to say, those who wrote in Arabic, arrived at the limits of the Ptolemaic system in some points[139] and even discussed the postulate of parallels found in Euclid.[140] After all, they were living in a historical and cultural context that had now become quite different from the Hellenistic-Alexandrian one. Apart from the new language, their sacred book, the *Koran*, even prevailed over the sciences, orientating them in different ways. In addition to what has already been mentioned above, we may add here the calculation of the waxing moon, in order to made a precise start to the ninth month of the Islamic lunar calendar, that is to say, *ramadam* [being very hot].

And yet the Greek heritage continued to loom large, for better or for worse, over this new, different scientific culture. Why? Music helps us to find the answer. We have seen that the relative theory, both in al-Farabi and in Ibn Sina, did not admit the use in mathematical formulas, either of the algebra of fractions, or, even less so, of *asamm* numbers [surds], or irrational numbers. Many practical suggestions made by musicians led to the feeling that even variants to the Pythagorean numerical scheme were too rigid and inadequate a strait-jacket. But then, not even the continual Aristotelian recourse to the sensibility of the ear, arriving at the explicit proposal of a real scientific experiment with the new instrument devised by al-Farabi, did succeed in breaking open the cage of the more orthodox Greek theories, to allow them to escape in the direction of Aristoxenus. al-Farabi would have liked to "demonstrate" what the best tuning systems were for Arabic instruments, following the scheme of Euclidean "propositions". The *Elements* of the highest classical Greek mathematics remained for everybody the source of accuracy and truth. Although, as *falasifa* [philosophers] their constant, favourite reference point in many questions

[138] Benoit & Micheau 1989, p. 165.

[139] Saliba 2002, pp. 198–236.

[140] Houzel 2002, pp. 341–347.

was Aristotle, only the weight of the *Koran* represented a higher authority for them than Euclid.

Umar al-Khayyam was undoubtedly the one who went furthest away from Euclid, with his ratios and irrational numbers, attempting even to demonstrate his axiom of parallel lines by means of a principle attributed to Aristotle.[141] In the light of Persian music, which was so rich in micro-intervals, could not he, at least, have sustained a non-Pythagorean theory? Only two texts dealing with music are at present extant by this mathematician, astronomer and poet. In the first one, *al-Qawl a la adjnas allati bi'l-arba'a [Discussion of the Genres Contained in a Fourth]*, he divided the tetrachord following the general Greek theory, which distinguished the three genres as *qawi* or *tanini* [strong, diatonic], *mulawwan* [coloured, chromatic] or *mu'adil* [medium] and *rakhw* [weak, enharmonic] or *ta'lif* [compound]. al-Khayyam inherited this classification from al-Farabi and Ibn Sina, who had contemplated various ways of dividing the ratio of the fourth, 4:3, into three others. Practically speaking, for a musician, it is a question, in the modern Western $do - fa$ interval, of tuning re and mi at different intervals, so as to obtain melodies with different characteristics of sounds and emotions.

The Persian mathematician listed ten "strong", seven "coloured" and four "compound" genres. Almost all of these are already found in al-Farabi and Ibn Sina. He added one "coloured" genre to those already known, represented numerically by

48, (6:5) 40, (40:39) 39, (13:12) 36

and one "compound" [enharmonic] genre,

100, (5:4) 80, (40:39) 78, (26:25) 75

Another "compound" genre, which he believed to be new, is also found in the works of two other theoreticians.

Some of these were judged by al-Khayyam to be good, but others unsuitable for musical practice, as if he had tried them himself on the *tunbur*, called *tar* in Persian, or at least he had listened to them with his own ears. He also took the liberty at times of criticising his two preceding masters. He considered the classic Pythagorean composition of the fourth (9:8) (9:8) (256:243) to be the closest to current practice, and the most widespread at the time, in his world.

This poet and natural philosopher seemed to be particularly happy he had invented genres with micro-intervals that were smaller than semitones. Perhaps because these were appreciated also by his Persian fellow-countrymen? He knew that he could derive many others, but then – he wrote – he would be moving away from practice and from the ear. Was this a sign that he often visited places where music was made, or even where he played himself?[142]

In Book III of his *Sharh ma ashala min musaradat Kitab Uqlidis [Comment on the Difficulties of Certain Postulates in the Work of Euclid]*, our Persian philosopher

[141] Youschkevitch & Rosenfeld 1970, pp. 329–330. Rashed & Vahabzadeh 1999, pp. 271ff.

[142] Barontini & Tonietti 2010. Cf. Shiloah 1979, p. 296. Youschkevitch & Rosenfeld 1970, p. 326.

composed ratios not only for geometry and trigonometry, but also for the theory of music. "The *tarkib al nisba* [to make up the ratio] on which some parts of music are based is numerical." For the composition of the ratios, al-Khayyam also used the term *ta'lif*. As the terminology seems to be the same as the preceding text, we might think of a link between them. On the contrary, our mathematician quoted another text. "We have already partly mentioned this subject in the *Sharh el Mushkil min Kitab al Musiqa [Comment on the Problematical Passages in the 'Book of Music']*". But this text is not available today. As its title is extremely similar to the preceding book on Euclid which quotes him, *Sharh* .... *Kitab* ..., it would seem to be reasonable to advance the hypothesis that in this case, likewise, al-Khayyam was not only referring to the book of al-Farabi, but was also goign beyond Euclid's *Sectio canonis [Division of the monochord]*. It is likely that the *Discussion of the Genres Contained in a Fourth* is a fragment of this, or maybe an initial version.

In the *Shahr .... Uqlidis*, one page after the section on the ratios for music, as already mentioned above,[143] having chosen the unit of measurement, the Persian mathematician represented magnitudes even in an incommensurable ratio by means of numbers. Then, did he ever arrive at the point where he criticised in this sense the Greek Euclidean-Ptolemaic rational-numerical theory for music? We would really like to have an opportunity to verify this in those pages to which he himself referred, but which we only know to a small extent nowadays.[144]

So far, unfortunately, the limited extant passages about music by al-Khayyam tend to confirm that he, too, remains in the stream of variations of Pythagorean ratios between whole numbers. A similar choice to abstain from what would easily have become conceivable, above all for a mathematician who had no scruples about thinking in terms of irrational numbers, thus deserves to be examined to discover the reason behind it, if any. He, too, was considered a heretic, and was once defined as a "snake that poisons the *sharia* [law of the Koran]" and as "a miserable materialistic philosopher and naturalistic" on another occasion. The implacable al-Ghazzali did not spare him, either. About that, there can be little discussion. It is enough to read his famous *Robaiyyat [Quatrains]* abundantly washed down with wine, traversed by cosmic doubts, and sustained on an eternal Nothingness, so extreme as to appear irremediable, even for any mystic god: where the only certainty appears to enjoy himself, as long as this ephemeral life is possible.

Is the figure before us, then, a pessimistic, pantheistic, sceptical, ironic natural philosopher? And yet in other official books of his, written at the request of those in power to whom he owed his survival, we find a different al-Khayyam. Here, he praised the duty of prayer, as if he desired to escape from the accusations brought against him by orthodox Muslims.[145] When, and to what extent, was he pretending? Perhaps he made an effort every now and then also in his writings on mathematical

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[143] See above, Sect. 5.4.

[144] Youschkevitch & Rosenfeld 1970, p. 327. Rashed & Vahabzadeh 1999, pp. 376–379. Barontini & Tonietti 2010.

[145] Khayyam 1952. Khayyam 1956. Youschkevitch & Rosenfeld 1970, pp. 330–331.

sciences, equally close to his official engagements for his protectors, to appear to be faithful to tradition, not expressing his thoughts fully: "A pearl that I cannot thread, due to my trembling with fear."[146] Thus he would seem to adopt the same approach also as regards the theory of music. Or maybe, rather, would not he himself offer us the best indication that life, lived for such a short time on this poor earth of ours, goes well beyond any claim of truth and consistency? There is no doubt that it would be useless to search for consistency and truth in history, if it is narrated by paying attention to the lives lived.

On the basis of secondary literature, the division of the octave into perfectly equal quarter tones was only proposed much later in the Arabic context. The Syrian mathematician and theoretician of music, Muhammad ibn Husayn Attarzade (1764–1828) sustained this view in his *Rannat al-awtar … [The Sounds of Strings]*. But, even though a pupil of his advertised it, the book remained unpublished, and the division of the two octaves into 48 intervals was still generally rejected in the twentieth century. In practice, these musicians preferred to leave it up to each player to choose the micro-intervals to be played.[147] Yet melodies had already been plucked on the *tunbur* of Baghdad, known also as the *tunbur mizani* [measured], which was tuned in quarter tones, dividing the neck into 40 "equal" parts. But unfortunately, this went back to the pre-Islamic period, and consequently was considered pagan, and was condemned by the orthodox.[148]

Within its complex variety, therefore, Arabic music included, and includes, micro-intervals. However, these were not easy to justify on the basis of the most commonly followed, and best-known theories. Apart from a few sporadic episodes to be verified, the theory of Aristoxenus about quarter tones, which would seem to be the most natural one in this case, remained without followers, even though it was as well-known as the Pythagorean-Euclidean idea. It would have been relatively easy to arrive at it through the translation of the *Elementa Harmonica* into Arabic, under the title of *Kitab al-ru'us [Book of the Beginning]* and a book on rhythm such as *Kitab al-iqa [Book of Rhythm]*, or by reading Ptolemy's book on the subject.[149] The historical situation appears to be interesting, because unlike Greek scientific orthodoxy, these Arabs, on the contrary, seemed to be losing the prohibition against irrational numbers and non-exact square roots, in their new algebra. As a result, they would have had at their disposal a more suitable calculating instrument for the development of a good mathematical theory of quarter tones. The event that they did not do so depended on the dominant values of their culture. Which values?

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[146] Khayyam 1952, 1956, p. 47.

[147] Shiloah 1995, pp. 116–117. Righini 1983.

[148] See above, Sect. 5.6. Farmer 1975, pp. 498 and 508. The division of the neck into so many equal parts could only give a particularly unsatisfactory approximation of equal quarter tones. Whether we reason in terms of the height of sounds or the length of strings, because it is a question of constructing a geometrical progression, and consequently, in order to maintain equal intervals, the frets need to be placed at increasingly close distances from one another moving towards the acute notes, that is to say, towards the *ponticello*.

[149] Farmer 1930, pp. 327–328. Farmer 1975, p. 510.

On the one hand, the separation between the practice of musicians and the theory of mathematicians/philosophers, present here as also in the Greek case, seemed to be less difficult to overcome, as it was for al-Farabi. Yet on the other hand, in actual fact it was aggravated by the religious attitudes that were contrary to music, or at least to that kind of freedom in music which most appreciated micro-intervals. Furthermore, the case of music reinforced the idea that algebra was not yet, in that period, the versatile, universal instrument that it has become for us today. In the Arabic scientific culture of the period, we are about to meet those values that created a hierarchy; consequently, Greek tradition and that of algebra could not stand on the same level. As a result, Aristoxenus could not have enjoyed the same prestige reserved for Euclid and Ptolemy.

All the same, there are some elements to sustain a singular alliance between Islamic theology and Greek Euclidean geometry. In the second part of our history, we shall come across encounter one of the most famous names of the European scientific revolution of the seventeenth century, Leibniz, who defended a similar alliance and summed it up *mutatis mutandis* in himself. Here, the similarity is borne out by the event that both found a common space in the same scholastic institutions, which were set up for all subjects, including sciences, around the Koranic *madaris* [schools].[150] Again, music provides another argument in favour.

Sinners intoxicated with wine were charmed by the large bistred eyes that sparkled in the faces, more beautiful than the moon, above those supple, smooth bellies narrated in *The Thousand and One Nights*. But they had to suffer the frowning, withering glares of the bearded guardians of orthodox religion. The kind of music which appeared to them to be "sweet and feminine", thanks to the play of lascivious micro-intervals, must have been judged extremely dangerous. History books record episodes like that of the year 933. "Al-Kahir ordered the banishment of *kiyan* [slave-girl singers], the prohibition both of wine and of the *makhanith* [effeminate singers], the imprisonment of all musicians, and the destruction of musical instruments."[151] Had there been a little exaggeration in the realisation of the musical ideals present in the *Koran* or proposed in Plato's *Republic*?

As a result, a theory of music that regulated it, reducing it to the more appropriate "strong, masculine" genres, would have been highly appreciated. This is exactly what we have read in al-Farabi, for the music played on the *tunbur* of Khorasan. The discipline was established on the basis of the orthodox Greek theory, although his musician's ear and his Aristotelian sensibility could equally well have led him to other conclusions. These *falasifa*, who undoubtedly also knew Plato's *Republic*, could not accept any kind of music quite freely.[152] They, too, did not want to corrupt the souls of young men, who had to be ready rather for war. The famous theologian,

[150]Ihsanoglu 2002.

[151]Shiloah 1979, p. 219. Shiloah 1995, pp. 31–44 and 63–64. Cf. Farmer 1975, pp. 478, 490 and 493.

[152]*Plato's Testament for the Education of Young Men* was translated into Arabic; Rashed 2002, p. 36.

Abu Hamid Muhammad al-Ghazzali (1058–1111), distinguished seven purposes for which music could be made, and five cases in which, on the contrary, it was prohibited. Among the former, we read: "... 2) To incite to battle, 3) To inspire courage on the day of battle ... 6) To elicit love and longing in circumstances that permit singing and 7) to evoke love for God." Among the latter: "1) When produced by women under certain conditions ... 3) When the song's contents are not compatible with the spirit and precepts of religion; 4) When the listener is ruled by lust; or 5) If one listens to music for its own sake."[153]

al-Farabi seems to be in search of a balance, and appears to wish to construct a universal system, suitable for every kind of music, but as a result, he showed us a hierarchy. This is why Euclid still remained so important for them, in spite of all the other scientific novelties that were under their nose. Suspended between a transcendent God and the arts of *iblis* [Satan], the part of music that philosophers preferred was in all cases the theory. "... musical praxis never became an integrated part of the official educational system."[154]

This argument is generally valid. We have seen many heterogeneous elements flow into the great basin of the *Dar al-Islam*. Greek elements arrived from the North, along the Tigris and the Euphrates, whereas the Indian elements arrived from the South, across the Indian Ocean, or the Afghan mountains, or else from the East, coming around Lake Aral and the Caspian Sea. I was almost forgetting that following this route, some Chinese craftsmen were taken prisoners in the battle fought at the River Talas (751), and subsequently taught the inhabitants of Samarcanda how to make paper. But we should not get the idea that Arabic scholars, whose mother tongue was frequently Persian or Syriac, mixed everything together in an undifferentiated eclectic melting-pot. Their culture possessed the values which gave rise to a new hierarchy.[155] For them, Euclid was more important than algebra, the *Almagest* of greater significance than Indian astronomic tables, the letters of the Arabic alphabet, used as numbers and as notes, were more widespread than Indian numbers. The mathematical science historians who sustain the opposite commit an error of anachronism, projecting on to that completely different culture modern European values in which algebra has, in the meantime, arrived to rule geometry, and sciences are expressed in symbols, using Indo-Arabic numbers.

In the end, among the values exalted by the Arabic historical context, we also find those by means of which we are describing our different scientific cultures. As al-Farabi clearly stated: "The purpose of every theory is to help us to reach the truth." Islam is a *din al-haqq* [religion of the truth]. The truth lies in the material and immaterial word of God. Before the truth, which is absolute, we should be *muslim* [submissive]. Only then can we be faithful, otherwise we would become infidels. In this way, the *Koran* equates religion and truth, which become the Law. History was

[153] Shiloah 1995, pp. 43–44. Tauzin 2007.

[154] Shiloah 1995, p. 60.

[155] Nasr 1977, p. 22. In the classifications of sciences, prevalence is usually given to Koranic sciences, with some exceptions, like Ibn Sina; Jolivet 2002.

subject to this Law, and was substantially cancelled, because the events of human life aimed to reconstruct the ‘truth’, as in court cases. It thus became relatively easy to pass from religion to jurisprudence, and even to mathematical sciences, as we have seen.[156]

Whether its nature was religious or geometrical, we can have little doubt that the position of the truth was on high, very high. It must have been guaranteed either by the prophet Muhammad or by Euclid, because that truth transcended sensibility, experience and common discourse. It should not have been difficult to annex the Greek geometrical tradition and to make it compatible with the *Koran*, seeing that both were justified by their transcendence of the world of senses. Even al-Ghazzali wrote: “Those who imagine that Islam can be defended by rejecting the mathematical sciences are guilty of a serious offence against religion; there is nothing in revealed truth that is opposed to these sciences, by way of negation or of affirmation, and there is nothing in these sciences that is opposed to the truth of religion.”[157]

In pursuing transcendence, many of the dualistic categories of interpretation typical of orthodox Western scientific culture were created. Let us continue to read only al-Farabi, even though a lot of other pertinent material could be found elsewhere. The conditions of the perfect theoretician, for him, had to include “. . . 2) possessing the ability to deduce the necessary consequences of these principles, 3) being able to answer erroneous theories . . . to discern the true from the false . . .”.[158] This philosopher and musician presented a man as if he were split between sensibility and rationality.[159] In turn, sensibility might be “normal”, like that possessed by the Arabs and the Byzantines, or “abnormal”, the kind found among the Ethiopians, Turkish nomads and Slavic races, “above all that of the peoples of the extreme North”.[160] We have already dwelt on the dualism between musical genres that are described as “sweet . . . feminine, which recall the sweetness of the female” and the “strong . . . masculine” kind.[161]

In one page, the current dualism assumed a philosophic depth, and the discussion became general. “Among the sciences, there are some where contrary data and opposites are treated in the same way and considered on the same level. In arithmetic, for example, no distinction is made between even and odd numbers; the ones are not considered more important than the others. On the contrary, other sciences have as their object the particular study of one of the opposites; . . .”. In music, al-Farabi divided sensations into “natural or repugnant”. The musical

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[156]al-Farabi 1930, p. 158. Cheddadi 2002, pp. 106–108. The ‘truth’ of Islam was used by al-Biruni to judge both the Greeks and the Indians. He partly saved the former, like Socrates, but condemned the latter; Nasr 1977, pp. 190–192.

[157]Nasr 1977, pp. 250 and 253.

[158]al-Farabi 1930, p. 2.

[159]al-Farabi 1930, pp. 7, 9, 11, 30.

[160]al-Farabi 1930, pp. 38–39.

[161]al-Farabi 1930, p. 61.

product might be the result of “nature” or of “art”, like “geometry, which is at times theoretical and at times practical.”[162]

And he was not content with “. . . a probable opinion, because a probable opinion is a belief which may be false.”[163] On the contrary, he searched with conviction for “. . . certainty, clarity . . .”. For this reason, he could not limit himself to sensations that were subject to error, but entrusted all judgement to intelligence. “If it were not so, there would be no knowledge of certainty or conviction.”[164] This scholar had a greater interest in “distinguishing what is true or false in the sayings of the different theoreticians.”[165] Thus formulated, this “distinguishing” easily became the discriminating found in the laws: “. . . we have established the laws that govern the production of notes . . .”.[166] To refute the objection of others, to exclude possible alternatives, to silence adversaries, to condemn music that was unpleasant, or for some reason unsuitable, what better means existed than having the Law come down from on high?

Unlike the Chinese culture, the Arabic one was and is the culture of the Law. It was and is so, even more than the Indian culture, where the single transcendent *brahman* was still always obliged to the forms of thousands of swarming visible incarnations amid ancient, humid jungles. In the *sura* [chapters] of the *Koran*, the basic transcendental truths were incarnated. “We will show them our portents on the horizon, and within themselves, until it is clear to them that it [the *Koran*] is the truth.”[167] The Arabic language of the prophet Muhammad had proved to be capable of expressing it. Therefore it would appear sufficiently powerful in its universal generality to cover every other kind of discourse, including scientific subjects. This language, which was no longer the one spoken by a minority of nomadic tribes scattered across the desert, but was used by the populations of a vast empire supplied “the Clement and Merciful God” with the necessary symbols, “Power and strength exist only in the Most High God, the Infinite One.”

At the centre of Arabic culture, there was, and is, the *kalam* [speaking, discourse, reminiscent of the Graeco-Latin *calamus*, the reed, or pen for writing]. Its general characteristics can thus be revealed by its language too, and the relative rules fixed for it by scholars. Thus we find a dualism with strong moral connotations in the terms used to divide sentences into *hasan* [beautiful, good] or *qabih* [nasty, bad], *mustaqim* [straight, correct] or *muhal* [deviated]. Grammar was made prescriptive by *usul* [both general principles and rational bases], proposed by a famous school

[162] al-Farabi 1930, pp. 27–28. On the dualism of the Ikhwan al-Safa, see above, Sect. 5.3, and Nasr 1977, p. 129.

[163] al-Farabi 1930, p. 31.

[164] al-Farabi 1930, p. 30.

[165] al-Farabi 1930, p. 162.

[166] al- Farabi 1930, p. 115.

[167] Quoted in Nasr 1977, p. 20. In the *Koran*, Arabic became a sanctified language, an untouchable, eternal, official written expression whose historical evolution was practically blocked; Choubachy 2007.

created at Basra, which considered the language to be fixed, and not subject to change. In contrast with them, another school at Kufa opposed an Arabic language seen rather as subject to transformation, and rich in exceptions. In this dispute, the stakes were the highest possible, because it was a question of fixing criteria to delimit the *corpus* of religious and juridical texts. The scholars of Basra prevailed, and around them the orthodoxy of the Law was formed. The others were left only with the historical role of "non-conformists".

Contrast was also inevitable with scholars who had been influenced by Hellenistic Greek ideas. Among them, we find not only our musicologist, al-Farabi, but also a current of rationalistic theologians known as *mutazilits*. With their translations from Greek, they changed the terminology. Now *mustaqim* [straight] came to mean *orthos* [correct case], and together with *kalam* [discourse] inspired by the practice of speakers, also a more formal *gumla* [proposition] appeared. Even abstract metalinguistic terms were expressed, such as *ismiyya* [nominality] and *filiyya* [verbality]. The dualism followed an explicit pattern between *sura* [form, morphology] and *madda* [matter, content, meaning]. This derived from Aristotle, who had made the whole universe spring from an undifferentiated primordial matter distributed in the various forms of bodies, a theory called hylomorphism (matter/form). Above all, "... *itbat* [affirmation], used both in theology and in philosophy to affirm the existence of something and to indicate the result of a demonstration, appeared in the place of *igab*, with the meaning of 'affirmative assertion'".[168] Unlike Chinese, therefore, Arabic predicates the properties both of concrete objects and of abstract entities relatively easily.

Also in this second case, the discussion was not a trifling matter. Was Arabic to remain the only language of the Koranic revelation, or, like all the others, would it be subject not only to God, but also to the logic with universal claims that derived from Greek? Ibn Faris (???–1005) sustained that "... as Arabic is the means of expression of every authority, and of every religious conscience, Muslims have the duty of governing its rules used to deduce, interpret and apply the Law correctly." The dream of the most orthodox Arabic grammarians came true in the twelfth century, when there appeared among the precepts of the Islamic religion also the principle that "... leading members of the community had the religious obligation to be masters of the Arabic language."[169]

Dictionaries of Arabic were organised in a particularly peculiar way. They contained not only the words actually *mustamal* [used] in the historical language, but also those that could possibly be formed by means of combinations and permutations of letters, but remained *muhmal* [unused].[170] The language was thus represented as the corpus of all the possible combinations and permutations of letters, whether present in Arabic or not. In this way, based on the roots of words, they arrived at numbers as high as 12,305,412. How would they obtain them? By

[168] Carter 2002, pp. 78, 79, 81.

[169] Carter 2002, pp. 81, 82.

[170] Carter 2002, p. 84.

what processes of reasoning? Did they patiently count them one by one? In any case, we see that this idea of the language also extended to theories of music, like that of al-Farabi. "In music, notes are the first element; they play the same role as phonemes in poetry." Our Persian philosopher thus combined notes and, if necessary, intervals, as poets combined phonemes, feet, and metres.[171] The initiator of Arabic lexicography, the exponent of the school of Basra Halil ibn Ahmad (eighth century), distinguished between vowels and consonants on the basis of periodic, musical sounds and irregular aperiodic sounds.[172]

Based on a finite number of well distinct, i.e. discrete letters, within a hierarchic culture centred around purportedly divine discourse, the characteristics of a language like Arabic were variously received also in other sectors. The *mutazilits* theologians, also from Basra, developed a general philosophy based on *kalam* [speaking], which had now become discourse about God, or theology, and innumerable *mutakallimun* [speakers, theologians] flocked around them for centuries in eternal, incessant religious and philosophic disputes. They, too, had been influenced by Hellenistic Greek readings, but, to the credit of the Koranic word, they preferred the atomistic, non-Aristotelian side of the Ancients. Thus in their works, we find a defence of Euclid's ideas, such as indivisible geometrical atoms, in contrast with the infinite divisibility sustained by their Aristotelian rivals.

As we know from the ancient Greek age, this was connected with the controversy about movement, with all its paradoxes. To defend it, the theologians of Basra found an original solution, which was clearly dangerous for their adversaries: "… movement supposes a sort of renewed miracle by which God causes the moving object to skip some spaces filled with an infinity of points." But the curious response of the *mutazilits* to Aristotle's criticism of Zeno from Elea,[173] in turn aroused the opposition of Ibn Sina. This Persian doctor-philosopher considered the discontinuous process theorised by Arabic philosophers and theologians as simply "absurd", because on the contrary, it defended the continuous movement of the Stagirite. And as a better justification, he almost went so far as to write on actual infinity for instants of time. But with their theories of the movement, the infinity, the falling of heavy bodies, and the void, did the *mutakallimun* decree the end of the Aristotelian cosmos largely in advance compared with the more famous and better studied Giovan Battista Benedetti (1530–1590), Galileo Galilei and even Leibniz?[174]

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[171] al-Farabi 1930, pp. 26–27, 59, 127.

[172] Rashed 2002, pp. 86–88. This Egyptian from Cairo, who moved to Paris, where he came into contact with an algebraic-Bourbakist environment, would undoubtedly be interested in the musical side of the history: in order to enrich the topic of how Arabs loved calculating combinations, arrangements and permutations, perhaps as a prelude to modern combinatorial analysis.

[173] See above, Sect. 2.5. Cf. Sorabji 1983.

[174] Marwan Rashed 2002, pp. 49–72; however, she has forgotten Zeno from Elea, and unfortunately she does not say anything about their possible (Pythagorean-discrete?) theory of music. Rashed 2002, pp. 630–635.

We leave this problem to historians who still have a little faith in scientific progress. From our different viewpoint, focusing on cultural diversities, we prefer to consider the characteristics of the ‘continuous/discrete’ dispute that are typically present in the Arabic historical context. Despite a widespread Aristotelianism, here religion and language, in some interesting cases together with Pythagoreanism in music, seem to have shifted the balance towards a conception that preferred to exalt the corpuscular structure of the world.[175] Also in this respect, the Arabs are very different from the Chinese, and some more some less were drawing closer to that part of Western scientific evolution which was destined to become majority, and orthodox with the scientific revolution of the seventeenth century.

Writing in Arabic, our scholars thought they disposed of an extremely powerful language, as it was capable of expressing their single God through its letters, seeing that His representation with images was strictly forbidden. The task of reaching the transcendent truths of nature, music and mathematical sciences should thus have seemed a bit easier to them. The sentences, words and letters of Arabic must have appeared amply sufficient for these purposes. It would have been useless to develop another symbolic language, necessarily devoid of prestige or charisma, besides that of divine revelation. Our symbols of mathematical sciences were thus to be invented in another age and in another culture, likewise transcendent, but in a different way, as well as with distinctive problems of language.

> Never did my intellect move away from science,
> There are few secrets that have not been revealed to me yet,
> And night and day I have reflected for seventy-two long years,
> And the only thing I learnt is that I have never known anything.
>
> Umar al-Khayyam, *Robaiyyat [Quatrains]*

[175] Nasr 1977, p. 250.

Chapter 6
With the Latin Alphabet, Above All

They arrived at a village, where it was necessary to search for an algebraist [surgeon] who cured that wretched Samson.

Miguel Cervantes, *Don Quijote da la Mancha*

Understandably, therefore, all living creatures are captivated by music, because the heavenly soul that animates the universe took its origin from music.

Ambrosius Macrobius, *Commentarii in Somnium Scipionis*

6.1 Reliable Proofs of Transmission

An edition still exists of a Latin translation made from the Arabic version of the *Magiste Syntaxis Mathematica [Greatest mathematical composition]* by Ptolemy. The original Greek title will be forgotten, seeing that from then on, for everybody, this famous book became the *Almagest*. How little the passage from one culture to another can be ignored it is clear from the comments of Nicolas Copernicus (1473–1543) on his own copy.[1] In the previous chapter, we saw the importance of the characteristics of the various cultures coming into contact: Greek, Indian, Persian and Arabic. But we should not overlook the role played in our history by the heretics, against the background of an orthodoxy reconstructed each time around the places of political and religious power.

Under the banner of "In hoc signo vinces" [In this sign [the cross] you will triumph], which appeared in the dreams and on the bellicose ensigns of Constantine, an alliance was established between the new Christian religion and the declining Roman Empire. Having thus been consolidated, the Church condemned Arius, at

[1] Benoit & Micheau 1989, p. 162.

T.M. Tonietti, *And Yet It Is Heard*, Science Networks. Historical Studies 46, DOI 10.1007/978-3-0348-0672-5_6, © Springer Basel 2014

the council of Nicea (325), for not conceding any divinity to the historical Christ, proclaiming him, on the contrary, consubstantial with God the Father. At the Council of Ephesus (431), the Church also freed itself of Nestorius, who had, it is true, admitted the two natures of Christ, human and divine, but wanted to keep them separate. At the council of Calcedonia (451), in the end, also the third possibility of a Christ who was only divine, sustained by the Monophysites of Eutyche, was declared heretical. We have already spoken above of the closure of Plato's Academy at Athens (529), and we should not overlook the Sabaeans of Harran, either, who looked back to the patriarch Enoch in the *Bible* and other astral religions.

All these heretics, for various reasons, became vagabonds, and wandered around in search of new places where they would not be persecuted, or better, they would be appreciated for their skills. Thus we find the Nestorians widespread in the East, in Persia and even in China, where there is still today a stone column at Xian that mentions them. The Monophysites arrived in Egypt (copts), in Syria (Jacobites) and in Armenia. These developments are not only interesting for the history of religions, but also for the history of sciences, seeing that we find that these heretics made good use of their culture and linguistic knowledge to transfer the Hellenistic Greek scientific heritage to the Arabic Empire.

At this point, there is a change in the dominant language to Latin, and with the return to Europe, we draw closer to what Eurocentric scholars consider the central event of all history: the so-called scientific revolution of the seventeenth century. And yet, *mutatis mutandis*, even here at the threshold of modern science, we will see that the role of heretics cannot be ignored. Everything was now translated into Latin, commented on in Latin or, wherever possible, written *ex novo* (just) in Latin. Such a language, even though the Roman Empire had declined and disappeared long before, had remained the dominant one in a vast area of Europe, because it had become the means of expression of the Roman Christian Church. The *Bible*, the *Gospels* and the liturgy were recited in Latin. In the conversion of various heterogeneous populations to Christianity, these were taught, wherever possible, to read and write in Latin letters. For centuries, European culture will remain firmly anchored to Latin. In the case of mathematical science, the link to it lasted even longer: as late as the early twentieth century, a mathematician like Georg Cantor (1845–1918) continued to write at times in Latin. Up to a few decades ago, a famous physics journal, *Il nuovo cimento*, accepted articles written in Latin, as did also the *Archive for History of Exact Sciences*. In Eastern Europe, on the contrary, writings were in Greek, but this was because the schism with the orthodox Christian church had taken place (in 1054), and their monks travelled around all these countries. The theological discussion now regarded the place of the Holy Spirit in the trinitarian hierarchy with God the Father, and God the Son.

All this should prepare us to find, also here in Europe, a constant, powerful presence of religion even in the mathematical sciences. In Greece, Plato and Aristotle, in China, the classic books of Confucius, in India, the *Veda*, in the Middle East, the *Koran*, here in Europe we find the *Bible*. A large number of the characters we shall meet are in various ways religious figures, with some heretics, or suspected of being such. Even the few who had not received a solid training in the various convents and seminaries were to take up a position before the Church of Rome. Last of the consequences, but not at all the least, we note that this cultural context was creating "A world without women" for the sciences.[2]

The scientific innovations of the period sprang from groups of texts from two main sources. The first, also chronologically, was the Arabic group; subsequently, especially in the fifteenth century, scholars searched for, and found, though not always, the original direct Greek roots. We have already spoken of Ptolemy, and as regards Euclid, we must add that the early Latin versions were translations from the Arabic editions. Among the most active translators, we find Gerardo from Cremona, who worked at Toledo.[3] Dusty Greek books sometimes popped up in some library or convent, or carried 'to safety' from the East, under the threat of Arabic expansion. Other times, nothing was found and nothing arrived. The case of Archimedes is emblematic: some of his treatises are no longer extant, and we lack them completely, whereas others are known only by virtue of their translations into Arabic.[4]

However, Euclid and Ptolemy enjoyed all too much respect, and as far as they were concerned, writers only dared to produce commentaries and compendia, which in the end proved not to be very important, without any new ideas. The greatest novelty was contributed by Leonardo da Pisa, called Fibonacci, in 1202 with his *Liber Abbaci*. Thus we find the "figuras indorum",[5] that is to say our numbers which were consolidating their position in Europe, in a book written by the son of merchants. Yet it was not easy to weld the tradition of calculations taught at the abacus schools for the efficient management of commercial activity together with that of geometry. As with the Arabs, the latter continued to prevail over the former in the learned environment of scholars, creating a hierarchy. In order to be considered authoritative, good scientific reasoning had to be formulated in Latin, and make reference to Euclid, as the source of all certainties and incontrovertible truths.

6.2 Theory and Practice of Music: Severinus Boethius and Guido D'Arezzo

In Latin Europe, we have not to make any effort to find our precious theory of music. Immediately after *De Institutione Arithmetica*, hard on its heels, came *De institutione musica* by Severinus Boethius (c. 480–526). The book became an inescapable reference point, the most authoritative source to draw on for this subject, for better or worse. It formed the hub of the orthodoxy of the Pythagorean model, which lasted, intact and unchallenged, among theoreticians for a 1,000 years, at least until the sixteenth century, and in some cases even later. Its success was

[2]Noble 1994.

[3]Benoit & Micheau 1989, pp. 171–172.

[4]Boyer 1990, pp. 159–164.

[5]Arrighi 1994.

undoubtedly favoured by the tragic events that occurred to its author. The moral interpretation of his work as a philosopher was Christian indeed, and he had a brilliant political career. But as a result of this, he was accused of betrayal before the court of Theodoricus, and consequently he was executed.

In the field of music, Latin scholars hardly translated anything from Arabic. Although treatises of algebra and medicine by them were put into the new dominant language, they ignored the pages about music which were sometimes attached. "... in their translations of various Arabic treatises, they omitted the portions that regarded music! As a result, the section about music in the *Šifa* by Avicenne was never translated."[6] The neo-Platonic tradition was directly followed and maintained fresh by Marcus Tullius Cicero (106–43 B.C.), Ambrosius Theodosius Macrobius (early fifth century), and Proclus (410–485). At the end of his work *De Republica*, the famous orator, polemicist and politician of the pre-imperial period added a *Somnium Scipionis*, where he described the music of the heavenly spheres.[7]

"... 'quis est, qui conplet aures meas, tantus et tam dulcis sonus?' 'Hic est' inquit 'ille, qui intervallis coniunctus imparibus sed tamen pro rata parte ratione distinctis, impulsu et motu ipsorum orbium efficitur et acuta cum gravibus temperans varios aequabiliter concentus efficit. Nec enim silentio tanti motus incitari possunt, et natura fert, ut extrema ex altera parte graviter, ex altera autem acute sonent. Quam ob causam summus ille caeli stellifer cursus, cuius conversio est concitatior, acute et excitato movetur sono, gravissimo autem hic lunaris atque infimus. Nam terra, nona, inmobilis manens una sede semper haeret conplexa medium mundi locum. Illi autem octo cursus, in quibus eadem vis est duorum, septem efficiunt distinctos intervallis sonos, qui numerus rerum omnium fere nodus est. Quod docti homines nervis imitati atque cantibus aperuerunt sibi reditum in hunc locum, sicut alii, qui praestantibus ingeniis in vita humana divina studia coluerunt. ...' "[8] ["... 'what is this sound, so great and sweet, that fills my ears?' 'This' he replied 'is the sound which is produced by the impulse and the movement of the orbits, by tempering together intervals that are unequal, yet calculated precisely in their ratios; by mixing the acute with the deep, various harmonies of sounds are similarly created. Such vast movements cannot take place silently indeed, and nature causes the extremes at one end to sound deep, and at the other end, acute. For this reason, that supreme starry course in the skies whose change takes place most rapidly moves with an acute, stimulating sound, whereas the deeper sound accompanies the lower course of the Moon. The Earth, the ninth heavenly body, which remains immobile, and confined to one spot, occupies the central position in the world. Instead, the other eight courses, in two of which the same force resides, create sounds that may be distinguished into seven intervals, whose number is the essence of almost everything. By reproducing this with singing and strings, wise men opened up the way to return to this place,

[6] Burnett 1990, pp. 87–88.

[7] Cicero 1992.

[8] Macrobius 1893, pp. 657–658.

like others who cultivated studies of the divine with outstanding intellects during their human life' ...".]

In the Christian environment, readers disregarded the ancient context of the Punic Wars between Carthage and Scipio Africanus, who appeared to his descendant Scipio Aemilianus in a dream. They preferred to interpret the story in favour of the immortality of the soul, which returns to heaven. The most famous commentary on this work is that of Macrobius, in which we can find some of the most permanent occurring *topoi* that accompanied Pythagoreanism in music.

"Deinde tonus per naturam sui in duo dividi sibi aequa non poterit. Cum enim ex novenario numero constet, novem autem numquam aequaliter dividantur, tonus in duas dividi medietates recusat ...".[9] ["Therefore, due to its own nature, the tone cannot be divided into two equal parts. As it is formed by the number nine, and the number nine can never be equally divided, the tone refuses to be divided into two halves ..."]. The conclusion presents us with the same prohibition of dividing the tone into two equal parts. But now, far from the sources of Euclid and Ptolemy, or incapable of following the reasons partly justified by mathematics, the argument used became worse than mistaken, unless the text is corrupt, or errors have been made by the copyists. Macrobius seems do not understand that dividing an interval of a geometrical progression means extracting the square root in order to obtain the proportional mean. Thus 9 would give the excellent whole number 3, whereas it is 8 that does not produce the whole number, but contains $\sqrt{2}$. Furthermore 9, which clearly cannot be divided into two whole numbers like 8, all the same contains two rational parts that are equal. In other words, Macrobius is confusing the geometrical progression with an arithmetic one.

Ancient Pythagoreanism was declining to pure verbal rhetoric, losing its bearing mathematical skeleton. Only a showy Platonic dress was left visible, which however continued to perform its advertising function effectively. As in Plato's *Republic*, now through the end of Cicero's *De Republica*, the choice of the intervals for scales and the modes of music made it possible to justify the melodies considered to be moral and suitable for that particular social organisation, refusing the others. Mathematical sciences again served for the prohibition; whether they were right or wrong did not seem to be of any importance to them.

"... cum sint melodiae musicae tria genera, enarmonium, diatonum et chromaticum, primum quidem propter nimiam sui difficultatem ab usu recessit, tertium vero est infame mollitie, unde medium, id est diatonum, mundanae musicae doctrina Platonis adscribitur."[10] ["... seeing that there are three genres of musical melodies, enharmonic, diatonic and chromatic, the first has actually become detached from use due to its difficulties, whereas the third is shameful because of its effeminacy and its licentiousness; as a result, the intermediate one, that is to say, the diatonic, is attributed by Platonic doctrine to the music of the universe."] Now music's task was to open up to the soul the doors of the heavenly paradise, whether this was Platonic

[9] Macrobius 1893, p. 586.

[10] Macrobius 1893, p. 598. Cf. Flamant 1977, pp. 351–381.

or Christian. But wasn't there a risk that all those different micro-intervals, used in the lascivious modes and admitted by the rival theory of Aristoxenus, might become extremely dangerous keys? They were definitely ready to open up quite different paradises, visible and tangible, since they were earthly ones. Macrobius despised all the arts of the senses; he condemned the art of singing, and dancing, and even cookery. Honest men and women should not practise them. He believed that ears should be closed to the temptations of a sensual melody. For him, the only music should be the heavenly kind of the spheres, abstract, transcendent and inaudible. He had largely taken his inspiration from Plato.[11]

Compared with Macrobius, Proclus was more important for the mathematical sciences, above all in view of his commentary on Euclid's text, though he has not left us anything dealing with music. As has already been said, this scene was dominated by Boethius, in whom we find the commonplaces of Pythagoreanism, together with the relative errors of Macrobius.

In the tradition of the *quadrivium*, Boethius now placed *Arithmetic* before *Music*. He wrote all the numbers in Roman numerals, and used them to present the few notions that were needed to define the usual Pythagorean ratios. Rapidly freeing himself from the task, he indicated the difference between music and the other mathematical disciplines. "... cum sint quattuor matheseos disciplinae ceterae quidem in investigatione veritatis laborent, musica vero non modo speculationi verum etiam moralitati coniuncta sit." ["... as there are four mathematical disciplines, while the others attend to investigations into the truth, music, on the contrary, is not only concerned with speculation, but also with morality."]

In customs, the extreme positions were a "lascivus animus" [lascivious spirit] and an "asperior mens" [more rigid, proud mind]. The former enjoyed lascivious melodies, and "when he heard them, he often became effeminate, and weak". The latter, on the contrary, preferred rousing tunes, and "took strength from those that were more stimulating". These were, respectively, the Lydian and the Phrygian modes.[12] Besides these ideas, Boethius also followed the other *topoi* of Pythagoreanism and Platonism. He discussed the variants, like those of Ptolemy, and refused the opposite position of Aristoxenus.

Unlike human instrumental music, the music of the heavenly spheres did not arrive at man's ears. Nevertheless, the sound was defined as "... percussio aëris indissoluta usque ad auditum" ["... the connected striking of the air as far as the organ of hearing"]. Some movements were faster, others slower, some more frequent, others more spaced; this was the derivation, respectively, of the sounds that were more acute or deeper. "In quibus autem pluralitas differentiam facit, ea necesse est in quadam numerositate consistere." ["In which the multiplicity is different; this is necessarily composed of certain numbers"].[13] The ratios for music were thus

[11]Flamant 1977, pp. 355–356.

[12]Boethius 1867, pp. 179–180.

[13]Boethius 1867, pp. 189–190.

made of numbers; in this way, they distinguished between the various consonances and dissonances, as we have seen in the Greek tradition.

Boethius repeated, and established through the centuries, the legend of Pythagoras, who "... divino quodam nutu ..." ["... as a result of some divine will ..."] is said to have understood that the different sounds of hammers in a smithy depended on their different weights. Such indisputable authority made it superfluous for him to verify whether it was really the double weight that produced a sound an octave lower, and whether the other ratios of consonance were obtained by changing the weights proportionately. All the consonances were then fixed by the Roman numerals XII, VIIII, VIII, VI. Pythagoras is said to have investigated what "ratios corresponded to all the consonant relationships".[14] "Nunc quidem aequa pondera nervis aptans eorumque consonantias aure diiudicans, nunc vero in longitudine calamorum duplicitatem medietatemque restituens ceterasque proportiones aptans integerrimam fidem diversa experientia capiebat." ["He got the most genuine certainty by means of different experiments, at times adapting suitable weights to the sinews, and judging their consonances with the ear, and at times creating pipes twice or half as long, and preparing other ratios."][15]

There would be too many earthly sounds, because they form a *continuum*. "Continuae enim voci terminum humanus spiritus facit, ultra quem nulla ratione valet excedere." ["Thus the spirit of man places a limit to the continuous sound, beyond which it is not appropriate to go for any reason."] In order to explain how sound arrives at the ear, Boethius made a comparison with the ripples produced by a stone in water. Some codices also included a figure (Fig. 6.1) to describe the ratios of music.[16]

The belief that, according to this theory, the tone could not be divided into two equal parts was repeated several times. "Rursus tonus in aequa dividi non potest ..." ["Again, the tone cannot be divided into equal parts ..."]. "... quod tonum in gemina aequa diceremus non posse disiungi." ["... we will say that the tone cannot be broken down into two equal parts."] "... ut tonum sequioctavam facere proportionem [9:8] eumque in duo aequa dividi non posse, sicut nullam eiusdem generis proportionem, id est superparticularis [n+1:n];" ["... the tone is composed of the sesquieighth ratio, and it cannot be divided into two equal parts, seeing that no ratio of the same kind exists, that is to say, superparticular;"].[17]

Consequently, for this orthodox tradition, "Non est igitur diapason [octave] consonantia constans sex tonis, ut Aristoxenus arbitratur. Quod in numeris quoque dispositum evidenter apparet." ["The consonance of the diapason, therefore, is not composed of six tones, as Aristoxenus believes. That is clearly apparent also in the order of numbers."] Having defined the ratio of the *comma*, the quantity by which six tones exceed the octave, as DXXIIII.CCLXXXVIII:DXXXI.CCCCXLI

[14]Boethius 1867, pp. 197–198.

[15]Boethius 1867, p. 198.

[16]Boethius 1867, pp. 199–201.

[17]Boethius 1867, pp. 202–203 and 223.

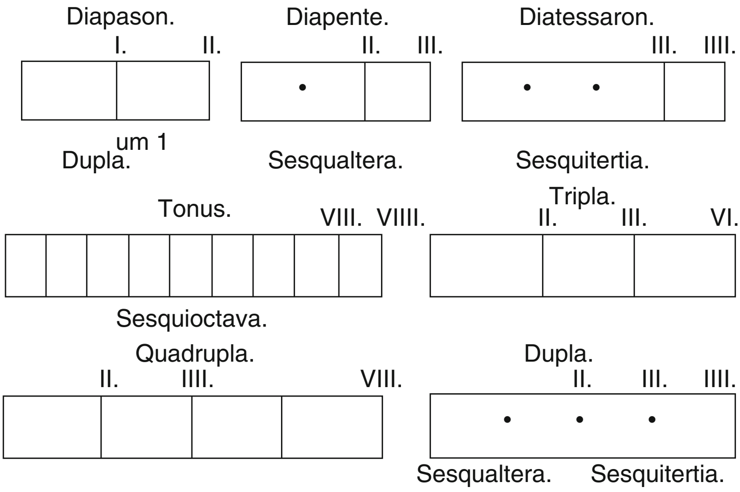

Fig. 6.1 The Roman numerals used by Boethius for musical intervals (Boetius 1867, p. 201)

[524,288:531,441], Boethius insisted: "Sed de his, quid Aristoxenus sentiat, qui auribus dedit omne iudicium, alias commemorabo. Nunc voluminis seriem fastidii vitator adstringam." ["But I will recall another time the beliefs of Aristoxenus, who entrusted all judgement to the ears. Now, to avoid tedium, I will limit the series of the volume."][18]

And yet our philosopher dedicated another two chapters, in Book 3, to the subject "Adversus Aristoxenum", and one to the proof of Archytas that a ratio of the kind 9:8, that is to say, superparticular, could not be divided into two equal parts. In the first chapter, he stated that the tone 18:16 could be separated into the ratios 18:17 and 17:16, but these are not the same. Thus he selected only rational numbers as legitimate, without mentioning them by name, though.[19] He always limited himself to these, when he again wrote that Aristoxenus was wrong in sustaining that the interval of the fourth was composed of two tones and one semitone, or that the octave was six tones. And he again indicated by how much they exceeded the octave, i.e. the value of the *comma*.[20]

For the demonstration that a superparticular ratio, such as 9:8, could not be divided into two equal ratios, he even appealed to Archytas: "Superparticularis proportio scindi in aequa medio proportionaliter interposito numero non potest. Id vero posterius firmiter demonstrabitur. Quam enim demonstrationem ponit Archytas, nimium fluxa est." ["The superparticular ratio cannot be broken up into equal parts by inserting a proportional number in the middle. This will proved

[18]Boethius 1867, pp. 265–267.

[19]Boethius 1867, pp. 268–270.

[20]Boethius 1867, pp. 273–275.

with certainty below, seeing that the proof offered by Archytas is too weak."] "... quoniam minimi in eadem proportione sola differunt unitate, quasi vero non etiam in multiplici proportione minimi eandem unitatis differentiam sortiantur, cum plures videamus esse multiplices praeter eos, qui in radicibus collocati sunt, inter quos medius terminus scindens aequaliter eandem proportionem possit aptare." ["... as minimal ratios in the same proportion differ only by one unit, as if minimal numbers did not share also proportionately the same difference of unit as multiples, when we see that there are many more multiples, excepting those expressed in roots, among which the median term may be adapted which divides the same ratio into equal parts."][21] This sentence appears to be obscure. Perhaps he meant that the proof can be based on a ratio in which the terms, like 9:8, differ by only one unit, when the same relationship could be written in many other ways with the multiples 18:16, 36:32, etc. What is, however, clear is that roots are excluded, even if Boethius knew, therefore, that they could successfully divide the tone into two equal parts. He later tries to justify this by resorting to consonances and geometry.

After thus criticising Archytas, Boethius for the umpteenth time calculated the *comma*: the difference between the two unequal parts (the *apotome*, or greater semitone, and the *diesis* the lesser semitone), in which the tone could be divided. In his opinion, that was the only legitimate way. Then he tried to calculate how many times those intervals contained the *comma*. But instead of considering them, as he had done so far, as ratios between numbers, he took the difference between the terms. Thus, instead of being represented by the two considerable figures, 524,288:531,441, the *comma* was reduced to a much more manageable VII.CLIII [7,153].

He proceeded in the same way with the terms of the other musical intervals, modified here for convenience in their proportions, obtaining, for the difference of the tone LVIIII.XLVIIII [59,049], for the difference of the *apotome* XXXIII.DCCLXXVII [33,777], and for that of the lesser semitone XXVI.DCXXIIII [26,624]. He undoubtedly facilitated the comparison between the intervals: the scholar thus rapidly concluded that the lesser semitone was more than three *commas* and less than four, that the *apotome* naturally became more than four and less than five, and lastly, that the tone contained more than eight, but less than nine. To arrive at this result, he simply multiplied 7,153 by 3, 4, 5, 8, and 9, and compared the product with the relative differences of the preceding musical intervals (Fig. 6.2).[22] For example, $7{,}153 \times 3 = 21{,}459$, which is less than 26,624.

The proof of Archytas using semitones may have been *fluxa* [weak], but this one of Boethius using *commas* seemed to be worse, because it was *errata* [mistaken]. In other words, seeing that the notes are fixed in the Pythagorean model by numbers arranged in a geometrical succession, and *not* in an arithmetic succession, adding or taking away an interval means multiplying or dividing the terms (appropriately), and not adding or subtracting. The differences between numbers in a geometrical

[21]Boethius 1867, pp. 285–286.

[22]Boethius 1867, pp. 291–299.

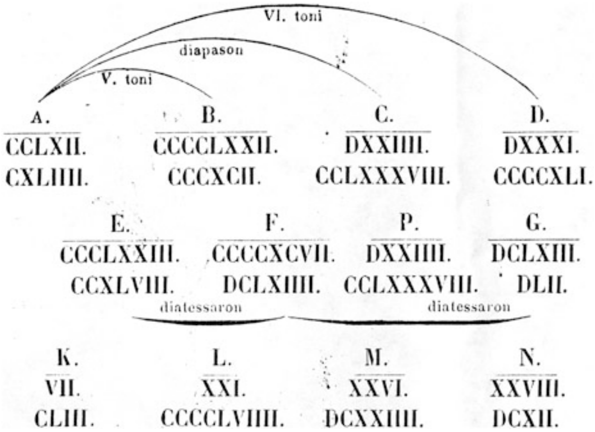

Fig. 6.2 How Boethius calculated the *commas* contained in the Pythagorean tone (ibid., p. 295)

succession increase proportionately, and do not remain the same. Boethius should have multiplied his fine Roman six-figure numerals nine times together, and not simply by nine, and then have compared the result.

In the tradition of Pythagorean arithmetic, so fond of whole numbers, the *comma* was making the theory of music uncomfortably asymmetrical, and registered the incompatibility between the fifth of 2:3 and the octave of 1:2. But could the *comma* at least have become, for music, that kind of basic unit, father/mother of the rest, as 1 was interpreted at the time for all the numbers? The treatises tried at the time to measure the other intervals by the *comma*, as the Greeks had done, and as Boethius was repeating. Yet in general, they only gave the results, without proving them adequately. Though the procedure in theory appeared to be clear, in practice it became almost impossible to follow without a good technique of multiplication, and considerable patience. The temptation to take easy short-cuts, which led to the same results, must have been irresistible. Thus we shall see the passage of more than 1,000 years, before we find a good, brief, elegant proof, without the need to perform all these calculations.[23]

At the end of Book III, Boethius proclaimed that he had proved what he had promised.[24] However, the search for a *solidus* [solid] argument about the impossibility of diving the tone into two equal parts remains vain. Here we find, rather, a clear case of how the arguments and reasons proposed, even in mathematical sciences, derive critically from general rules and principles, often accepted tacitly, as they are widely believed and taken for granted. Actually, the presumed proof depended on what number was considered to be at the time, and what relationship was thought to exist between the different disciplines under study.

[23]Tonietti 2006b, pp. 153–156; see below, Sect. 6.5, and Appendix C.

[24]Boethius 1867, p. 300.

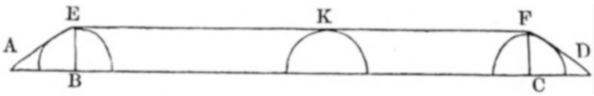

Fig. 6.3 The monochord of Boethius (ibid., p. 349)

In the pattern of the *quadrivium*, music followed arithmetic, and the latter dealt with discrete quantities, that is to say, whole numbers and fractions. Continuous quantities, like incommensurable magnitudes and roots, were considered to be parts of geometry. Nobody could have taken the liberty of using them for music without departing from orthodox theory. Thus remaining exclusively inside these limits, it would be impossible to divide the tone into two equal parts: because the proportional mean between 8 and 9 is an incommensurable magnitude, which can be measured only by means of the (irrational) square root of 2.

While the tone could be safely divided into two equal parts for heretics like Aristoxenus, or for those who admitted $\sqrt{2}$ among the numbers, orthodox scholars were using whole numbers to deny the heretics this possibility that they claimed. The grotesque aspect was that in this period, orthodox Pythagoreans were not even capable of expressing the question in correct mathematical terms. But right this event made the nature of the conflict even clearer. Not that this musical heresy was really *impossibilis more mathematico* [impossible according to mathematics], but consequently, it appeared to be *prohibita* [prohibited]. Many centuries were to pass, and other events are to be told, before the prohibition was allowed to drop.

In that period, the reasons of orthodox Pythagoreans were sustained by a series of affirmations, which Boethius dutifully noted down. Having reduced sounds to the numbers of ratios, depending on whether these were either multiples or superparticular, the note of music were as a result heard to be consonant in some cases, and in others dissonant. Consonances were then those notes "... quae simul pulsae suavem permixtumque inter se coniungunt sonum." ["... which, struck at the same time, mingle together in a pleasant, integrated sound"].[25] Boethius took pains to repeat to us the well-known musical ratios of the Greeks, calling the notes by both their Greek and their Latin names. He distributed their scales among the three genres, diatonic, chromatic and enharmonic, derived from the relative divisions of the monochord in accordance with the respective tetrachords.

In order that the ears could judge consonances with certainty, he explained how to construct a monochord with *regula* [ruler], *magadas* [hemispheres] and *nervus* [sinew]. The various vibrating lengths of the string, necessary to generate the different notes, were obtained by shifting the central hemisphere (Fig. 6.3).[26]

After stating that something confused arrived from the ears, whereas the *ratio* [reason] could judge the purity of them, perceiving even the smallest differences,

[25] Boethius 1867, pp. 301–302.

[26] Boethius 1867, pp. 348–349.

Boethius concluded that it was impossible to trust the ears completely. This *ratio* guided and corrected the erring senses. And yet, there was considerable disagreement among theoreticians about the harmonic rules. The followers of the Pythagoreans preferred reason, while Aristoxenus, on the contrary, judged through the senses. Lastly, to Ptolemy did not seem that harmony could be contrary neither to the senses nor to reason. We know that he tried to reconcile both together. But then the latter, like the Pythagoreans, placed the differences between sounds in the quantity, that is to say, in whole numbers, while Aristoxenus was left alone to decide on the basis of quality.[27]

In the arguments offered, the following dualistic opposition assumed greater mathematical significance. Like the colours of the rainbow, for which it cannot be said with certainty where they finish, if sounds for music formed a continuous spectrum, whether deep or acute, they would not have a designated place which would fix them precisely. Only discrete sounds, like distinct, unmixed colours, occupy an exact, fixed place. "Continuae quidem non aequisonae voces ab armonica [sic!] facultate separantur. Sunt enim sibi ipsis dissimiles nec unum aliquid personantes. Discretae vero voces armonicae [sic!] subiciuntur arti." ["Then the continuous notes, not in unison, are far from the possibility of harmony. They are different from themselves, and are not capable of sounding in some unit. Indeed discrete notes are subject to the art of harmony."][28] On this basis, both the Pythagoreans and Ptolemy had established the consonances, even though the former excluded one like the octave united to the fourth (as its ratio 8:3 is not superparticular), which was admitted by the latter.

The conclusions of Boethius are all against Aristoxenus, who did not use numbers, "... sed aurium iudicio permitit ..." ["... but allowed the ears to judge ..."].[29] Now taking his inspiration from Ptolemy, our medieval philosopher believed that he could in this case represent the error of Aristoxenus more convincingly (Fig. 6.4).

The strings represent the octave and its division into the six Pythagorean tones. But to us this appears to be just a translation into geometrical segments of previous calculations with ratios between numbers. Of course the string long GP (the six tones) would sound more acute, because it is shorter than the octave HR.[30] But why prevent Aristoxenus from using six other strings of different lengths 'equally' distributed between AK and HR? Nobody could make any geometrical objections to this operation, because these lengths exist, and can be constructed relatively easily, even if they are naturally incommensurable magnitudes that can only be measured by means of roots. The argument of Boethius based on the figure thus became double-edged, and was reduced to the affirmation that continuous magnitudes did

[27]Boethius 1867, pp. 352–355.

[28]Boethius 1867, pp. 356–357.

[29]Boethius 1867, p. 363.

[30]Boethius 1867, p. 364.

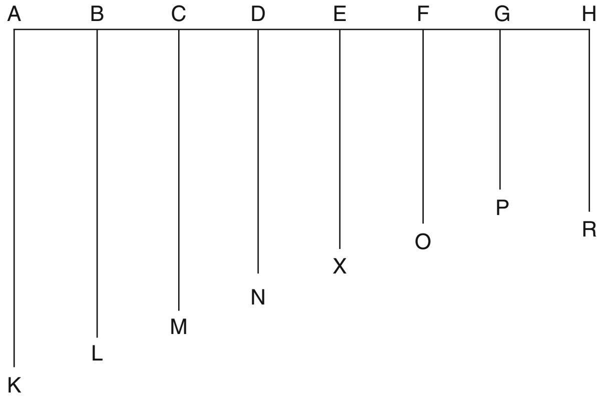

Fig. 6.4 How Boethius proved that the octave sounded deeper than six tones (ibid., p. 364)

not produce consonances. Was he making a prophecy, rather, about the music of the twentieth century?[31]

Aristoxenus went so far as to make divisions into quarters of a tone, which he distributed in the genres like the enharmonic *mollius* [softer, more effete] and the diatonic *incitatius* [more stinging, stimulating]. The former was also called *spissum* [slow]. Boethius reported that Aristoxenus had been blamed by Ptolemy for proposing divisions that could not be perceived by the hearing. In fact, the latter seemed to be his favourite theoretician of music.[32] After being converted to Christianity, he translated some books by Aristotle and the NeoPlatonics into Latin, trying to reconcile them, as they would have done in Arabic countries. With him, the concept became famous that distinguished music into three species. The highest, *mundana*, was the music of the spheres, generated by heavenly bodies and not discernible with the ears, but only with the mind and the soul. The intermediate, *humana*, species descended inside us. Whereas the *instrumentalis* music appeared inferior to him because it was created by the hands of players of instruments.

Whether they were appreciated or not, the orthodox theories of music were becoming a stiff, empty slough that prevented growth or change. The characters like Boethius are interesting because they are capable of showing us the dominant opinions of the period, but they do not tell us much that is new. Five centuries later, on the contrary, to cut a long story short, a novelty arrived which was to produce a profound change in musical activity, in the direct practice of composers and musicians. Guido D'Arezzo (early 1000–c.1050), the famous monk, re-elaborated, perfected and modified previous ideas in a form of notation that was destined to establish itself in view of its effectiveness in the writing of music, which up to that time had essentially been limited to memory and oral transmission. As we are

[31]Tonietti 2004. See Part II, Sect. 12.4.

[32]Boethius 1867, pp. 365–366 and 370–371.

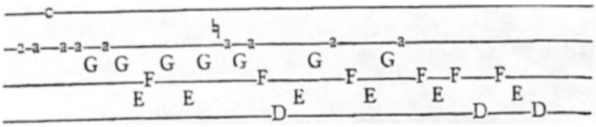

Fig. 6.5 How Guido D'Arezzo wrote the notes of music on the lines (Guido D'Arezzo 1963, p. 10)

narrating, we know quite a lot about the musical theories of the Greeks and of the Arabs, but their ancient music is largely unknown to us, and can only be described by means of words and with a great effort of imagination.

How was the tempo to be indicated? For music, it is necessary to write not only the height of the notes, but also the way in which they run one after the other, creating the tempo. Gradually and slowly, the modern stave was now born. From Fig. 6.2, we can see that Boethius indicated the notes by means of the letters A, B, C, D, ... The Latin letters were to go on being used in the Anglo-Saxon musical world. Whereas the Latins, following Guido, were to adopt new names, such as *ut* (subsequently *do*), *re, mi, fa, sol, la, [si]*, taking the first syllables of a hymn dedicated to **s***[ancte]* **i***[oannes]*: **Ut** *queant laxis* **re***sonare fibris* ... [In order to be able to resound with relaxed strings ...][33] The hymn-book presented to Pope John XIX and the *Micrologus* contained the most famous novelty, destined to spread all over the world. The succession of notes was represented on parallel lines read from left to right (Fig. 6.5).[34]

Here, time became space; it transformed itself into a geometrical line capable of stringing together like pearls the heights of the sounds, one after the other, in a regular, rhythmic succession. At last, it was possible to put down on paper even the most important characteristic of music, the most difficult one to seize: *musica sine lineis est sicut puteus sine fune* [music without lines is like a well without a rope].[35] Guido was interested in teaching young people how to sing. His notations are useful and intuitive to succeed in intoning the notes and remembering them. If *A* indicates a note (*la*), *a* indicates the octave above it (another *la*). Three tones one after the other, as there are between *fa* and *si*, are difficult to sing, because they sound rough, since they form one of the hardest dissonances. Actually this tritone was called the *diabolus* [devil] *in musica*. Then the note creating the roughness, *si* natural, was represented with the square symbol ♮, while between this note and *la*, another round note was introduced, indicated by *b* (♭), would sweeten the interval *fa* − *b*, making it now equal to the fourth *do* − *fa*.

Our monk recognized that music was appreciated thanks to its variety, like colours for the sight, smells for the sense of smell, or tastes for the sense of taste.

[33]Guido D'Arezzo 1963, p. 45.

[34]Guido D'Arezzo 1963, p. 10.

[35]Guido D'Arezzo 1963, p. 31.

It possessed an enormous power. By singing, the doctor Asclepias had healed a madman. “Et item alius quidem citharae suavitate in tantam libidinem incitatus, ut cubiculum puellae quaereret effringere dementatus: moxque citharoedo mutante modum voluptatis poenitentia ductum recessisse confusum.” [“And similarly, a certain other person was excited to such a state of lust by the sweetness of the lyre that, like a madman, he tried to break into a girl’s bedroom: but immediately, as the citharist had changed the voluptuous mode, he was brought to penance again, and withdrew, upset.”][36]

The theory of music followed by Guido was the one that started with the legend about the hammers that Pythagoras heard, beating out the notes. From this, the ratios contained in the numbers XII, VIIII, VIII, VI were obtained. Subsequently, Boethius was to explain, and to demonstrate that these were the substance of the wonderful and difficult harmony of this art. With those notes, Pythagoras was to put together the monochord. “… in quo quia non est lasciva, sed diligenter aperta artis notitia, sapientibus in commune placuit, atque usque in hunc diem ars paulatim crescendo invaluit, …” [“… in which the knowledge of the art was generally appreciated by wise people, because it is not lascivious, but crystalline if well considered, and furthermore, up to this day, by growing slowly,the art acquired value, …”].[37]

The Church of Rome has always shown a great interest in music, and was particularly careful that it was practised in accordance with its rules. In 341, the Council of Antioch against Arius, who denied Christ’s deity, even defined a musical agreement between the three persons of the Trinity, Father, Son and Holy Spirit, referring to it in the Creed as a $\sigma\upsilon\mu\phi\omega\nu\iota\alpha$ [sounds in harmony]. Even those who were most worried about controlling the direct exercise of music in liturgical chants, like Guido D’Arezzo, ended up by falling back, with Boethius, into the group of Pythagoreans. A few centuries were to pass, and another historical context was to develop, before Aristoxenus returned to the scene, receiving a certain consideration. And then the mathematical sciences were to be invited to change to a great extent, as we now go on to relate.

6.3 Facing the Indians and the Arabs: Leonardo da Pisa

In the preceding section, we have seen that numbers were written with the Roman symbols which are no longer used today, except in rare, or particular, cases. Even though he was not the very first, the greatest contribution to the spread of Indo-Arabian numbers in Europe came from Leonardo, who was born at Pisa, and belonged to the Fibonacci family, which descended from the late (fu) Bonaccio. Leonardo da Pisa, or Leonardo Fibonacci (perhaps 1170–c. 1250), was a member of the merchant class, and operated in North Africa; his father, in particular, kept

[36]Guido D’Arezzo 1963, p. 14.

[37]Guido D’Arezzo 1963, p. 24.

the official accounts on behalf of the Republic of Pisa at Bugia (nowadays Bejaia, in Algeria).

"… inspecta utilitate et commoditate futura, ibi me studio abbaci per aliquot dies stare voluit et doceri. Ubi ex mirabili magisterio in arte per novem figuras Indorum introductus, scientia artis in tantum mihi pre ceteris placuit, et intellexi ad illam, quod quicquid studebatur ex ea apud Egyptum, Syriam, Greciam, Siciliam et Provinciam cum suis variis modis, ad que loca negotiationis postea peragravi per multum studium et disputationis didici conflictum. Sed hoc totum etiam algorismum atque arcus Pictagore quasi errorem computavi respectu modi Indorum. Quare amplectens strictius ipsum modorum Indorum, et attentius studens in eo, ex proprio sensu quedam addens, et quedam etiam ex subtilitatibus Euclidis geometrice artis apponens, summa huius libri, quam intelligibilius potui, in XV capitulis distinctam componere laboravi, fere omnia que inserui certa probatione ostendens, ut extra, perfecto pre ceteris modo, hanc scientia appetens instruantur, et gens Latina de cetero, sicut hactenus, absque illa minime inveniatur." ["… here, in view of its future usefulness and convenience, I would stay and learn for several days the abacus. There, following my introduction, as a consequence of marvellous instruction in the art, to the nine digits of the Indians, the knowledge of the art very much appealed to me before all others, and for this, I realized that all its aspects were studied in Egypt, Syria, Greece, Sicily, and Provence, with their varying methods; and at these places thereafter, while on business, I pursued my study in depth, and learned the give-and-take of disputation. But all these algorithms, as well as the bows of Pythagoras [abacus], I considered as almost a mistake, with respect to the method of the Indians.[38] Therefore, embracing more stringently that method of the Indians, and taking stricter pains in its study, while adding certain things from my own understanding, and inserting also certain things from the niceties of Euclid's geometrical art, I have striven to compose this book in its entirety as intelligibly as I could, dividing it into fifteen chapters. Almost everything which I have introduced I have displayed with exact proof, in order that those further seeking this knowledge, with its pre-eminent method, might be instructed, and further, in order that the Latin people might not be discovered to be without it, as they have been up to now."][39]

"Almost a mistake", compared with the way of calculating of the Indians, was what he called the others, which had become the subject of disputes and were spread over the countries that he had visited. Thus, in 1202, he wrote the *Liber Abbaci [Book of the abacus]* to teach the Latin people the numbers of the Indians, and their ability to calculate, with the improvements of the "niceties of Euclid". "Novem figure indorum he sunt 987654321. Cum his itaque novem figuris, et cum hoc signo 0, quod arabice zephirum appellatur, scribitur quilibet numerus, ut inferius demonstratur." ["These 987654321 are the nine figures of the Indians. And thus any number is written with these nine figures and with the sign 0, which in Arabic is called 'zephirum' [void, zero]"]. The first number represents the units, the second

[38]Cf. the interpretation of André Allard in Fibonacci 1994, p. 85.

[39]Fibonacci 1994, p. 15.

the tens, the third the hundreds and so on, each is the decuple of the previous one. Be careful, though. "Primus gradus in descriptione numerorum incipit a destera." ["The first level in the sequence begins on the right"]. Thus the number is read from right to left, like Arabic writing. "Cum quattuor namque a mille usque in decem milia, ut in sequenti cum figuris numeris super notatis ostenditur." ["With four [figures] from one thousand to ten thousand, as is shown in the following numbers indicated by the previous symbols."] "MI 1001, MMXXIII 2023, ..., MMMMCCCXXI 4321."[40]

The book presented problems to be solved, for which appropriate rules were given, without presenting them separately in an abstract form. The only symbology adopted (also subsequently) was the line between the numbers of fractions. In Chap. XV, "De questionibus aliebre et almuchabale" [On problems of algebra and almucabala], equations were written with the words *census* for the square root of the unknown and *radix* for the unknown.

He wrote in Latin, so that he would be read not only by the merchants of his own country, but also by the educated classes of various other lands. In time, the *Liber Abbaci* was to inspire numerous manuals in the vernacular (Italian) called "Commercial practice", and naturally, the books for the "Abacus schools", where, in spite of the name, the counting-frame had been abandoned, and the teaching of calculation followed the new method, using pen and paper. Together with the dominant problems of trading, with exchanges, currencies, partnerships, interest, usury, ... Chap. XII contained one which today would seem to be more suitable for farmers.

"Quidam posuit unum par cuniculorum in quodam loco, [...], ut sciret quot ex eo paria germinarentur in uno anno: cum natura eorum sit per singulum mensem aliud par germinare; et in secundo mense ab eorum nativitate germinant........"[41] ["A certain person put a couple of rabbits in a certain place, [...], to discover how many would be born from them in one year: if their nature were to give birth to another one every single month; but they [only] reproduce starting from the second month after their birth. As the above-said couple reproduce in the first month, they will double, and in one month there will be two couples. One of these, that is to say, the first couple, reproduce [also] in the second month; and thus there will be 3 couples in the second month; of these, in one [more] month, two will remain pregnant; and in the third month, 2 couples of rabbits will be born; and thus in this month there are 5 couples; of which 3 couples will remain pregnant in the same month; and in the fourth month, there are 8 couples; ... if we add to these also the 144 couples that are born in the last month, there will be 377 couples; and this is the number of couples that the aforesaid couple will have generated in the said place after one year. In this limit, it may be seen how we have worked with this, that is to say, we have added the first number to the second, i.e. 1 to 2; and the second to the third; and the third to the fourth; and the fourth to the fifth, and so on, until we

[40]Fibonacci, *Liber Abbaci*, Chap. I.

[41]Fibonacci 1994, p. 209.

have added the tenth to the eleventh, i.e. 144 to 233 and we have obtained the sum of the desired rabbits, that is to say, 377; and thus it is possible to proceed through the infinitely numerous series of months".][42]

In modern symbols, the number N_{m+1} of couples present in the month following m is obtained by summing the number of couples present in the month N_m, that is to say, the total number of the month, with the pregnant couples, namely the number of couples present in the previous month N_{m-1}, in the hypothesis that they take 1 month to become sexually mature.

$$N_{m+1} = N_m + N_{m-1}$$

The succession of whole numbers defined by this relationship was to be called "Fibonacci's": $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots, 144, 233, 377, \ldots$ In this way, Leonardo from Pisa tried to foresee the number of couples of rabbits by means of a numerical calculation. He did not consider either the arithmetic or the geometrical succession. Rather, he had placed limits deriving from considerations of a biological nature: the duration of rabbits' pregnancies and the time in which they reach sexual maturity. He had arrived at a new law of growth, never considered by anybody before him, by reasoning on a case that was 'not very noble' like the mating of rabbits. The stars in the sky and music had already inspired some mathematical models, as we have seen, but now these were flanked by another one, generated by animals of the earth. Would more successful results come from down here, or from on high? In their diversity, was there any contrast or competition between them?

The *Liber Abbaci* did not deal with music, nor did *Practica geometriae [The practice of geometry]*. And yet we shall see, in Sect. 8.3. of Part II, that Kepler was to prove to be capable of imagining relationships even between the sexes, musical notes, the golden section (which in the meantime had become the divine proportion) and Fibonacci's succession. Depending on its modes and conceptions, seeing that it was suspended between heaven and earth, music could be made to ascend coldly on high among the absolute ideas, or to come down beneficial to the depths, among the sensibilities of life. In fact, this makes it a little more bearable for us.[43]

On the contrary, the problems of the *Liber Abbaci* escaped from the classical Greek *quadrivium* then dominant, and addressed us common mortals. As they were used to perform calculations, and no longer to debate an abstract theory, these numbers of Leonardo da Pisa endangered the Pythagorean patterns: fractions appeared, square roots moved at least slightly further away from geometry, to draw timidly closer to arithmetic. "Radix quidem cuiuslibet numeri est numerus qui, cum in se multiplicatur, facit ipsum numerum. [. . .] Nam quidem numeri habent radices, et vocatur quadrati: et quidam non; quorum radices, que surde dicuntur, cum impossibile sit eas in numeris inveniri, qualiter in quantum plus possumus fere, ad

[42]Tonietti 1999b, pp. 215–216.

[43]"Ohne Musik wäre mir das Leben ein Irrtum, eine Strapaze, ein Exil" ["Without music, life would be an error, a strain, an exile"] Nietzsche 1888.

ipsa radices venire demonstravimus. [...] Secundum vero geometriam, non numero, sed mensura, radix cuiuslibet numeri invenitur".[44] ["To be precise, the square root of any number is a number which, multiplied by itself, gives that same number. [...] Because, clearly, [some] numbers have roots, and they are called squares: but certain others do not; the roots of these [are] called surds, when it is impossible to find them using [whole] numbers, as we have shown that we can approximately draw as near as possible to these roots. [...] In truth, according to geometry, the roots of any number can be found, not by means of numbers, but with measurements."]

Even if Leonardo Fibonacci here affirmed the traditional idea of numbers, which excluded irrationals, subsequently, when he went on to operate with roots, he employed different terminologies, such as "... ratiocinatum et inratiocinatum numerum ... per numeros ratiocinatus et inratiocinatus ..."[45], which we find no difficulty in translating into our modern "rational and irrational numbers". Thus square roots were gradually becoming numbers that could be used, like the others, and not just represented as geometrical magnitudes. Based on how he calculated the approximate value for the solution of a third-degree equation, he was going beyond the limits placed by the Pythagorean sects and Euclid.[46]

The Pisan scholar and merchant had thus introduced into Europe even the move to supersede Greek tradition as regards irrationals, which the Arabs had started to realise. Relations changed between geometry and arithmetic. "Et que arismetrica et geometria scientia sunt connexe, et suffragatorie sibi ad invicem, ..."[47] ["Which sciences of arithmetic and geometry are intertwined, and sustain each other, ..."]. Thus a new attitude appears here, with respect to the recurring use of geometry to demonstrate the results of algebra for a certainty with the relative subjection of the latter to the former, raising the prospect even of an inversion. Now that the submission also to the god of the Arabs had lost importance, in the direction which was most profitable for the merchants, what new results would be obtained? Though we shall see that Euclid's prestige, as the source of every authoritative argument, was to remain very rarely debated for other centuries, until the famous seventeenth one.

Leonardo da Pisa was not the only one who introduced the novelties of the Arabs into Europe. Together with other contributions, *algorithms*[48] were spreading, thanks to the considerable advantages they offered in calculations. But the cultural, religious and social grounds were full of obstacles. With the direct return of Greek tradition during the Renaissance, the problems of the mathematical sciences were to come to maturation.

> Also Epistemone bought a picture, in which were represented to the life the ideas of Plato and the atoms of Epicurus. Another, bought by Rizotomo, represented Echo *au naturel*.
>
> François Rabelais

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[44]Fibonacci 1994, p. 105.

[45]Fibonacci 1994, p. 106.

[46]Boyer 1990, p. 299.

[47]Fibonacci, "Prologus" of the *Liber Abbaci*.

[48]Fibonacci 1994, p. 84.

6.4 Constructing, Drawing, Calculating: Leon Battista Alberti, Piero della Francesca, Luca Pacioli, Leonardo da Vinci

At variance with the teaching of the *quadrivium*, which was addressed in the Middle Ages to the formation of the learned, together, in general, with the *trivium*, above all for theological, juridical and literary questions, the abacus schools usually ignored music.[49] With the current of Boethius, this was to continue, in the fifteenth century, within the general renewal, nourished by the newly-discovered Greek and Latin texts, which we conventionally continue to call the Renaissance. To cut a long story short, seeing that I am where I am, we may start from the books written by the famous architects and painters of the period. For their profession, it was natural to try to fix rules for proportions, harmony and beauty, in Latin *concinnitas*. Pythagoreanism and Platonism reappeared everywhere, but at last we will even find a rival school for music, in the general mingling again of ideas.

Leon Battista Alberti (1404–1472) expressed himself as follows in Book IX of his *De re aedificatoria [On building]*: “Clearly, those same numbers which make the harmony of voices seem very pleasant to the ears of men, are the same ones that also fill the eyes and the spirit with marvellous pleasure. Thus we will obtain the whole rule of completion [the ratio between magnitudes] from musicians, to whom these numbers are perfectly well known: . . . ”. When the master-builders and executors of his projects did not follow his instructions, he complained because “. . . there is a discord in all that music” that the architect had carefully put into the construction.[50]

He thus adopted the ratios of music in his drawing of the plans: fifths 4:6:9, fourths 9:12:16, the octave 1:2:4 and the tone 8:9. Linked with these, he used arithmetic, geometrical and harmonic means. He received from the most classical antiquity his natural source, the *De Architectura libri dece [Ten books on architecture]* by Marcus Vitruvius Pollio (first century B.C.), who had also dealt with musical harmony in Book IX. In various editions and translations of the sixteenth century, the work met with great success. Variously justified by the rediscovery of classical Graeco-Roman culture, the search for harmony and proportion in the things of the world became a general characteristic of the age. For this reason, according to the interpretation given in the Pythagorean sects, music enjoyed a particular fortune.

Also Andrea Palladio (1508–1580), among others, wrote in his *I quattro libri dell’architettura* [The four books of architecture] that the best ratios between the length and the width of rooms were the traditional Pythagorean ratios encountered

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[49]No trace of music is found in the first 16 Books of the Centre for the Study of Medieval Mathematics in Siena. Not even Paolo Dell’Abaco (c. 1281–1374) in his *Pratricha d’astorlogia* [practice of astrology] put the stars in a relationship with the notes, as was tradition. This diversity, becoming general and significant, might then characterise the abacus schools.

[50]Quoted in Borsi 1966, p. 97, from a translation into Italian of 1823.

several times for notes. For him, even the heights, in relation to the plane dimensions, had to follow the arithmetic, geometrical and harmonic means. Instead, he excluded $\sqrt{2}$:1 from these rules, because it was irrational. Wasn't the desire to avoid this particular ratio, which geometry and the design of magnitudes handled, on the contrary, without any particular problems, an indication that he was guided by music, which ignored it because it was compulsorily considered to be a dissonance?[51]

Among the different followers of Platonism, we choose, on the basis of his importance, Marsilio Ficino (1433–1499), who assigned music a significant place in Book III of his "De vita coelitus comparanda" ["How to obtain life from heaven"[52]] of the *De vita libri tres [Three books on life]*. "... singing is the most powerful imitator of all things. [...] the matter of singing is more pure and much more similar to heaven than the matter of a medicine. [...] Singing, then, full of spirit and sense, ..., corresponds to this or that star, ..., and transfers it into the singer, and from him into the person who listen to him closely; [...] by means of the song, it acts powerfully on his own body, ... spreading, it moves that of the next person immediately afterwards; [...] an admirable force is present in an excited spirit that sings, if you have granted to the Pythagoreans and the Platonics that heaven is a spirit that arranges all things with its movements and its tones. [It gives] ... to Saturn a slow, deep, raucous, querulous voice; to Mars, instead, a voice with the opposite characteristics; fast, acute, sharp and threatening, with intermediate qualities, then, for the Moon. To Jupiter, then, deep, severe, sweet, and constantly joyful songs. To Venus, on the contrary, we attribute voluptuous songs, full of lasciviousness and softness. To the Sun and Mercury, however, we attribute songs with intermediate qualities. [...] You, then, will conciliate every one of the four planets with the songs that are typical of each, above all if you add appropriate sounds to the songs. To the point that, when you have called them singing and playing in accordance with their custom, in the appropriate mode and tempo, they will be, as it seems, immediately ready to answer, either like an echo, or like a string that vibrates in a lyre, every time that another string with a similar tension vibrates."

For the celebrated humanist and doctor from Tuscany, certain peoples would be more sensitive than others to the influence of the planets, through the relative music. Then, warmed by the sun of Phoebus Apollo, the shining one, "... many inhabitants of the eastern and southern regions, above all the Indians, have a wonderful power in words, because they are largely solar. [...] those who in Apulia are touched by the tarantula are taken by wonder, and lie half-dead, until each one hears his determined sound. And then he starts dancing, following that sound, he sweats and then starts to get better. And if he hears a similar sound after ten years, he will immediately be

[51] Quoted in Borsi 1966, p. 106. Francesco Giorgi (1466–1540) followed musical ratios for the project of San Francesco della Vigna in Venice, Walker 1989a, pp. 72–73.

[52] Which may also be translated as "On life in a relationship with heaven".

driven to dance. In truth, on the basis of a series of pieces of evidence, I imagine that sound is Phoebean and jovial".[53]

In view of his ability both as a doctor and as a philosopher, Ficino also took his inspiration from Arabic literature, in particular Ibn Sina-Avicenne. Naturally, within the scope of his striking openness to other, faraway cultures, he still conserved some typically Platonic, and now Christian, sides about the sins of the flesh and of Venus. Among the things that prevented scholars from exercising their intelligence properly, he included sexual activity. "However, The first monster is the coitus to which Venus pushes us, above all if this goes, even slightly, beyond our forces. … And nothing more than this evil can be harmful to the mind. … And it is so damaging that Avicenne, in his book *On animals*, wrote: 'If, during sexual intercourse, a man produces more sperm that nature can suffer, this may be more harmful to him than if he lost a quantity of blood forty times greater.' " Following Venus, they would have been reduced to "an ancient body of a cicada by now worn out". "… they were struck by pleasure to such an extent that, continuing to sing, they ignore food and drink, and without realising it, they died". Thus Plato narrated the transformation into cicadas of the men seduced by the singing of the Muses.[54]

Ibn Sina, who had by now become Avicenne in the translations into Latin, was continually quoted, second only to Plato. When our doctor departed from the most precise translations, the Arabs, the "Bracmans" [Brahmans], the Indian philosophers, together with the Pythagoreans, the Egyptians, the Chaldeans and the Magi appeared to him the source of an extremely ancient wisdom to be recovered, which with him was becoming confused in the mists of exotic Oriental myths. For this reason, it has been written that one of the works he repeatedly read was the *Ghayat al-Hakim [The aim of the sage]*, translated as *Picatrix latinus* and attributed to al-Magriti.[55]

However, at least in the case of music, Ficino remained capable of obtaining effects that were also tangible. "Mercury, Pythagoras, Plato prescribe that the confuse or saddened spirit should be calmed and comforted with the sound of the lyre and with sweet, harmonious songs. Then, David, the sacred poet, freed Saul from his folly with the psaltery and the Psalms. I, too, if I may be allowed to compare the lowermost with the supreme, often experience at home how effective the sweetness of the lyre and singing can be against the bitterness of black bile."[56] Though mentioned here and there, on the subject of heavenly harmonies, our

[53]Ficino 1995, pp. 271–274. The present writer, in about 1980, saw with his own eyes "tarantula-bitten" women dancing at Galatina (province of Lecce, South of Italy). Though by now masked by the white robes of a Christian saint, like Saint Paul, in spite of the positive experiments with chemical fertilisers and their poisons in agriculture had killed it, the culture of a Greek god was still felt. But in the land that still today uses ancient *grico* as its dialect, the dance induced by the tarantula's bite and the rhythmic music (also known as *pizzica* [sting], and similar to the *tarantella*, derived from the iambic trimeter) to cure it seem to be guided more by Dionysus than by Apollo.

[54]Ficino 1995, pp. 164–165.

[55]Ficino 1995, passim; Nasr 1977, p. 44.

[56]Ficino 1995, p. 118.

humanist from Tuscany never quote Boethius directly. His main source perhaps may therefore have been the Ptolemy of *Harmonicorum sive de musica libri tres [On harmonies, or three books on music]*. "... for this reason, at the point where Ptolemy speaks about consonance, he says that, more than all the other heavenly bodies, Jupiter is in perfect symphony with the Sun, and Venus with the Moon."[57] Anyway, as in the Pythagorean tradition, he affirmed that medicine and music were closely connected together.[58]

After taking holy orders as a priest, our scholar and translator of Plato and Plotinus tried to make acceptable his desire to mix, in the Florence of the Medici and Savonarola, the wine of Bacchus with the milk of the Holy Roman Church, sweetened by Oriental honey. Thus we see him searching for a common denominator between Plato and Aristotle, between Titus Lucretius Carus and Saint Thomas Aquinas, between Greece and Persia, between Florence and Rome, between astrology and Christianity, between the body and the soul, between earth and heaven. In the general mingling again of values during the Renaissance, he would have liked to keep everything together. Did he succeed? Was music sufficient to demonstrate the harmony of the cosmos? Was it really a help to avoid touching the fresh cheeks of the beautiful Florentine ladies, in order to obtain a life which, though not eternal, would at least be long? Greek, Arabic, Latin and Indian writings (the Chinese were absent) would seem to offer him some possibilities. For a while, an elect multitude of scholars were to do likewise, using a large variety of means and arguments. His *De vita* enjoyed great success, with several editions, reprints and translations, as well as developments.[59]

But in the seventeenth century, other natural philosophers were to abandon these all-inclusive projects, to follow different methods: distinguishing between the things to study, instead of looking for the links; exalting dualisms, in order to reduce them by means of positive sciences, instead of integrating them into the *anima mundi* [soul of the world]. Reasoning by analogy was to be considered particularly doubtful, compared with the certainties attributed to the deductions of the mathematical sciences. Modern scientific orthodoxy was to be revised within a couple of centuries, condemning the ideas of the Florentine humanist in the end as heretical eccentricities without any future. For the moment, in 1490, the Roman Curia tried to put him on trial, accusing him of wizardry, and of course, of trading with the devil, in spite of all his caution and his apologies. Under the protection of Cosimo de' Medici, Ficino inspired a kind of Platonic Academy. A pupil of Leon Battista Alberti even imagined a dialogue between the architect and the Christian humanist, at Camaldoli.

We shall not follow the developments of all this towards the *De occulta philosophia [On occult philosophy]* by Agrippa von Nettesheim (1486–1535)[60] or

[57]The treatise will be printed at Venice in 1562. Ficino 1995, p. 201.

[58]Ficino 1995, p. 95. He also wrote, in 1484, a "De rationibus musicae"; Massera 1977, p. 27.

[59]Ficino 1995, pp. 68–71. Cf. Walker 1989b.

[60]*La magia naturale nel Rinascimento* 1989.

the *De arte cabalistica [The cabbalistic art]* by Johannes Reuchlin (1455–1522).[61] The battlefield on this topic already appears to be very crowded, and I am not at all attracted by the idea of delivering or parrying lance-thrusts in favour of, or against, the usual rationalities, irrationalities, astronomy, astrology, chemistry, alchemy, science, magic and so on, from a dualistic opposition to another. I leave the whole field to those who still take pleasure in some philosophy of progress or of history. Instead of moving immediately to the cold, impenetrable mist of the North, suspended on high, let us stay for a while in the Tuscan countryside illuminated by the noonday sun and return to visit the artisan workshops. The next characters to consider worked in those of Borgo Sansepolcro, near Arezzo, and of Florence.

We will not find any music in Piero della Francesca (c. 1420–1492), or in Friar Luca Pacioli (c. 1440–1517), or in Leonardo da Vinci (1452–1519). We can understand why, because they came from the tradition of the abacus, which no longer followed the *quadrivium*. However, we must not complain about two of them, because they succeeded in filling our lives with harmony, if not our ears, in truth our eyes. We are sorry that the previous two northern scholars avoided it. Through the notes of music, their philosophies, as had happened to the Pythagoreans, would have been clearer, and their arts more comprehensible to common mortals.

In any case, we witness a general search outside Europe for new elements through which to reinterpret the past in order to construct a different future. Pieces of Arabic, Jewish, Persian and Indian culture were juxtaposed and mixed with the fragments recovered from the Greek and Latin heritage. However, everyone recombined them to form different pictures, following their own inclinations and their own linguistic and technical abilities. In the heat of the argument, they easily surrendered to the temptation to believe that anything they desired was real. Pythagoras increasingly appeared to be hidden in myth, and scholars insisted on following even those sentences of his which were worse than doubtful, termed *Golden verses*. Everyone, without exception, had to make his affirmations compatible with the orthodoxy of the Christianity imposed by the Church of Rome; for this reason, they sought the roots of everything in a Biblical land in the Middle East, and in the innumerable subsequent transplants. Those who did not succeed in doing so met with serious problems, whether they lived in Florence, Cologne or Paris. We have already seen an example with Ficino. And also Reuchlin was persecuted for his attempt to reconcile together Judaism, Christianity, and Islam with Pythagoreanism, under the colours of the cabbala. He opposed the burning of Jewish books, but unfortunately, were burnt not only those not appreciated papers written by heretics. Whether excited or not by music, women were often treated worse, as is known.[62] In that period, people who wanted to part themselves from Rome enjoyed greater success, like Martin Luther (1483–1546) in 1517.

Even for mathematical sciences, however, it is not possible to divide up on the blackboard the good from the bad. What might be useful? Perhaps would be those

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[61] Reuchlin 1995.

[62] Ginsburg 1989.

anachronistic filters constructed ad hoc by certain philosophers to present some advertising for the present-day scientific community, and to justify themselves? We will subsequently find the most occult Pythagoreanism in Newton, the most cabbalistic combination in Leibniz, the music of the spheres and astrology in Kepler, as in Galileo Galilei.[63] Only by not discriminating (a posteriori) will we witness the insemination, and the birth of modern mathematical symbolism, which in time was to pass to a dominant position in all the sciences. Therefore, we cannot even observe, as Descartes wished to make us believe, that those events took place far away, and separately, from the 'bad' magical, astrological, alchemistic part. Like others born from featherless bipeds in the past, the greatest scientific novelties of the seventeenth century were to come to light *inter faeces et sanguinem*. Out of modesty, these, too, were to remain hidden from the majority of people, because, besides the Arabic-Indian language of numbers, natural philosophers would invent and use new words and symbols, with meanings different from the current ones in Latin and in common languages. Also in this, they were to be the heirs of the Pythagorean sects, where discourses were addressed to the initiated, remaining (deliberately?) incomprehensible for others. The belief that mathematical sciences were (or are) open to everybody is a somewhat unrealistic pious illusion, circulated by the followers of the Enlightenment, promoted by the philosophers of progress, nourished sparingly in schools, and maintained alive with drip-feeds, for the interests of professionals of popularization.

Those who succeeded in making themselves understood a bit better, even when the subject was mathematics, practised professions in contact with the land, with popular life, and not just with paper. Today they are only barely mentioned in science histories, like Aristoxenus, and they are now famous rather as painters and artists. Piero della Francesca has not left us only his colours, the spaces and the terrestrial harmonies of his frescoes for the *Storia della Croce* [History of the Cross] in Arezzo, but also books on mathematics: the *Trattato d'abaco [Treatise on the abacus]*, the *De prospettiva pingendi [On the perspective of painting]* and the *De quinque corporibus regularibus [On the five regular bodies]*. Recently, another manuscript of his was found, dedicated to no less than Archimedes.

The *Treatise on the abacus* presented rules and exercises of algebra that were "necessarie a' mercanti" ["necessary for merchants"]. Piero generally supplied solution procedures, which he always calculated explicitly by means of numbers and roots. "Et sono alcuni numeri che ànno radici discreta la quale se po' intendere; et alcuni sono che l'ànno indiscreta, la quale è dicta sorda, le quali è imposibile trovare." [And there are some numbers that have a discrete root which can be understood; and there are others for which it is not discrete, but is called surd, and which is impossible to find.] Was this a tragedy, then? Not at all, because Piero calmly added: "ma in che mo' vi se po' approssimà ... lo mostrarò." [But how we can approximate this ... I will show.] As Archimedes had done, he approximated π with $\frac{22}{7}$. But with "... statue de marmo e de metallo, cioè figure de animali

[63] See Part II, Sects. 8.2, 8.3, 10.1 and 10.2.

rationali et inrationali" [statues of marble and metal, that is to say, figures of rational and irrational animals] he took a watertight wooden box, filled it with water, and calculated the increase in volume when he immersed the statue.

Piero, who quoted Euclid, Archimedes and Ptolemy, also used the classic Greek method of dealing with irrational magnitudes. In a picture, if he had to calculate the length AB of the pentagon, knowing the chord EB, he used the result AB = HB, where HB is obtained from the ratio EH:HB = HB:EB. This ratio, "avente il meçço et doi estremi" [having one middle and two extremes], was immediately put into an equation, arriving at an irrational root.

In his *De prospettiva pingendi [On the perspective of painting]*, he explained that, for him, perspective was visible geometry. He then had to adjust the classic definitions of Euclid for the point, the line and the surface to his models, which remain those of a painter. Whereas, according to geometricians the point and the line are imaginary, as they do not appear but to the intellect, "io dico tractare de prospectiva con dimostrazioni le quali voglio siano comprese da l'ochio." [I say that perspective should be handled with proofs which I want to be realized by the eye.] In his practice as a painter, Piero brought to the light the conflict between geometrical reasoning and the senses. On the one hand, in certain cases, he succeeded in giving effective geometrical proofs of how to draw objects in perspective. On the other, he constructed his results (as also Euclid had sometimes done) by resorting to visual rays, representing the lines through the eye. Actually, in order to deal with "corpi più difficili" [more difficult bodies]: "Nel puncto .A. se ficchi il chiodo, o vuoli uno acho con filo di seta sutilissimo, siria buono uno pelo di coda di cavallo, spitialmente dove à a fermarse su la riga." [If you plant a nail at point .A., or you prefer a needle with thread of extremely thin silk, a hair from a horse's tail would be good, especially where it has to stop on the line.] By means of such constructions, Piero succeeded in being more faithful to Euclid than those academics, like Luca Pacioli, who gave excessively Platonising reading of a religious character.

The interest in construction is again found in the treatise on the five regular solids: tetrahedron, cube, octahedron, dodecahedron, icosahedron, also called Platonic solids. Piero presented a version "for arithmeticians": he calculated numbers that represented lengths of the sides, diameters, measurements of surfaces and volumes. In the painter's mathematical models, we find the problems typical of his mercantile social context and his workshop, where various arts were practised. For this reason, he was interested in the Arabic algebra imported into Italy by Leonardo Fibonacci, and the geometry of the eye. With such a mathematics arguing meant *constructing* the result by means of a wholly explicit procedure, which, in order to arrive at numbers, did not hesitate to resort even to empirical means, and to the physiology of sight.[64]

[64]Piero della Francesca 1913; 1942; 1970. On Archimedes, see manuscript 106 in the Riccardiana Library in Florence, published in a facsimile by VIMER.

Piero della Francesca's books give us a good representation of the conflict present in the age. On the one hand, there was the algebra of numbers, developed for merchants, who preferred the "discrete" aspects of the world for their monetary calculations; on the other, rather by drawing, they practised "continuous" geometry, which represented the world by analogy. The relative tension mixed again the mathematical sciences in their traditional relationships. Approximations were now sought, overcoming the obstacles raised by the Greeks, in order to measure even incommensurable magnitudes.

It would be a mistake to take Piero for a Pythagorean because he dealt with numbers, as it would be to take him for a NeoPlatonic *à la* Ficino because he wrote about perfect regular polyhedra. The artisan workshop where he worked, on the contrary, gave a completely different direct, practical sense both to his algebra, and to his geometry, considered in close contact. Our Tuscan painter lived on the earth, and even if he looked up, he kept his feet firmly planted down here. His painting shows us this, as well. His Divinities, his Madonnas, his angels rose up over the altars and the walls, leaving fields and villages below. The colours are those of the earthly world, and do not transcend it, like the golden background of the Gothic period. On the head of the Madonna, now at the Brera Museum, Piero suspended an egg, and not the Platonic dodecahedron. He chose as a symbol that unit where organic life was concentrated, not that of the ethereal, eternal quintessence.[65]

In the altarpiece, the second saint from the right is considered to be the portrait of Luca Pacioli. This friar had been one of the painter's direct pupils, learning algebra and geometry in the workshop. Friar Luca succeeded at the same time in becoming wholly similar to his master, and totally different from him. His *De divina proportione [On the divine proportion]* and *Summa de arithmetica, geometria, proportioni et proportionalità [General treatise of arithmetic, geometry, proportion and proportionality* were derived, all too clearly, from the books by Piero, and in some points, they were close copies. Without too much effort, Giorgio Vasari (1511–1574), Piero's first biographer, was able to sustain that he had plagiarised them, because he never even mentioned the name of his master. In his *De divina proportione*, part III simply translated the painter's *De quinque corporibus regularibus* [On five regular bodies] into Italian.

After all, they both came from the same environment, and village, that hamlet in the upper valley of the Tiber, uncertain whether to come down the valley to Rome and the Church, or move over slightly to the Arno, arriving at Arezzo and Florence, but with the other alternative of passing through the Apennines to Urbino. Brother Luca preferred to follow the stream, which led him to become a Franciscan minorite, that is to say, a professional also of the written word, as skilful as his master was with the paint-brush. He held lessons all over Italy: we find him several times in the universities, palaces, and courts of Venice, Rome, Perugia, Florence, Naples, Milan, Padua, Assisi, Urbino and Bologna.

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[65]Tonietti 2004a, pp. 75–81.

Among the other epochal events that took place in 1492, including, to quote the less well-known, the expulsion of the Arabs and the Jews from Spain, and the death of Lorenzo il Magnifico, we must add here the death of Piero. Of the following generation, Friar Luca could then exploit that invention, which was not European, but Chinese, which had shortly before started to enjoy great success also here: printing. In that period, the codices and manuscripts that the values of the age considered most important began to be circulated in numerous copies, thus making them relatively easier to obtain. We know that the *Bible* was undoubtedly printed. Natural philosophers and scholars of mathematical sciences thought of Euclid's *Elements*. The most recent translator into English, and promoter of Euclid wrote: "1482. During this year, the first printed edition of Euclid appeared, which was also the first printed mathematical book of a certain importance."[66] However, that is not so, but rather a demonstration of the power of Eurocentric prejudices. We know, from Chap. 3, that in 1084, the *Zhoubi* was printed, and that a copy of its Song edition, of 1213, still exists in Shanghai.

Pacioli printed his *Summa de arithmetica* in 1494, and in 1509 his *De divina proportione* and the translations of the *Elements* both in Latin and in Italian. In the *Summa*, he put everything that could be of practical use in his environment, and thus in the tradition of workshops and abacus schools. The book was the first to formulate in a printed form that method of accountancy known as double-entry bookkeeping, and is thus considered to be the beginning of financial mathematics. But the invention cannot be attributed to him, as he had been preceded by others which had enjoyed less success.

His interlocutors were by now professors and religious dignitaries like himself. With him, the same matter of the Tuscan painter changed its meaning in depth, even if it appeared alike. For our friar, regular polyhedra represented, as in Plato, the essence and order of the universe: the tetrahedron was thought to correspond to fire, the cube to the earth, the octahedron to air, and the icosahedron to water; whereas the dodecahedron was just reserved for the Fifth Essence, that is to say, the heavenly virtue conferred by God. And the formation of the dodecahedron was also considered the fifth of the "convenientie" [properties], which induced Luca to call his treatise *De divina proportione*. The ratio called by Euclid "il mezzo ed i due estremi" [the mean and the two extremes], which Piero had patiently used to construct the figure of the pentagon and some solids, now allowed Luca to arrive at heaven. This is why it was called "divine", "holy", "exquisite". It also possesses the other attributes of God. It is one, "... lei fia una sola e non più" [it is only one, and not more] and triune, because it is fixed by three terms. Furthermore, it is unchanging and eternal. Above all, "... commo Idio propriamente non se po diffinire né per parolle a noi intendere, così questa nostra proportione non se po mai per numero intendibile asegnare, né per quantità alcuna rationale exprimere, ma

[66] Euclid 1956, p. 97. 1492, 1482, continuing to follow the succession of dates, if only to support the memory, in 1472, the most ancient bank in the world, among those that are still active today, started its existence in Tuscany. Ah, money! To believe advertising.

sempre fia occulta e secreta e da li mathematici chiamata irrationale." [... as God cannot be properly defined, or understood by us in words, so this ratio of ours can never be assigned by means of an understandable number, nor can it be expressed by any rational quantity, but is always hidden and secret, and called irrational by mathematicians].

For Brother Luca, irrationals measured the distance between the earthly world and the heavenly, divine world, for the very reason that they cannot be put into words. Their effect were likewise considered essential, and were variously described as singular, ineffable, admirable, unspeakable, priceless, excessive, supreme, most excellent, almost incomprehensible and most meritorious. The divine ratio represented its main geometrical model, in the truest Euclidean tradition. In time, it was to change its name, becoming the golden section; every epoch possesses the cultural models that it deserves.

If he also addressed painters and educated artisans, it was to show them that his learning would be useful for their work, which he would like to be guided by the abstract, general principles of his mathematics. The "sollazzi" [amusements] that he received from his stonecutters appear to be particularly significant in this connection. These 'ignoramuses' believed that it was possible to construct other regular solids, besides the magnificent five inscribed in the sphere. Pacioli compared this impossibility to that of defining God. Cicero had said the same thing in his *De natura deorum [On the nature of the gods]*, where, in turn, the Latin orator made fun of the presumption shown by the poets.

Even Albrecht Dürer (1471–1528) might have succeeded in learning something from the mathematics of our Franciscan friar, but for the precise reason that the latter had learnt them from another painter. Thus the effectiveness of mathematical sciences in their applications to perspective did not seem to derive so much from the truths guaranteed by the heaven of Plato or Thomas Aquinas, as, rather, from the mathematical models of Piero. But how could Luca have recognised this without giving up his own role?

We can now understand better why Brother Luca Pacioli did not recognise his debt to Piero della Francesca. Historians took sides from the beginning, some with Piero and others with Luca. But their argument in doing so, were excessively influenced by their various academic professions, those of the field of art opposed to those of mathematics. Of course, at that time, it was not compulsory to quote one's sources, nor had footnotes come into use yet, nor were there the bibliographies that afflict us today.

The reasons for his failure to acknowledge Piero are more significant. Brother Luca could not recognise the paternity and the merits of Piero without losing prestige. He would have lowered the divine truths of heaven to the inferior level of a painter too much in contact with the earth. Doubtless, he filled up his books with exercises with numbers, but to demonstrate that the earth was subject to heaven. Thus, to use words that have become familiar, it was more a question of two incommensurable paradigms than of ingratitude or plagiarism.

The event that Pacioli did not deal with music in the *Summa* depended because the subjects derived from the abacus schools. Only in the initial letter of dedication to Guidobaldo, Duke of Urbino, after “strologia” [“astrology”] and the architecture of Vitruvius and Leon Battista Alberti, did he include a very brief mention of music, almost as if to apologize for overlooking it. “La musica chiaro ci rende lei del numero, misura, proportione e proportionalità esser bisognosa. De la cui melodia Vostra ornatissima corte de ogni genere virtutum è piena. Maxime per lo culto divino. La celeberrima capella de dignissimi cantori e sonatori. Fra li quali el venerabil padre nostro conterraneo e per habito frate Martino non immeritatamente è connumerato.” [Music shows us clearly that it needs number, measure, ratio and proportionality. Of its melody, your court highly ornate of all kinds of virtues is full. Especially for divine worship. The most renowned chapel has extremely worthy singers and players. Among whom the venerable father from our same region with the same habit, Friar Martino, deservedly is included.] The friar went on to quote other applications, such as the military arts, medicine or poetry, which all need numbers and ratios. How it is possible not to suspect that he did not consider music, although he quoted that of Boethius at the beginning, because Piero had not taught it to him? For this reason, he did not write anything about it in the *Divina proportione*, either, where he would have found a better means to rise up to heaven, as Cicero had done.

Piero’s books were forgotten in the private libraries of his patrons; his work as an artist was to be fully enhanced only in the twentieth century. Brother Luca, on the contrary, was successful. Moving around skilfully in the palaces, whether religious or not, and ably exploiting the printing-press, writing in Italian, he publicised the novelties of algebra and the good old details of geometry which had come back into the stage. Forgetting his Franciscan humility, he proved to be an excellent adman for himself. Forgetting his vow of poverty, in his various wills, he left his possessions to members of his own family. He entered into the evolution of mathematical sciences, even though I would not be able to indicate his particular merits, apart from his success in seizing his opportunities.

In his manual on *Algebra*, Raffaele Bombelli (1526–1572) considered him the first scholar, after Leonardo da Pisa, who took an interest in this science, but judged him to be a “sloppy writer”. Effectively, one century after Dante Alighieri, the Italian of Pacioli is almost illegible. In any case, he did not invent anything to solve algebraic equations in general. The new solution formulas for radicals of third- and fourth-degree equations were to be discovered by Scipione del Ferro (1465–1526), Nicolò Tartaglia (c. 1500–1557) and Ludovico Ferrari (1522–1569). In the meantime, algebra became established in Europe as a discipline in *Ars magna, sive de regulis algebraicis* by Girolamo Cardano (1501–1576).

In brother Luca, the hierarchy, between the geometry of the Greeks and the algebra of the Arabs, was still hiding excessively the new tension that the abacus schools had created between the two disciplines. In him, the two points of view are present at the same time, without finding any point of integration. The conflict has to be solved in favour of algebra only later, in the seventeenth century. If we wish to look for a role to present Pacioli in the history of scientific culture, it would be that, with him, the figure of the university professor was defined. He displayed two of the typical qualities, which later became usual. His name was always included in a clear position on the work, so as to attribute its paternity to himself. In this way, he took care to distinguish himself from the artisans, even if he had to censure their merits in order to do so.[67]

The edition of his *De divina proportione* has the great value of containing the figures reproduced by Leonardo da Vinci. The two Tuscans had met, working together for the Sforza court in Milan. Later, in Florence, they were even to live together in the same house. Unlike Piero, Leonardo placed his polyhedra directly in the world, with shades and colours: he cut them, hollowed them out and stuck them on top of one another, drawing a manifold variety of cases, and creating more than 60.

Leonardo did not leave us any treatises on mathematics, indeed, as everybody knows, he did not write treatises about anything. However, he, too, studied Euclid, with the help of brother Luca, who acted as a translator for him, though he did not transcribe it. Rather, as if taking notes, he drew small geometrical figures, and among these the theorem of Pythagoras. He had proposed an experimental geometrical method to duplicate the cube. It seems that he copied the multiplication table and the classification of ratios from Pacioli's *Summa de arithmetica*. When it was necessary, he undertook numerical calculations, which now and again he got wrong. He was undoubtedly more at his ease with geometry than with algebra, so much so that he extracted square and cubic roots of whole numbers by means of geometrical designs. Other problems of geometry interested him more. "De trasmutatione d'equali superfizi rettilinie in varie figure curvilinie, e così de converso." [On the transmutation of equal rectilinear surfaces into various curvilinear figures, and vice versa]. Leonardo performed these exercises, and dealt with mathematical problems, while he was working on projects for war machines and fortifications, studies on anatomy and nature, and sketches of pictures and frescoes.

To this famous painter, the world seemed to be a self-consistent organism, which functioned harmoniously. To understand it, therefore, it would be necessary to connect everything to everything else, following their reciprocal influences. Leonardo understood a phenomenon when he was able to put it in its place in the overall organism. For this reason, he constructed numerous analogies. Hair is like a river, rivers are like the veins of the earth, the sea is the lake of blood. Leonardo's machines were inspired by imitations, many of which imitated living organisms, like those of flight. In turn, the organs of the body are studied in their matter, flesh, bones, nerves, without remorse. As the veins are like rivers, and bones like stones, what is effective for the earth is valid also for living man. Leonardo's physics also dealt with arms and legs, represented by means of levers, balances and inclined planes.

[67] Pacioli 1509. Tonietti 2004a, pp. 82–84.

In the *Codex Atlanticus*, he designed a series of gears. We can see on the sheet 24 axes, each of which has, at one end, a large gearwheel, and at the other, a small one. Each large wheel engages a small one, which goes round 10 times for every revolution of the big one. As the small wheel is integral with the large one at the other end of the axis, and as this, in turn, engages the next small wheel, the second one will complete 100 revolutions for every turn of the first axis. And so on, until the last wheel, which will complete 10^{23} revolutions. In our great age of computers, the multiplying machine easily reminds us of a calculator, but Leonardo reasoned by analogy. By means of it, he imitated something that goes round and heats up by friction. He reproduced what goes round in the sky and emanates heat: this is the model of the sun. "Il sole che scalda tanto mondo [. . .] e che in 24 ore fà si gran corso, a comparazione di questo strumento, parrà, il sole, senza moto e freddo." [The sun, which heats up such a large world, and which goes such a long way in 24 hours, compared with this instrument, will seem to be, the sun, motionless and cold.]

We are faced with an analogical mathematical model of movement. As movement produces heat by friction, fire was also modelled. He imitated the heartbeat by means of the oscillations of a pole around a point of equilibrium. The valves at the cusp of the aorta distend and deflate like the sails of a ship.

Leonardo constructed all his numerous models by means of drawings, in order to represent the analogies in the purest form. He "dimostrava" ["demonstrated"], in the sense of showing, that is to say, making evident to the sight, which in his opinion, was the most noble and surest of the senses. "La vera cognizione della figura di qualunque corpo consiste nel vederlo in diversi aspetti [. . .]. Per rendere dunque chiara la figura di qualunque membro dell'uomo, [. . .], io osserverò la regola predetta facendo di ciascun membro quattro dimostrazioni, una per ogni lato, e delle ossa ne farò cinque." [The true knowledge of the figure of any body consists of seeing it from different points of view [. . .]. In order to make clear the figure of any member of man, [. . .] I will observe the above rule, giving four representations of each member, one for each side, and as regards the bones, I will make five.]

The 'demonstrations' are drawings of anatomy, which are generally precise and particularly fine, sometimes accompanied by small geometrical charts. Leonardo reached the apex of his 'demonstrative' ability, when he was studying the movement of waters. Here and there on the sheets of the various codices, he drew those shapes that waters assume in front of, or after, obstacles, coming out of bottlenecks, falling from on high, swirling round the pillars of bridges, along riverbanks, on the seashore. Leonardo noticed the differences and followed their variety, changing the profile of obstacles and channels. He also considered waves, but he appeared to be fascinated above all by whirlpools and turbulences in their formation. For whirlpools, he was even the first who gave the quantitative law of movement. "Li retrosi delle acque han le sue parte tanto più veloce quanto elle sono più vicine al suo centro ." [Eddies in waters have parts that are faster, the closer they are to its centre.] It was interesting that he considered water a continuum with essentially different properties from those of sand. Above all in these cases, we could understand the nature and the effectiveness of Leonardo's mathematical model. With him, it could be not suitable to make too many distinctions between his ability as a painter, his mathematical culture and his knowledge of natural phenomena.

Leonardo lived in a world made of flowing water, flying birds, wind and clouds. He sought a meeting-point between experience and mathematics. “Quelli che si innamorano della pratica senza la scienza sono come i nocchieri che entrano in naviglio senza timone o bussola, che mai hanno certezza dove si vadano. Sempre la pratica deve essere edificata sopra la buona teorica, della quale la prospettiva è guida e porta, e senza questa nulla si fa bene.” [Those who fall in love with practice, without science, are like helmsmen that enter into a waterway without any rudder or compass, and never have any certainty of where they are going. Practice must always be built on sound theory, of which perspective is the guide and the door; and without this nothing is well done]. Unlike Piero della Francesca, Leonardo proposed perspective even through a panel of glass, as he drew in the margins of a codex. However, it is likely that Leonardo did not often use this perspectograph which is difficult to control in drawing large environments. Perhaps it mainly had a didactic aim. The technique, however, arrived as far as Dürer.

“Nessuna umana investigazione si pò dimandare vera scienza, s’essa non passa per le matematiche dimostrazioni.” [No human investigation may be called true science, if it does not pass through mathematical demonstrations.] But to reach experience, these were not to remain closed in the mind. “... in tali discorsi mentali non accade esperienza, senza la quale nulla dà di sé certezza.” [... in these mental discourses experience is not acquired, without which nothing gives any certainty of itself.] Therefore, above all, “... ricordati, quando commenti l’acque, d’allegar prima l’esperienza e poi la ragione.” [... remember, when you comment on the waters, to present first experience, and then reason.] Leonardo treated painting as a meeting point between experience and mathematics, that is to say, as science. “Se tu sprezzerai la pittura, la quale è sola imitatrice di tutte le opere evidenti di natura, per certo tu sprezzerai una sottile invenzione, la quale con filosofica e sottile speculazione considera tutte le qualità delle forme: mare, siti, piante, animali, erbe, fiori, le quali sono cinte di ombra e lume. E veramente questa è scienza e legittima figlia di natura, perché la pittura è partorita da essa natura.” [If you despise painting, which is the only imitator of all the evident works of nature, you will undoubtedly despise a subtle invention, which with philosophical and perceptive speculation considers all the qualities of the forms: sea, sites, plants, animals, herbs, flowers, which are surrounded with shadows and light. And truly, this is science, and the legitimate daughter of nature, because painting is generated by nature itself.] For him, painting was thus a true instrument of knowledge, more universal than language itself, and better than philosophy.

Despite the event that today he is famous above all as a painter, in that period Leonardo built fortifications and equestrian monuments, and organised musical festivities, which he filled with scenery and wonderful machines. The passion of his life remained projecting canals and repairing rivers, both in the Arno basin and in that of the Adda. He captured the variety of the world, from that of faces to that of landscapes, from leaves to rocks, from clouds to floods. He was interested in the movement of water, which, more than any other, offered him bizarre varieties.

Not unrelated to this, his preferences were certain feminine sensibilities, and sexual attitudes which, in their obscurity, and the risk even of being put on trial for homosexuality, did not correspond to the social norm. His friendship with Friar Luca Pacioli has already been mentioned.

He sought an immediate, sensible adhesion, direct contact, almost an immersion in the phenomena that common sense had put before him. Of the world, he loved the shapes and the qualities which are the visible signs of variety, not the laws, which instead express uniformity and constancy. Possessing the talent of a painter that we know, Leonardo did not demonstrate with philosophical arguments through the words of sentences, but rather represented through the lines of his drawings. Thus the phenomenon acquired the intuitive evidence of what happens directly in front of our eyes. For this reason, in mathematics, he preferred geometry that is seen, and not numbers that are calculated.

His attention for the organisms living on the earth, which he transformed into a general interconnected model of the world, has an Aristotelian flavour for us today. But clearly, not the Aristotle revised by the Scholastics and commented by the Peripatetics, if anything it looks like sensual variety of Avicenne. In any case, by now, after the static, finite geometrical constructions of Piero, and the divine mathematics of Luca, in the drawings of Leonardo the world moved. "Universalmente tutte le cose desiderano mantenersi in sua natura onde il corso dell'acqua, che si muove, cerca mantenere il suo corpo secondo la potenza della sua cagione e, se trova contrastante opposizione, finisce la lunghezza del cominciato corso per movimento circulare e ritorto ." [Universally, all things desire to maintain themselves in their nature, with the result that the course of water, which moves, tries to maintain its body through the power of its cause, and if it finds a contrasting opposition, it finishes the length of the course that it has begun with a circular, twisted movement.] Leonardo looked not towards the sky like the Pythagoreans or the Platonics, but, like the Chinese, down at the earth. Here, he observed the stability of the movement of water, that is to say, its ability to maintain its own course, resisting (small) disturbances. Shortly afterwards, the movement of bodies was to be brought into focus in the scientific revolution of the seventeenth century, but unfortunately, the property of stability was to be put aside for following centuries.[68]

Piero, Luca and Leonardo should be counted among the characters of the European scientific and cultural world between the fifteenth and sixteenth centuries. Luca has left his mark in histories as a mathematician, whereas Piero and Leonardo in time became practically only masters of the figurative arts. And yet, even as a mathematician, Piero appears to be more interesting than Luca, who prevalently transcribed other people's texts. Furthermore, some models described by Leonardo, especially those for water, represented the phenomenon in such a pertinent way that

[68]Leonardo da Vinci 1974; 1975; 1979a; 1979b; 1982a; 1982b. Tonietti 2002a. Leonardo considered music to be the "sorella della pittura" ["sister of painting"], which "compone armonia con la congiunzione delle sue parti proporzionali" [composes harmony with the union of its proportional parts]; Massera 1977, p. 28.

they were to be used in the eighteenth century to make a start to hydrodynamics. Positively, his law of vortices, stating that (as written above) the speed of water is inversely proportional to its distance from the centre, fully stands comparison with those of Newton. The English scholar made spheres and cylinders rotate in fluids, in order to try to make the vortices of planets, defended by Descartes, impossible.

But then, why was Leonardo's visual abstraction, which had proved to be so effective in these cases, forgotten in time as a scientific model? Nowadays hardly anybody seems to be ready to see any mathematics in it. Why have such important natural phenomena as the flow of waters and the atmosphere been the object of so relatively little study and dramatically unrecognized? We already know Michel Serres reply.[69]

We have now entered into a historical period of vortices and general turbulence, which shuffled the cards of a game whose rules were changing. In the comparison and the tensions created between the criteria of rigour deriving from the Euclidean tradition, and the ease of calculation offered by the new algebra which had arrived from the Arabs, various possibilities of evolution opened up for sciences. Who would prevail? In the following section, we shall see that the cultural context again favoured the Greek classical tradition. And sure we find once again the Latin language and the music of the Pythagorean sects.

De la religion tutta la forma
Consiste in vesti, ornamenti e pitture,
Campane, candelier, cantare a torma.
[Of religion all the appearances
Consists of robes, ornaments and paintings,
Bells, candlesticks, singing in large groups.]

Francesco Maurolico.

6.5 The Quadrivium Still Resisted: Francesco Maurolico, the Jesuits and Girolamo Cardano

In the course of his long life, Francesco Maurolico (1494–1575) wrote books and compendia about everything men of the period had succeeded in recovering from the Greek scientific heritage. As he worked on an encyclopaedic project organised in accordance with the pattern of the classic *quadrivium*, with him we again find our music.[70] But most of his efforts remained confined in manuscripts, while complex historical episodes, and editorial vicissitudes tossed his papers here and there, causing some of them to be lost.[71]

[69]See Sect. 2.8, and Part II, Sects. 9.2 as well as 10.2. Serres 1980, p. 76. Tonietti 2004a, pp. 84–95.
[70]Maurolico 2000. Maurolico 201?. See Appendix C.
[71]Tonietti 2006b.

His theory of music is well represented by the numerous tables disseminated throughout the main manuscript of his *Musica*. From the numbers, we can see what ratios Maurolico posited between notes. Arranged in a geometrical succession, musical intervals were fixed by the Pythagorean ratios seen above: diapason (octave) $4\frac{1}{2}$:9 (that is to say, 1:2), diapente (fifth) 6:9 (2:3), diatessaron (fourth) $4\frac{1}{2}$:6 (3:4). The Greek names were maintained for the single notes, such as *proslambanomenos* (added), *hypate* (principal), *parhypate* (next to the principal), *lichanos* (index), *mese* (mean), *paramese* (next to the mean), *paranete* (next to the acute), *nete* (acute), *anthypate* (opposed to the principal). These were made to correspond to the modes, like *Hypodorian*, *Dorian*, *Mixolydian*. They were also designated by means of the letters Γ, A, ..., e, f, g, A, in such a way that the same letter indicated notes separated by an interval of an octave. The distance between one note and the subsequent one could be of one tone, or a semitone (Chart 43, Appendix C).

The notes were imagined as arranged along a scale, which rose up from the lowest Γ, A, to the highest (the most acute) f, g, A. This ascent was also from Earth to Heaven, because the single notes were made to correspond to the heavenly bodies: Earth, Moon, Mercury, Venus, Sun, Mars, Jupiter, Saturn, the Firmament. In the manuscript, we find the traditional symbols of these planets, here substituted with their names. Thus, this natural philosopher of the sixteenth century believed in the music of the spheres, according to which the heavenly bodies would generate music in their movement. In turn, the stars were correlated with the Greek musical modes, in order to explain how the spirits of people were influenced by them. For example, Mars corresponded to the *Phrygian* mode, which according to Greek *ethos* "incites", "exasperates" and is "irascible". Whereas a very different effect is obtained by the *Hypolydian* mode, governed by Venus: "compassionate and tearful, it sympathises and gladdens."

For music, Maurolico declared that he had taken inspiration from Cicero, Boethius, Guido D'Arezzo and Faber Stapulensis (c. 1455–1537), who are quoted several times in the text. Together with Boethius, Maurolico returned to the Pythagorean tradition. But he overlooked the new variants to Pythagoreanism which had already been advanced at the time, as we shall soon see. In music, Maurolico appeared to be a conservative, just as it was clear from the arrangement of the planets in the tables that he adhered to the geocentric system. "Toleratur et Nicolaus Copernicus, qui solem fixum ac terram in girum circumverti posuit: et scutica potius aut flagello quam reprehensione dignus est." ["We tolerate also Nicolas Copernicus, who has kept the sun fixed, making the earth revolve around it, which is more deserving of the rod and the scourge than of being confuted."] He considered the followers of heliocentrism to be simply crazy.

However, it would be a mistake to consider him only one who condensed and copied other more or less famous books (Euclid, Archimedes, ...) in order to compile his own. At least in his style, his ability to understand and some interesting contributions, he played his role in his environment. This can also be seen in the field of music.

Maurolico had detected an error in the *De institutione musica* by Boethius. The result appeared to be correct, but the mathematical procedure used to arrive at it was decidedly mistaken. Having calculated the *comma* to be 524288:531441, what greater desire could there be (for a Pythagorean) than to produce at least a kind of basic interval, a kind of musical atom, to which all the others could be broken down, and consequently from which they could be obtained? Breaking them down and recomposing them naturally, by means of operations admitted as valid. But our experienced mathematician might begin to suspect that this was not possible. Maurolico wrote: "Constabit etiam quod diesis maior est, quam tria commata minor autem quam quatuor. Apotome autem maior, quam .4^{or}. commata, minor quam .5. Unde et tonus excedet .8. commata et minor quam novem commata nascitur quae omnia ex longo et multarum figurarum calculo constari possunt lege Boetium et Fabrum in musicis elementis." ["It is also found that the diesis interval is more than 3 *commas*, yet less than 4. The interval of the *apotome*, on the contrary, is more than 4 *commas* and less than 5. Thus the tone exceeds 8 *commas*, but is born less than 9. These are all things that can be verified by means of a long calculation with several figures; read the musical elements of Boethius and Faber."]

The number of *commas* contained in the intervals was often quoted, but usually not the calculation. Maurolico realised that the calculation of Boethius was wrong; perhaps in order to save effort, this medieval character had considered as equal the differences between the terms of the geometrical progression, treating it as if it were an arithmetic one.[72] So our scholar from Messina invented the first correct demonstration, known today, of this property. And what is more, he did not perform long calculations, but arrived at the result in a few lines, using the arithmetic mean, which is always more than the geometrical one.[73]

Maurolico was a member of that scientific culture which modelled the world, and not only music, above all by means of ratios and proportions. Among other things, he had the conviction of those who, like Cicero, 'heard' the farthest planets emitting the most acute notes. He could justify this by their greater speed. However, Jupiter and Saturn are bigger and farther away from the Moon, Mercury and Venus. Consequently, they might have had deeper voices. The question remained controversial. Boethius sustained this other music of the spheres. What descended from heaven, like what was sung and played on earth, was very varied, just as the sciences also appeared to be far from monolithic, unique and eternal.

Maurolico did not go beyond the limit of the Pythagoreans, who had sustained that the pitch of notes grew in a linear manner also with the tension of the string, just as their depth increased with their length. "Et nervus remissius gravius: intensus acutius." ["And the sinew that is more relaxed [sounds] deeper, the one that is tighter is more acute"].

Within the *quadrivium*, music was dealt with by means of whole numbers and fractions, because it was a 'discrete' discipline like arithmetic, different from the 'continuous' geometry and astronomy. And yet, in his *Musica*, our Sicilian scholar made calculations, even with roots. Thus he knew the alternative to Pythagoreanism.

[72]See above, Sect. 6.2.

[73]Tonietti 2006b, pp. 153–156.

"Non solum in proportionibus commensurabilibus, sed etiam in praecipuis numeris consistunt vocum musicarum consonantiae. Quoniam incommensurabiles proportiones (quoniam irrationales et ignotae sunt) semper faciunt dissonantiam: quoniam voces in tali proportione constitutae, propter incommensurabilitatem, non per ordinatos ictus sed semper diversos (quae diversitas parit discordantiam) invicem sibi respondet." ["The consonances of musical notes consist not only of commensurable ratios, but also of particular numbers. As incommensurable ratios (since they are irrational and unknown) always form only dissonances: seeing that the notes fixed by those ratios, as a result of their incommensurability, do not respond to each other in strokes that are orderly, but always different (a difference that generates discord)"].

Maurolico justified consonances by commensurable ratios, and the latter by the more frequent correspondences of the strokes transported by the two notes through the air to the ear. Before arriving at the notion of sound as a wave, vibrating strings were thought to make the air, carrying the strokes to the ear, tremble. Thus, a note arrived at the ear with a certain number of constant rhythmic strokes; on the basis of the ratio 1:2, another note one octave higher would generate twice as many. Consequently, every stroke of the first note would coincide with one out of every two strokes of the second note: the octave gave an excellent consonance. Ignoring all the questions of phase, that is to say, when the air started to strike in the ear, good correspondences, and therefore good consonances, would also be found with the other ratios. Whereas, necessarily, no strokes would correspond any longer with the ratio $1{:}\sqrt{2}$. In this case, all the strokes would prove to be out of time, thus generating, according to the theory, only horrific dissonances.

The theory of strokes enjoyed success in that epoch, and we shall often find it again, even as late as the eighteenth century. This is explained for the first time, and can be read in the papers of Maurolico. In this way, the rival theory of Aristoxenus would seem to have been liquidated for physical reasons. "Tonum non posse dividi per aequalia: quandoquidem toni ratio sesquioctava non est quae quadrati ad quadratum numerum: et perinde medium proportionalem numerum, qui proportionem per aequalia secet, non suscipit. Sic non datur locus Aristoxeno tonum per aequalia secari debere, asserenti." ["The tone cannot be divided into equal parts, seeing that the ratio 8:9 of the tone is not that of a square with the square of a number; thus it does not admit a proportional mean number which divides the ratio into equal parts. Thus no consideration is given to Aristoxenus, who asserts that the tone can be divided into equal parts."]

And yet our mathematician of the sixteenth century knew full well how to find the proportional mean between 8 and 9. He even wrote it down on a sheet of paper: "9 . r72 . 8", where "r" was the symbol, at that time, of the square root. As $\sqrt{72}$ contains $\sqrt{2}$, the Pythagorean prejudice against irrationals continued to make itself heard. But now, at the time of Maurolico, another deeply felt, even more decisive prohibition was added to the ancient one. The 'without *logos*', 'without discourse' of the Greeks had been transformed into 'inaudible' by the Arabs; the irrationals were thus called 'surds' [deaf] in the abacus schools. For our mathematician from

Messina, irrationals still remained 'unknown'. On the contrary, whole numbers, with their ratios, appeared to be intelligible, because they were finite. Irrationals were not, and could not be used by him as numbers. "Solus enim Deus infinitus." ["For only God is infinite"]. Maurolico had first taken holy orders as a priest, then he had become a Benedictine abbot at Castelbuono, near the Abbazia del Parto [Abbey of Childbirth], between Messina and Palermo. For a religious figure by profession, like him, the argument was conclusive. For those men projected into the divine realm, like Brother Luca, handling irrationals required all possible care, in order to avoid problems.[74]

Moving with greater sureness among Euclid's proofs than among calculations (every now and then he also made mistakes), Maurolico preferred reasoning to these, as we have seen with the *commas*. He gave general rules in the form of charts, in order to combine the ratios of music together. He defined a *Regula compositionis* and a *Regula subtractionis* by means of letters arranged as in the following figure (Chart 15, and 16 Appendix C).

Given two ratios, they can "continuare", be ["continued"], by obtaining from the two couples of terms three other terms. These latter contain both the starting ratios and the new one sought, which describes its composition. The rule makes it possible to obtain the ratio for the addition of the two musical intervals. Two ratios can also "subtrahere", be ["subtracted"], by means of the second chart, which makes it possible to obtain the ratio for the difference of musical intervals.

In the period in which the events of the epoch passed by *in freto siculo*, through the straits of Messina, this religious figure, wholly dedicated to the sciences, took part to them when necessary. Don Juan of Austria, the son of Charles V, consulted him, during a stay of his fleet on his way to the victory over the Turks at Lepanto in 1571. "... gli domandò il parere, e giudizzio intorno al tempo, ch'era per seguire nella partita ad affrontar l'armata Ottomana insino all'arsenale di Costantinopoli (se tanto fosse possibile) al cui compiacimento, e contemplatione, havendo egli calcolato il tempo [atmosferico] con l'osservatione fatta di tutto il viaggio verso Levante, e datoglilo in nota, seguì appuntino senza preterirsene un iota. Onde al ritorno glorioso e trionfale per l'havuta vittoria, non si satiavano quei Prencipi della lega, insieme con l'Altezza del Signor Don Giovanni, di lodar l'ingegno ed ammirar la dottrina, che parea signoreggiar i Cieli, ed haver in mano la briglia de venti e del Mare."[75] [he asked his opinion and judgement about the weather that would follow in the attempt to face up to the Ottoman fleet as far as the arsenal of Constantinople (if it were possible to do so much), and having calculated, to his satisfaction and contemplation, the weather with the observation of all the eastward voyage, and given a note of it to him, he followed it in detail, without going a jot further than that. Consequently, on their glorious, triumphant return because of the achieved victory, the Princes of the Alliance, together with their leader, Don Juan, did not

[74]On the passage between numbers that were *alogos*, *asamm* and *surds*, see Wymeersch 2008.

[75]Baron della Foresta 1613.

cease from praising his intelligence and admiring his doctrine, which seemed to be lord of the Heavens, and to have in his hand the reins of the winds and the Sea.]

He did not participate directly in the Council of Trent, but wrote a letter *Ad Tridentinae Synodi Patres*. Maurolico always had contact with the Jesuits, receiving also benefits from this. The most eminent nobles and the highest Sicilian dignitaries were his constant interlocutors. When the manuscript of *Sicanicarum rerum compendium [Compendium of Sicilian things]* was printed, many passages were omitted which would have caused him several problems, if they had been published in 1562. We know both the passages that were censured, and the authors of these cuts: the Jesuits. They very probably took similar action with the manuscript of *Musica*, as well. Actually, they only published a part of it, for their own didactic purposes, in 1575, immediately after the death of the author. What is missing from the printed text is, in particular, the original contributions of our man from Messina, with respect to the tradition of Boethius, those that have been highlighted here.[76]

When, in the North of Europe, the Church of Rome lost the Anglicans, the Protestants and the Calvinists, at the time of the Counter-Reformation, the liturgy of the Mass gave importance, not only to the Latin language and the relative interpretation of the Holy Scriptures, but also to music. It is well known that music was present in the discussions among the fathers of the Council of Trent, above all to limit polyphony, which was making the canonical text even more incomprehensible. It had become necessary for the Catholic Church to defend the orthodoxy of the ideas that it sustained, in cases considered important. Among these, both mathematical sciences and music were included. The Jesuits involved Maurolico in their projects concerning teachings to be offered in their colleges. Even Christophorus Clavius (1537–1612) visited him and consulted him, to organise the manuscripts for this purpose. This Jesuit was to become much more famous than him, for contributing, in Rome, to the reform of the calendar under Gregorius XIII in 1582, as well as printing some Latin editions of Euclid.

On the side of the Reformation, the mathematician and friend of Martin Luther, Michael Stifel (1487–1567) observed that: "... musicians speak of certain irrational proportions [*proportionibus quibusdam irrationalibus*]." In his *Arithmetica integra* [Integral arithmetic] (1544), he clearly included also the division of the tone, 8:9, into two equal parts. "Et toni dimidiatio praecise ponitur sic: 8 $\sqrt{72}$ 9 ..." ["And the division of the tone into two precise parts if posited as follows: 8 $\sqrt{72}$ 9 ..."]. "And because any part you please of the afore-mentioned halvings consists of a certain term or rational [number], and a term that is uncertain and unknown, or irrational, therefore also the parts themselves individually are uncertain and unknown, or irrational, ratios." And yet, in the end, like Francesco Maurolico, he refused the irrationals that were necessary for the numerical division into two equal parts, because they involved infinity. "It is properly debated whether irrational numbers are true numbers or fictions. ... They show us we are moved and compelled to admit that they [irrational numbers] really exist from their effects, which we perceive to

[76]Maurolico 1575. Maurolico 2000. Maurolico 201?. Tonietti 2006b.

be real, sure, and constant. ... On the other hand, other things move us to a different assertion, namely that we are forced to deny that irrational numbers are numbers. ... Just as an infinite number is not a number, so an irrational number is not a true number, and is hidden behind a sort of cloud of infinity."[77]

It might be relatively easy to control, and purge, those who lived under the cloak of the Catholic Church. But what about the others, when publishers promised to spread their ideas? Girolamo Cardano wrote literally everything that passed through his mind, and fairly often published it: at Milan, Nürnberg, Paris, Basle, Leiden. He embodied a figure that was the opposite of that of Maurolico. The religious dignitary from Messina lived a life of peace and quiet, just as the humanist and doctor from Pavia went through all kinds of experiences. Both revealed encyclopaedic interests. And yet Maurolico would like to find an order based on the Greek classics, while Cardano exalted the tumultuous variety of the world, like Leonardo da Vinci, who his father had met: from animals to dreams, from illnesses to gambling, from the movement of bullets to that of water, from rain to fossils, from algebra to horoscopes. The one for Christ cost him an accusation of heresy and imprisonment. Like Arius, had he demoted Christ to a simple prophet? Did he even consider himself to be one? His books were undoubtedly appreciated by protestants like Andrea Osiander (1498–1552). Yet he was to succeed skilfully in winning over authoritative characters of the Council of Trent like Giovanni Morone (1509–1580), in the end convincing the Church to pay him a stipend.

Also music was among Cardano's universal interests. We find it in the *De musica liber [Book on music]*, in the "Proposizione 166" of the *Opus novum de proportionibus numerorum, motuum, ponderum, sonorum, aliarumque rerum ... [New work on the ratios of numbers, of movements, of weights, of sounds and other things ...]* and in the *Della natura de principii et regole musicali* [On the nature of musical principles and rules].[78] This last book is considered by some scholars to be spurious. Essentially, it deals with the "mano musicale" [musical hand], which singers needed in order to intone notes, and how these were arranged in a scale along the icosichord of Guido D'Arezzo. It is interesting that in the end, in order to tune the lute by means of the monochord, he divided the tone into nine parts, five in the greater semitone, and four in the lesser one. Here, they were called "crome" [quavers] and not *commas*, and they were not represented by means of ratios between numbers. The author described as 'very common' this division of the tone, which was practised habitually and traditionally without any theoretical justification.

Instead, in the perhaps safer *De musica*, Cardano explained the ratios of Pythagorean tradition, with the tone at 9:8 and the ditone, consequently, at 81:64. He criticised Ptolemy for certain ratios which did not sound consonant to his ears, and quoted Aristoxenus, but only as a source for the inventor of the enharmonic

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[77] Tonietti 2006a, pp. 118–121; Tonietti 2006b. Pesic 2010, pp. 507–510.

[78] Cardano 1663, IV, pp. 105–116 and pp. 548–552; X, pp. 621–630.

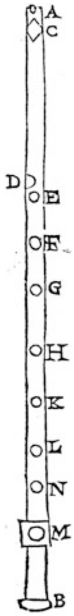

Fig. 6.6 How Cardano distributed the notes between the holes of the recorder (Cardano 1663, vol. X, p. 111)

genre. The book dealt above all with the art of constructing and playing the recorder (Fig. 6.6).

This must have been a broader project, which underwent various rewritings on different specified dates, the first in 1547. Thus it preceded the brief chapter in *De proportionibus* of 1570, which gave as the ratios for the ditone and the semiditone 5:4 and 6:5 (the new thirds), as well as 8:5 and 5:3 for the minor and major sixths. These were thus new, but they were the same proposed in the *Istituzioni armoniche* of Zarlino,[79] already published in 1558, but not quoted by him. Here, rather, the natural philosopher from Lombardy often mentioned the books of Ptolemy, and now he also dealt with the effects of music on the human spirit: "… Doricus ad alacritatem pertinet, ad pugnam Phrygius, … ad voluptatem Lydius, …" ["… the Dorian mode serves for alacrity, the Phrygian for battle, … the Lydian for voluptuousness …"]. And a successful doctor like him could not overlook the curative effects on the body, including its effects on the bite of the tarantula, which we have already seen in Marsilio Ficino.

The theory of music set forth by Cardano would seem to limited to this, unless there have been some losses. Compared with the other preceding treatises, what has come down to us does not contain anything new, indeed, it offers less. And yet in the *De vita propria [On my own life]*, he openly wrote: "In musica novas voces, novosque ordines inveni, aut potius inventos in usum revocavi, ex Ptolomaeo et Aristoxeno" ["In music, I have discovered new notes and new orders, or rather, I

[79] See below, Sect. 6.6.

have brought back into use the discoveries made by Ptolemy and Aristoxenus"].[80] Furthermore, in the *De libris propriis [On my own books]*, he states that he had filled up as many as 170 sheets!,[81] when the *De musica* that we know adds up to a few dozen. We are very sorry that we cannot read them, if he really wrote them. In his apologetic autobiography, he affirmed that he had corrected the book in 1574.[82] Anyway, he admitted that his "novelties" came from Ptolemy and Aristoxenus. However, above all Zarlino enhanced the new ratios for the third. As for the scales of Aristoxenus, at variance with the Venetian musician, the Florentine Vincenzio Galilei focused attention on them again, as we shall discover shortly.

For sound, the doctor from Lombardy followed Aristotle's theory of strokes between bodies, the air and the ear, but he did *not* use it to justify consonances, as did Maurolico, or, as we shall soon see, also Benedetti. Musicologists appreciate the competence of Cardano as regards the recorder and organs, but he did not exploit his mathematical knowledge for their tuning. In Europe, no current theory existed at the time to which one could make reference; otherwise, the Pythagorean affirmations about the lengths of strings would have had to be changed, to adapt them to the lengths of the pipes, bearing in mind also their diameter. From Chap. II, however, we know that the Chinese had found the way.[83] Thus, in Europe, wind instruments and organs were generally tuned by ear, without entering into the theory of it.

Here we find the other limitation of Cardano: not uniting experience with mathematical theory. And yet, this volcanic, and in some ways sulphurous, natural philosopher of the sixteenth century displayed a significant direct practical knowledge of music. Had he gone to a music school, learning to play the recorder, or perhaps other instruments, which he regularly practiced? The only (partial?) manuscript of the *De musica* that has come down to us even contains a musical composition of his: a polyphonic, 12-part motet on the antiphon "Beati estis, Santi Dei omnes" ["Blessed are you, all the saints of God"]. However, he wrote: "... in music, I have been inept from the practical point of view, although I know the theoretical part well enough."[84]

He undoubtedly appreciated music, and thank goodness! But he was to end up by being ashamed of this pleasure. "In iuventa, rursus, mediocris habitus, mediocriter iracundus, laetus, voluptatibus deditus, musicae praecipue ..." ["In my youth, instead, mediocre in my qualities, fairly irascible, carefree, addicted to pleasures, above all to music ..."].[85] He recognized its deep, strong influence on all of us, and yet unfortunately, he ended up by judging its effects to be negative. He spoke fairly badly, above all of musicians and singers, who were covered with

[80]Cardano 1663, I, p. 39. Cardano 1982, p. 157.

[81]Cardano 1663, I, p. 108.

[82]Cardano 1982, p. 164.

[83]See above, Sect. 3.2.

[84]"... in Musica ineptus usui fui, contemplationi non impar ..."; Cardano 1663, I, p. 31. Cardano 1982, p. 129.

[85]Cardano 1663, I, p. 52. Cardano 1982, p. 198.

somewhat colourful epithets. "... ebrij, gulusi, procaces, incostantes, impatientes, stolidi, inertes, omnisque libidinis genere coinquinati. Optimique inter illos stulti sunt." ["... drunkards, gluttons, insolent, capricious, impatient, silly, idle, involved in all kinds of debauchery. The best of them are fools."] "Cur musici sunt adeo flagitiosi inter caeteros, incontinentes, ebrij, lascivi, petulantes, infidi, incostantes, leves, gloriosi, mendaces, malae consuetudinis nugaces?" ["Why are musicians, more than others, wicked, intemperate, drunk, lascivious, insolent, untrustworthy, capricious, superficial, boastful, deceitful, bad-mannered clowns?"][86] It's music's fault!

To him, the Pythagorean and Platonic theories of music must have seemed rather suitable to restrain such stallions on heat. So how could he have agreed to extend their melodic possibilities with other dangerous ideas following Aristoxenus? Anyway, as regards music, he did not include any trace in his writings of a possible using the new algebraic tradition which he himself was promoting. His mathematical work ended up by concerning music only indirectly. For, with his formulas for the solution of equations, also those numbers to which theoreticians had denied legitimacy for thousands of years now entered, at least, into practice: irrationals.

In distributing the frets on a lute, he seemed to be uncertain whether to calculate the semitone approximately by means of ratios, which he wrote as fractions, $\frac{18}{17}$ and $\frac{17}{16}$, or to use the true value $R_2\frac{72}{8}$, where $R_2 72$ stands for $\sqrt{72}$. "... volendo dimidium proportionis $\frac{9}{8}$ duc 9 in 8 fit 72, accipe latus seu radicem quae est $R_2 72$ & huius proportio ad 8 hoc modo $R_2\frac{72}{8}$ [sic!] est semitonium verè." ["desiring to obtain half of the proportion [ratio] $\frac{9}{8}$, calculate 9 by 8: it gives 72; take the side, that is to say, the root, which is $R_2 72$, and its ratio with 8 in this way, $R_2\frac{72}{8}$, is the true semitone."][87]

We know, more or less, the formulas of Cardano and Cardano's joint. He had not invented any of this directly, but he was capable of promoting it by means of printing. With him, as with Pacioli, the distance between the creators who produce and the sellers who draw the greatest profit, also in their reputation, can clearly be seen. He certainly knew how to obtain it. "Le arti sono molte, ma una sola è l'arte delle arti e consiste nel saper dire cose generiche, molte cose con poche parole, cose oscure con termini chiari, esprimere il certo per mezzo dell'incerto." [There are many arts, but only one is the art of arts; it consists of knowing how to say generic things, many things in few words, obscure things in clear terms, expressing the certain by means of the uncertain][88] His most original inventions, those for the calculation of probabilities, were stimulated by his passion for the game of chess and dice, but the *Liber de ludo aleae [Book on the game of dice]* was only published posthumously. He calculated in how many different ways he could put four items

[86] Cardano 1663, II, p. 214 and p. 647. Sabaino 2003, pp. 89–124. Schütze 2003, pp. 105–124.

[87] Boyer 1990, p. 332. Cardano 1663, X p. 108. Cf. Pesic 2010, pp. 510–512.

[88] Cardano 1982, p. 192.

of clothing on (one very light, another light, one heavy, and one very heavy), if he wore two at a time: he wrote 14. As the arrangements of 4 elements, two by two, total 12, if he wasn't mistaken, he may have worn only one in summer, or in the in-between seasons.[89] He also observed the movement of water with precision, but how can we avoid suspecting that his source was Leonardo da Vinci?

The modern, contemporary world was drawing closer, full of secrets, vanities, thefts, competition and violence, all this often in the name of some good god.

Violence? Cardano's son, accused of poisoning his adulterous wife, was beheaded. Tartaglia had received this nickname as a result of a sword-cut to his face, which caused him to stutter.

What secrets? The formulae for the solution of radicals were not revealed openly, but were hidden in enigmas.

What competition? Challenges developed to establish who would be capable of solving a series of equations.

What theft? Cardano published formulae without the author's consent.

What vanity? Take your choice, there's no need to insist.[90]

Inter Faeces et Sanguinem, Q.E.D

As regards the division of the tone into five parts, though without running the risk of doing it with numbers, Cardano was referring to Nicola Vicentino (1511–1572), who had constructed a harpsichord, full of keys, and therefore capable of playing even the enharmonic micro-intervals of classical Greece. Protected at the court of Ferrara and in Rome by the Este family, our musician from Vicenza made reference also to Aristoxenus, and might therefore have divided musical intervals into equal parts. "...; così la natura della divisione del genere Cromatico comporta che si rompi l'ordine del Diatonico, & che si facci d'un tono due semitoni & poiché si facci il grado del triemitonio incomposto; che tutti questi gradi non vanno secondo il naturale diatonico: & la natura dell'Enarmonico genere rompe l'ordine del genere Diatonico & del Cromatico, & comporta che si facci i gradi & i salti fuore di ogni ragione, & per tal cagione tal divisione si domanda proportione inrationale. Si ch'el Discepolo de' imparare à comporre di cantare questi gradi & salti sproportionati, ..., & che nelle compositioni sappia accordare et accompagnare con l'armonia ogni sorte di voci sproportionate, & inrationali; ...". ["... thus the nature of the division of the chromatic genre involves the breaking of the diatonic order, and the making of two semitones from one tone, so that the degree of the uncompounded trisemitone is created; all these degrees do not go according to the natural diatonic: and the nature of the enharmonic genre breaks the order of the diatonic and the chromatic genres, and involves the creation of degrees and jumps outside any reason, and for this cause such a division is called an irrational proportion. Thus the disciple must learn to compose and sing these degrees and jumps out of proportion, ... and in compositions, he should be able to tune and accompany with harmony all sorts of irrational notes and out of proportion."]

[89]"... fient enim quatuordecim coniugationes ..."; Cardano 1663, I, p. 14. Cardano 1982, p. 77.

[90]Cardano 1982, passim.

In general, Vicentino gave the intervals only on the stave, and repeated the numerical ratios of Ptolemy and Zarlino exclusively in the final "Book V", in Chaps. LX–LXV. Here, however, he limited himself to the classic rational numbers, together with other possibilities. For example, he gave a new ratio for the minor third: "… it is like from 4 and a half to 5 and a half [9:11]. This is rational." For his musical purposes, the conclusion of the book sounded particularly clear. "Dichiarazione sopra li difetti del Liuto, e delle Viole d'arco, et altri stromenti con simili divisioni. C.[apitolo] LXVI. Dall'inventione delle viole d'arco et del liuto fin hora sempre s'ha suonato con la divisione dei semitoni pari, et hoggi si suona in infinitissimi luoghi, ove nascono due errori, uno che le consonanze delle terze et in certi luoghi delle quinte non sono giuste, & l'altro errore è quando tali stromenti suonano con altri stromenti che hanno la divisione del tono partito in due semitoni, uno maggiore et l'altro minore non s'incontrano, di modo che mai schiettamente s'accordano quando insieme suonano." ["Affirmation regarding the defects of the Lute, and violas, and other instruments with similar divisions. C.[hapter] LXVI. From the invention of the viola and the lute until now, they have always been played with the division of equal semitones, and today they are played in an infinite number of places, where two errors develop, one that the consonances of the third and in some places of the fifths are not correct, and the other error is when these instruments play with other instruments that have the division of the tone into two semitones, one greater and one lesser, they do not meet, with the result that they are never perfectly in tune when they play together"].

Music was his profession, and this got him into trouble. He even arrived at a public debate on how best to compose: only in the diatonic genre or also in the chromatic and enharmonic ones, preferred by Nicola Vicentino? Thus he was condemned, and lost a wager with a rival. "… intende di qual Genere sia la compositione che hoggi communamente i compositori compongono, & si canta ogni dì, … Et per questo il detto Don Nicola dover essere condennato, come lo condenniamo nella scommessa fatta fra loro, …". ["… he knows what genre the composition is which composers usually compose today, and is sung every day, … And for this reason the said Don Nicola must be condemned, as we condemn him in the wager made between them, …"].

Thus in the Christian and papal Rome, discriminations continued to be made, preferring the musical genre which had been considered the most suitable since the times of Plato. This way of composing excluded irrational ratios, which opened up to the chromatic and enharmonic genres considered socially indecorous. On the contrary, Vicentino loved them and would have liked to theorise them, providing also the instrument capable of playing them: the archicembalo (a great harpsichord with six keyboards). But on this, the tuning was still too full of the traditional *commas* to be really practical. We shall soon find the composer and lute-player, Vincenzio Galilei, who fully returned to Aristoxenus and was to introduce more radical novelties with greater decision.[91]

6.6 Variants of Pythagorean Orthodoxy: Gioseffo Zarlino, Giovan Battista Benedetti

It is curious that destiny had led also the musical theory of Maurolico to arrive at the same publisher, Francesco De Franceschi, of Gioseffo Zarlino (1517–1590): the most renowned theoretician of music of the age. In the same years when Boethius was being printed,[92] the books of the musician and theoretician from Venice, among which we will recall the first *Istitutioni Harmoniche*, proposed novelties, compared with the former, that were destined to spread all over Europe, marking the course of music.

Maurolico had ignored Zarlino, notwithstanding his books were already in circulation before the dates left in the manuscript of his *Musica*. At times, however, the scholar from Messina wrote about major and minor thirds, instead of the Pythagorean ditone. Thus, he too adopted for the intervals the terminology introduced by Zarlino, and still used today in academies of music. But he only used the names, not the substance. He called the Pythagorean ditone a major third, and the ditone without an *apotome* a minor third. Our Sicilian natural philosopher then played with the combinations of *apotome* and diesis (the unequal greater and lesser semitones into which the Pythagoreans had divided the tone), breaking down all the other intervals to their level. He even modified them, both by 'increasing them', substituting the *apotome* in the place of the diesis, and 'diminishing them', putting, vice versa, the diesis in the place of the *apotome*.[93] He appeared to be interested in calculating all the possible combinations, and even reduced the musical scale to a series of such semitones. His taste for combinations and mathematical symbology came to the surface elsewhere, as well.

Others, too, like Marin Mersenne and Wilhelm Leibniz, were later to conceive of a music similar to the art of calculating combinations between notes.[94] From Johann

[91] Cardano 1663, III, p. 603. Vicentino 1555, pp. 66v, 98v, 144v and 146v. Massera 1977, pp. 124–133. Cf. Pesic 2010, pp. 522 e 516. Pesic translates "proportione inrationale" as "irrational ratio". This leads him into a labyrinth of contradictions, different meanings and shifts of meaning between ratio, reason, proportion, cause, and so on. It would be better to translate as "irrational proportion". The *ratio* recalls the $\lambda o\gamma o\varsigma$, which, for the Pythagoreans, is necessarily only a relationship (ratio) between whole numbers. Whereas, the "proportion" was suitable also for magnitudes and could therefore even be "irrational". Many were worried that music was capable of unleashing the most diverse passions. On this subject, read Palisca 2000a.

[92] Venezia 1492; Massera 1977, p. 23. On Zarlino, see Massera 1977, pp. 134–140, 145–147; Walker 1989c; Mambella 2008.

[93] See Appendix C, Chart 37.

[94] See Part II, Sects. 9.1 and 10.1.

Sebastian Bach to Arnold Schoenberg, the musical scale was to be considered as decomposed into so many semitones.[95] But we shall see that these semitones were to be different from the Pythagorean ones, because they were to be tempered, that is to say, made all the same as one another, at least approximately.

The variant of Zarlino was different from the combinations of Maurolico, and was destined to catch on among theoreticians of music; even if it, too, was born within Pythagorean theory, like that of Ptolemy. Instead of working on the interval of the fourth, as the Greeks and the Arabs had done, he decomposed the interval of the fifth as follows: in order to divide it, he calculated, between 3 and 2, both the arithmetic mean:

$$\frac{1}{2}(2+3) = \frac{5}{2}$$

and the harmonic mean:

$$2 \times \frac{(2 \times 3)}{(2+3)} = 2 \times \frac{6}{5}$$

which he introduced into the interval of the fifth, shifting them by an octave. The ratio of the fifth 3:2 was thus decomposed into two new ratios, 5:4, the major third, and 6:5, a minor third. These latter went outside the Pythagorean *tetractis*, because they also used the numbers 5 and 6. The innovation of Zarlino was thus called a *senarius*, and the two thirds were included among the consonances, whereas the Greeks had not admitted the ditone.

Thus, also the moderate innovators now appeared on the scene of the sixteenth century: though conserving the general mathematical approach determined by numerical proportions, they introduced other ratios, and consequently, other intervals. Therefore the classification of the intervals was enriched by the major third, which took the place of the Pythagorean ditone, the minor third and the sixth 5:3.

It should be underlined that the first serious variant to Pythagoreanism, partly inspired by Ptolemy, came from professional musicians. But for the sake of brevity, we cannot, unfortunately, digress on music. We will only recall that polyphony had reached the heights of Giovanni Pierluigi da Palestrina (c.1525–1594), Roland De Lassus (c.1530–1594) and other Flemings: like Josquin Després (1440–1521), Henricus Isaac (c.1450–1517), Adrien Willaert (c.1480–1562) or Cyprien De Rore (1516–1565), who was at home between Rome, Florence, Venice and Parma. We could not keep silent either about the Venetians Andrea Gabrieli (c.1510–1586) and Giovanni Gabrieli (1557–1612), or the Neapolitan Carlo Gesualdo, Prince of Venosa (c.1560–c.1613), not to mention the madrigalists, Luca Marenzio (1553–1599) and Adriano Banchieri (1568–1634). Pythagorean theories no longer seemed to be sufficient to discipline all this creativeness. In this way, the simple melodic

[95] See Part II, Sect. 12.4. Tonietti 2004.

linearity of the Greek conception, or of Gregorian chant, might perhaps have been handled, with their relatively few meetings between the notes. But now, with those complex polyphonic Masses with 4, 8, 16, or 32 voices, the meetings and clashes between notes were multiplied, with effects that invited reflection on consonances or not.

In view of these problems, the Franciscan monk and priest Zarlino, a pupil of Willaert and the successor of De Rore as master of the chapel at St. Mark's, constructed his proposal for the new consonances of the third. His declared aim was allowing variety. "La varietà dell'harmonia . . . non consiste solamente nella varietà delle consonanze che si fa tra due parti. Ma nella varietà anco delle harmonie . . .". ["The variety of harmony . . . does not consist only of the variety of consonances between two parts. But also in the variety of the harmonies . . ."]. Although the Venetian musician continued essentially to deal with vocal polyphony, the term 'harmony' in time entered into use with a precise technical meaning, starting precisely from his major and minor thirds, but going much further than he had imagined. In the end, 'harmony' was even to be proposed as an alternative to polyphony. However, it is clear that our musician from Venice desired to maintain the general Pythagorean-Boethian picture, introducing only that amount of change that was sufficient to take into account the current music of the period. However, other musicians were to propose more radical changes than his. We shall now also write about these, because they maintained more or less close relationships, but all clearly with the evolution of the sciences.

Giovanni Battista Benedetti (1530–1590) was another Venetian, and was thus a member of that cultural context which has always, up to the present day, been outstanding particularly for music. His knowledge on the subject, and of the people, are certified by the letter to Cyprien De Rore that he published in the book *Diversarum Speculationum Mathematicarum et Physicarum Liber [Book of various mathematical and physical speculations]*. It also contains a chapter (XXXIII) in which this natural philosopher raised objections against the music of the spheres, which went back to the Pythagoreans. He took his inspiration also from Aristotle, and he tried to base his criticism mainly on terrestrial observations. If there were no air in the sky, how could the stars emit sounds? If the orbits in contact were "politas ac lenas" ["smooth and soft"], how could they generate sounds, seeing that rubbing two smooth mirrors together, no sound can be heard. Only if there were some element of "asperitatis, aut inaequalitatis" ["asperity or inequality"] and not perfectly smooth (as many believed the stars to be), a rotating sphere would generate sounds. ". . . ut etiam experientia à corpore aliquo fluido, quod in alio velocissime moveretur desumpta fretus . . ." [". . . as also basing myself on the experience taken from a body that moves with great speed into some other fluid, . . ."].

Benedetti even noted that the music of the spheres lacked consistency, because the relationships between the musical intervals, as fixed by the ratios of the Pythagoreans, did not correspond exactly to the aspects of astrology: such as sextiles, trines, quadratures and oppositions. In spite of Ptolemy, difficulties would arise. "Quod autem attinet ad motus, ad magnitudines, ad distantias, & ad influxus, nihil est, quod hisce proportionibus conveniat, sed quia haec omnia dependent ab infinita, & divina providentia Dei, necessario fit ut istae velocitates, eae magnitudines, distantiae, & influxus, talem ordinem, & respectum inter se ipsa, & universum habeant, qualis perfectissimus sit." ["As regards movement, dimensions, distances and the influences [of stars], nothing of this conforms to these ratios. But as all these things depend on the infinite and divine wisdom of God, then necessarily the speeds, dimensions, distances and influences will have that certain order and relationship between themselves and the universe, in such a way to be the most perfect."]

Among the consonant musical ratios, our Venetian patrician placed also the sesquifourth (5:4) and the sesquififth (6:5), which Zarlino had introduced.[96] In his letter[97] to his beloved friend, Cyprien De Rore "musico celeberrimo" ["most renowned musician"], Benedetti denied "... quod aliquis recte possit intelligere rationes consonantiarum musicae, absque cognitione illarum mediante ipso sensu, ..." ["... that anyone can understand correctly the relationships of musical consonances without a knowledge of those [obtained] through the senses themselves, ..."]. And by "the senses", our natural philosopher from Venice intended also musical practice. Vice versa, however, the "pratico puro" ["pure practical man"] would not be able to understand the intervals properly, and thus "... ad comparandam perfectionem musicae necessarium sit, & theoriam & praxim addiscere." ["... to prepare the perfection of music, it is necessary to devote oneself both to theory and to practice."]

In order to explain to a practical musician the numerical ratios of intervals, he even knew how to write notes on a stave. He could thus face up to the central problem that remained also in the variant of Zarlino: that of tuning organs and harpsichords, avoiding undesirable dissonances. For this reason, the fifths needed to be suitably diminished and the fourths increased slightly. If they left the theoretical 3:2, with 27:8 every three fifths, diminished by an octave 27:16 ($do - sol + sol - re + re - la = do - la$), the la would be found not to be tuned in the same way as the consonant major sixth 5:3. The major thirteenth $do - la$ was considered: "... valde odiosa ... sensui auditus ... auribus valde inimica ..." ["... rather distasteful to the sense of hearing ... rather hostile to the ears ..."]. They used the term *comma*, like the Pythagorean one, for this other fastidious difference to be eliminated between $\frac{27}{16}$ and $\frac{5}{3}$, that is to say, $\frac{27}{16}:\frac{5}{3} = \frac{81}{80}$, called the *comma* of Didymus, or syntonic, since the times of Ptolemy. Zarlino had brought the matter up, because the new ratios of the third generated, together with the Pythagorean tone 9:8, a second minor tone $\frac{5}{4}:\frac{9}{8} = \frac{10}{9}$. The difference between the major tone and the minor one equals $\frac{9}{8}:\frac{10}{9} = \frac{81}{80}$. A similar reasoning should also be carried out also for the fourths, to avoid three of them ($do - fa + fa - si + si - mi = do - mi$) differing from the major third $do - mi$ by the same *comma*. In order not to offend the ear, then,

[96] Benedetti 1585, pp. 190–191.

[97] Written between 1558, when Zarlino published the book Benedetti quoted, and 1565, the year De Rore died.

musicians had to tune their instruments with suitably dropping fifths and increasing fourths.

Benedetti still reasoned with proportions, e.g. "81 to 80" and "10 to 9" or with the Latin words "sesquioctuagesima" [9:8] and "sesquinona" [10:9], without using fractions as we have done. Perhaps also for this reason, there are some errors in the text. He did not make any mistake, on the contrary, when he calculated how many *commas* 81:80 there were in the major and minor tones. He calculated that $(\frac{81}{80})^9 = \frac{150094635296999121}{134217728000000000}$ and $(\frac{81}{80})^{10} = \frac{12157665459056928801}{10737418240000000000}$, concluding that 9 *commas* exceed 10:9, and 10 *commas* 9:8. Less motivated, perhaps, by minor calculations, he did not repeat the mistake of Boethius, or the brilliant demonstration of Maurolico either, however.

He quoted the "Excellentissimus Zarlinus in secunda parte *Istitutionum Harmonicarum*" ["Most excellent Zarlino in the second part of the *Institutioni Harmoniche*"], adding: "Sed quia sensus auditus non potest exacte cognoscere debitam quantitatem excessus, vel defectus, intendendo vel remittendo chordas instrumentorum, ideo hanc viam sequutus sum." ["But as the sense of hearing cannot know exactly the due quantity of the excess or the defect, in tightening or relaxing the strings of instruments, for this reason I have followed this way."] Helped more or less by the calculations on the *commas*, the ear of Benedetti seemed to tolerate dropping fifths and increasing fourths quite well. With him, the 'perfect' Pythagorean ratios now came down to the earth, where they had to be content with approximations.

And yet here, the Venetian patrician also sought other justifications for the way of generating consonances: "... qui quidem modus fit ex quadam aequatione percussionum, seu aequali concursu undarum aeris, vel conterminatione earum." ["... which becomes, indeed, the measure by means of a certain equality of strokes, or by means of the equal convergence of waves of the air, or the joint termination of these."] Therefore, he carried out experiments "nella mente" [in his mind] with the monochord, shifting the *ponticello* so as to divide the string according to the ratios desired: one half, one third, two fifths. He thus obtained unison, the octave and the fifth. He described the movements of the vibrating strings well. "... quo longior est chorda, etiam tardius moveatur, quare cum longior dupla sit breviori, & eiusdem intensionis tam una quam altera, tunc eo tempore, quo longior unum intervallum tremoris perfecerit, brevior duo interValla conficiet." ["... the longer the string is, the more slowly it moves; thus, when the longer is twice as long as the shorter, and both the one and the other have the same tension, then in the same time that the longer completes one interval of vibration, the shorter completes two."]

When the vibration of the string becomes percussion in the ear, "... in qualibet secunda percussione minoris portionis ipsius chordae, maior percutiet, seu concurret cum minori, eodem temporis instanti, ..." ["... the greater strokes or converges with the lesser at the same instant of time, at every second stroke of the lesser portion of string, ..."], in proportion to the lengths. Thus the ratio of the lengths gave the ratio of the strokes, which corresponded better, the better the consonance was: all in unison, one every two for the octave, etc. But with these "percussiones" ["percussions, strokes"], what would prove to be in proportion? ". . . hoc est tempus maioris intervalli ad tempus minoris erit sesquialtera . . .[. . .] . . . eadem proportio erit numeri intervallorum minoris portionis ad interValla maioris, quae longitudinis maioris portionis ad longitudinem minoris . . ." [". . . that is to say, the time of the greater interval will stand in a sesquialtera ratio [3:2] to the time of the lesser . . .[. . .] . . . the same ratio as the number of intervals of the lesser portion compared with the intervals of the greater portion will be that of the length of the greater portion compared with the length of the lesser . . ."].

For Benedetti, sound was a wave in the air, though he was not clear enough about how to measure it. He wrote about "time", "numbers of intervals", "percussions"; if he had calculated the ratio, he would have obtained the wave frequency. Instead, he left us other numbers, with the idea of seeking what consonances maintained in common, when varying the length of the string that generated them. ". . . unde productum numeri portionis minoris ipsius chordae in numerum intervallorum motus ipsius portionis, aequale erit producto numeri portionis maioris in numerum intervallorum ipsius maioris portionis; . . ." [". . . hence, the product of the number for the lesser portion of the same string with the number of intervals for the movement of the same portion will be equal to the product of the number for the greater portion with the number of intervals for the same greater portion; . . ."]. Thus, the diapason (octave) was characterised by the number 2 (2×1), the diapente (fifth) by 6 (3×2), the diatessaron (fourth) by 12 (4×3), the major hexachord (major sixth) by 15 (5×3), the ditone (major third) by 20 (5×4), the semiditone (minor third) by 30 (6×5), the minor hexachord (minor sixth) by 40 (8×5). The better the consonance, the lower the number would be.

In the end, the patrician from the Veneto region, a music lover, concluded his open letter to his friend, the renowned polyphonist, as he had started it: with a hymn to sensuality. "Voluptas autem, quam auditui afferunt consonantiae fit, quia leniuntur sensus, quemadmodum contra, dolor qui a dissonantiis oritur, ab asperitate nascitur, id quod facile videre poteris cum conchordantur organorum fistulae." ["On the other hand, the pleasure that consonances procure for the hearing originates because the senses are sweetened by them, just as, on the contrary, the pain that stems from dissonances is born from roughness; you will easily be able to appreciate this when the pipes of organs are tuned properly."][98]

Historians have generally judged Benedetti comparing him with the future Galileo Galilei. The former described the fall of heavy objects through the air as independent of their weight, in contrast with Aristotle, but without giving the mathematical law in a void like the latter. For music, we have preferred here to narrate the results in relation to Pythagoreanism. Our Venetian natural philosopher, who had stayed at Parma, and ended up at Turin, moved away from that tradition, because he criticised the music of the spheres and declaredly used his ear with an empirical spirit foreign to that tradition. However, in an attempt to combine together a mathematical theory based on numbers with the practice of musicians,

[98]Benedetti 1585, pp. 277–283.

he remained linked to Pythagorean ratios, only enriched by the Venetian Zarlino, without taking into consideration the alternative of Aristoxenus. And yet somebody else was to do so: a musician or a natural philosopher?

> Musicorum et cantorum magna est distantia
> Isti dicunt, illi sciunt, quae componit musica.
> Nam, qui facit quod non sapit, diffinitur bestia.
> [The distance is great between musicians and singers]
> The latter say, the former know, the things that music composes.
> For he who does what he does not know is defined as a beast.
>
> Guido D'Arezzo

6.7 The Rebirth of Aristoxenus, or Vincenzio Galilei

Zarlino was criticised by the Florentine noble, Vincenzio Galilei (1520–1591) in his *Dialogo della musica antica et della moderna* [Dialogue between ancient and modern music]. A lute-player, composer and *connoisseur* of rival theories (he had even been a pupil of Zarlino), this musician from Florence immediately denounced "... la poca fede d'alcuni stampatori di Venezia ..." [the little faith of some printers in Venice] who are said to have boycotted him "... per compiacere ad alcuno il quale o tratto da invidia impediva che queste mie fatiche uscissero fuore ..." [to please someone who, moved by envy, prevented these efforts of mine from coming out]. As a result of these editorial intrigues, he consequently printed the book at Florence, and in Italian instead of Latin, under the patronage of Count Giovanni Bardi. Actually, Vincenzio Galilei was a member of the famous Camerata de Bardi, which was renewing musical style, moving away from polyphonic complications in search of a monodic, melodic simplicity, suitable to let the meaning of the poetic verses in music be understood: "recitar cantando" [reciting in song]. Our Florentine musician expounded his theory in the form of a Platonic dialogue between two characters, one of whom was his patron, the musician Giovanni Bardi.[99] It can already be seen, from the way the subject under discussion was presented, that even though he dealt with the numerical ratios for notes in detail, Vincenzio Galilei evaluated the theory on the basis of the requirements of musicians. Definitely, beside his calculations for the ratios, he always put a stave with the relative notes (Fig. 6.7).

To convince the reader, whether theoretician or musician, of his arguments, he described how to construct an instrument on which they could be verified. "Tirinsi sopra una piana superficie due corde all'unisono, d'un'istessa lunghezza, grossezza, & bontà; dividasi poi col compasso una di esse ... & chi volesse ancora udire qual si voglia intervallo in una sola corda, ...". [Let two strings be extended over a plane

[99] Vincenzio Galilei 1581, p. [iii]. As Fabio Fano wrote in the "Prefazione", the charge against Zarlino of having prevented the publication of the book at Venice became explicit in the *Discorso intorno all'opere di Messer Gioseffo Zarlino* of 1589, p. 8. Cf. Massera 1977, pp. 140–148.

Fig. 6.7 How Vincenzio Galilei assigned the relative ratios to the notes on a stave (Vincenzio Galilei 1581, p. 20)

surface in unison, with the same length, breadth, and good quality: then let one of them be divided with a compass . . . and anyone who wants to hear any interval on a single string, . . .].[100] "... in reprovare l'opinione" [in reproving the opinion] of "Reverendo M. Gioseffo Zarlino", the Florentine composer went back over the current theory, which had modified the Pythagorean order into Ptolemy's "syntone", and contained the (already seen above) new consonances of the third and the sixth in the new ratios of 5:4, 6:5, 8:5, 5:3. But, this other Galilei noted, "(... contro l'opinione del prattico)" [in contrast with the opinion of the practical musician], in the new scale, not all the minor thirds have the same ratio, because this depends on where they begin. On the contrary, the musician would like to play on the keys (or on the lute) in the same way the intervals indicated at the same distance on the stave. The problem derived from the difference between the Pythagorean major tone 9:8 and the minor one 10:9, called the syntonic *comma* 81:80, already seen in Benedetti. Or again, Zarlino's minor third, together with the major tone, exceeded the correct Pythagorean fourth by a *comma*. And so on, the book multiplied examples that put Zarlino in contrast with musicians, the "prattici moderni contrappuntisti" [practical modern contrapuntists], with the numerical ratios calculated by theory, and lastly artificial theory with "Nature", including and excluding those syntonic *commas*, which others cheerfully ignored.

Also Galilei father wrote of how many of these *commas* were contained in major and minor semitones and tones, though without offering any calculations. And as regards Pythagorean *commas*, instead, he trusted Boethius, who was honoured with a "very well", which we now know he did not deserve.[101] He represented Zarlino's *senarius* in a figure which contained its intervals (Fig. 6.8).

He insisted on pointing out that all this was not new at all, but had been taken from Ptolemy's *Harmonics*; in his *Quadripartite [Tetrabiblos]*, Ptolemy had even compared "gli aspetti de' pianeti alle forme degli intervalli musici ..." [the

[100] Vincenzio Galilei 1581, p. 3.

[101] Vincenzio Galilei 1581, pp. 9–10.

Fig. 6.8 How Vincenzio Galilei represented Zarlino's *senarius* (ibid., p. 10)

aspects of the planets to the forms of musical intervals.] The Florentine lute-player considered the names given at the time to musical intervals "corrotti & guasti" [corrupted and wrong], because the same word indicated different ratios. He rebuked practical musicians because, instead of their "... intelletto o il senso dell'udito" [intellect, or sense of hearing], they trusted (in reading music?) their sense of sight, when this deceived us, because "... ha nel distinguere i suoni quella o poca più parte che ha l'udito nel discernere le differenze dei colori; & particolarmente si ingannano i sensi tra le minime differenze dei comuni & dei proprii oggetti." [... in distinguishing sounds, it has the same role, or little more, as hearing has in discerning differences of colours; and particularly, the senses are deceived between the minimal differences of common, or personal, objects][102]

Vincenzio Galilei knew where to set the difficulty in dividing intervals into equal parts: "... per non potere dividere alcuno intervallo rationalmente de tre primi semplici generi di proporzioni in parti uguali, ..., intendendo sempre secondo la facultà Aritmetica." [... because it is not possible to divide any interval of the three primary simple genres of ratios into equal parts, rationally, ... meaning always according to the Arithmetical means.] To this, he added his repeated praise of the ancients: "... la musica prattica de tempi nostri non habbia quella facultà d'operare negli animi degli uditori alcuno di quelli maravigliosi & virtuosi effetti che l'antica operava." [the practical music of our times does not possess that faculty of creating any of the wonderful, virtuous effects in the minds of listeners that ancient music created.][103] Musicians, all dedicated to polyphony (which for him was the modern music of the period), believed that they were singing in accordance with the tradition of the Ptolemaic syntone, but they were wrong, because they did not take the *commas* into consideration. Then, the Florentine composer took as the lesser semitone the ratio 135:128, which, added to the semidiapente 64:45, completed the fifth 3:2, and, subtracting this from the tritone 45:32 gave the fourth 4:3 again.[104]

Thus we might take him for a better Pythagorean, who revealed the defects of modern theoreticians and of "modern practical contrapuntists" (like Zarlino), unable to understand the real scale of Ptolemy. But no: with him, also the theory of Aristoxenus, after being opposed for centuries, at last reappeared on the stage,

[102]Vincenzio Galilei 1581, pp. 16–17.

[103]Vincenzio Galilei 1581, pp. 15–16.

[104]Vincenzio Galilei 1581, pp. 25–26.

now reappraised and admitted like the others, and no longer rejected on principle by the authorities.

In order to overcome the preceding difficulties, musicians made sure that "... vengano le Quinte rimesse [abbassate], & per l'opposto le Quarte tese [innalzate] un poco più di quello che alle proporzioni converrebbe ...". [the Fifths were lowered, and on the contrary, the Fourths tightened [raised] a little more than was appropriate for the ratios ...]. Benedetti had made exactly the same proposal.[105] But now Vincenzio Galilei, in criticising Zarlino, was obtaining, for the current practice of players and composers like himself, a theoretical opening, and a historical justification which our natural philosopher from Venice lacked. "Fra gli strumenti di corde tengo che la Viola d'arco, il Liuto & la Lira con i tasti suonino il Diatonico incitato di Aristosseno: & muovemi a creder questo, il vedere & udire in essi l'ugualità de Tuoni ugualmente i due pari Semituoni divisi; ...". [Among the stringed instruments, I believe that the Viola, the Lute and the Lyre play, with their frets, the diatonic *incitatus* by Aristoxenus: and I am moved to believe this by seeing and hearing in them the equality of tones, and also the two equal semitones divided.] "Tengo che la Terza maggiore sia contenuta in una proporzione irrationale assai vicina alla Sesquiquarta [5:4], ma non già che i suoi lati (per così dirgli) siano il Tuono Sesquiottavo [9:8] & il Sesquinono [10:9]; ma si bene due parti uguali di detta Terza, tale quale ella è divisa al modo de Tetracordi d'Aristoxenus; ma non così esattamente." [I believe that the major third is contained in an irrational ratio very close to the Sesquifourth [5:4], but not in such a manner that at its sides (so to speak), there are the sesquieighth tone [9:8] and the Sesquininth [10:9] instead however there are two equal parts of this third, in the same way that it is divided in the Aristoxenus' tetrachords; but not so precisely] "Fuggendo sempre (...) l'inegualità de Tuoni ... noi al presente torremo principalmente secondo il modo d'Aristoxenus, per non potersi in altra maniera dividere in parti uguali alcuno intervallo superparticolare, ...". [Always fleeing from ... the inequality of tones, at present we will take away principally in accordance with the method of Aristoxenus, because we cannot divide any superparticular interval into equal parts in any other way.] Thus the Florentine composer stated that the lute and the viola should be tuned in accordance with the scale of Aristoxenus.[106]

"... diviso il Tuono in due parti uguali ..." [... having divided the tone into two equal parts ...], the Octave "... trovo havere divisa in dodici Semituoni & sei Tuoni, così detti da Aristosseno ..." ["... I find is divided into twelve semitones and six tones, so defined by Aristoxenus ..."]. For the tuning, in practice, he tried to approximate himself to the exact division by means of geometry. "Divido adunque la linea AB in diciotto parti & verso l'acuto (dal grave partendomi) dove quella prima parte termina pongo il primo tasto [sul manico del liuto]." [Thus I divide

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[105] See above, Sect. 6.6.

[106] Vincenzio Galilei 1581, pp. 30, 31, 33, 42. He even translated Aristoxenus into Italian. He was probably not satisfied with the translation into Latin published in 1562. A lot of his knowledge about Greek music derived from Girolamo Mei (1519–1594). Palisca 1989, pp. 168–169.

the line AB into eighteen parts, and towards the acute (starting from the deep) where that first part terminates, I place the first fret [on the neck of the lute]. Galilei father seemed to know that his procedure would be approximate. In fact, in trying to justify the number eighteen (6, 12, 18?), he referred to the compass, which would not enter precisely "… nel voler misurare in sei volte appunto la circonferenza del circolo con l'apertura di esso." [… in wanting to measure in six times the circumference of the circle with its opening.] Consequently, the "industrioso agente … con la sua discrezione e diligenza" [industrious agent … with his discretion and diligence] would have to "… ovviare a quella poca disconvenienza che è tra il misurante & il misurato." [make up for that little inconvenience that exists between the measurer and the thing measured.] And our musician from Florence did not miss the opportunity to criticise his Venetian rival theoretician, who tried to justify his senarius. "Vuole il Zarlino al c. 14 della I parte delle sue *Inst.[itutioni]* che l'apertura di esso sia appunto la 6. parte del cerchio; la qual cosa è falsa." [Zarlino says in Chap. 14 of Part I of his *Inst.[itutioni]* that its opening is precisely the 6th part of the circle; which is false.] Vincenzio Galilei then tried to translate into whole numbers "… quello che pur hora vi ho mostrato con la linea." [what I have just shown you with a line.] But he took good care not to use square roots, perhaps in order to avoid risking the current objections of natural philosophers, who, as he too must have known, prohibited their use in music.[107]

However, this lute player did not go so far as to propose the extension of the tuning of Aristoxenus (today known as the equable temperament) also to keyboard instruments, that is to say, those with a fixed tuning. That which to us would seem natural and convenient today was excluded by him. "… il Liuto ha diviso il Tuono in parti uguali, & lo strumento di tasti l'ha in parti disuguali separato …". [… the Lute has divided the Tone into equal parts, and the instrument with keys has separated it into unequal parts …] He was thinking of the organ and the relative great quantity of sound, as a result of which "… esso con violenza maggiore ferisce l'udito" [… with greater violence it offends the hearing] with dissonant chords. In that period according to this Galilei, therefore unlike the lute, certain tempered chords produced on keyboards, would have caused an "intolerable" effect. Only in Bavaria, when he was playing for the "Great Albert", had he encountered a keyboard instrument, with strings similar to those of the Lute and the Viola, which, tuned like these, made the "sweetest sound". Following his hearing, he noted that above all it was a question of getting used to it. As a result of playing the Lute with that equable tuning, he had "assuefatto il senso […] … per essere di già invecchiato in quel si fatto temperamento …", ["accustomed his sense […] … because he had already grown old in that temperament …"], a thing which had not (yet!) happened with keyboard instruments.[108] As a matter of fact, today I play a piano tuned (like the others) by following the equable temperament of Aristoxenus, without offending my ears with the relative chords.

[107] Vincenzio Galilei 1581, pp. 49–50.

[108] Vincenzio Galilei 1581, p. 47.

Our Florentine composer and musician knew full well what great advantages musicians would obtain if they were freed from the prohibitions and prejudices of orthodox theories. If a keyboard instrument is tuned in accordance with Boethius or Zarlino, with unequal tones and semitones, and all those fastidious *commas* popping everywhere, "... non può il suonatore di esso, quantunque prattico & perito, trasportare in questa & in quella parte per un Tuono & per un Semituono (in quelli dico che per lo più si esercitano) ciascuna Cantilena [melodia], come nel Liuto con tanto comodo & utile si trasporta." [... However skilful and experienced the person playing it may be, he cannot transpose, into one part or another by a tone or by a semitone (I mean among those who are most trained), each *Cantilena* [melody] as it can be transposed so easily and effectively on the Lute.][109] Instead Aristoxenus, according to Vincenzio Galilei who desired to remove "... calunnia da dosso" [slander from his back], "... diviso il modo Dorio in dodici parti uguali ...", ["... having divided the Dorian mode into twelve equal parts ..."], proposed and practised as many as 13 modes, instead of the customary well-known 7 or 8 (Fig. 6.9).[110] Thus, if not (mentally) Aristoxenus, too many of whose papers went astray, at least the father of the more famous Galileo Galilei represented, by means of the geometrical distances of intervals, the 12 scales of the modern minor mode. To see this, it is sufficient to identify the letter A of the Dorian mode as la, and to move upwards or downwards every time by one semitone. The thirteenth scale is the same as the first one, transposed to one octave higher.

Some of these things had actually been taken from Ptolemy, but he continued to insist throughout the book that the best had been offered by Aristoxenus. "... sapeva Aristoxenus d'havere a distribuire in parti uguali la qualità del tuono, & non la quantità della linea, corda, & spazio: operando allhora come Musico intorno al corpo sonoro e non come semplice Matematico intorno alla continua quantità." [Aristoxenus knew that he had to distribute the qualities of the tone, and not the quantity of the line, string, or space, into equal parts: working as a musician around a body making sounds, and not as a simple Mathematician around a continuous quantity.] Ptolemy "... lo riprende inoltre che il Tuono non si possa dividere in due parti uguali ..." [... rebukes him, furthermore, because the Tone cannot be divided into two equal parts ...] that is, having transformed the ratio 9:8 into 18:16, between these, 17 would divide the ratio into unequal parts. "... ma non così disse, né intese Aristoxenus; ma si bene nella maniera che vi ho dimostrato particolarmente nel mettere i tasti al Liuto, nella quale si può veramente dividere ciascuno intervallo musico in quante parti uguali si voglia, non altramente che con il mezzo del Monocordo; perché in quell'atto è considerato dal Musico il suono come qualitativo & non come quantitativo ...". [... however Aristoxenus did not say so, or mean that; but, on the contrary, in the manner that I have shown you, particularly in putting frets on a Lute, in which it is truly possible to divide each musical interval into as many equal parts you like, in no other way than by means

[109] Vincenzio Galilei 1581, pp. 47–48.

[110] Vincenzio Galilei 1581, pp. 51–52.

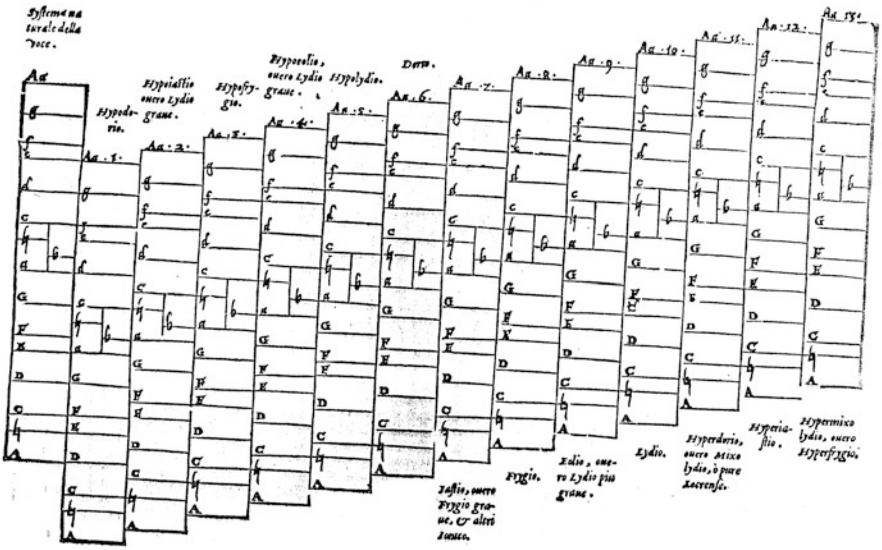

Fig. 6.9 Following Aristoxenus, Vincenzio Galilei distributed notes in scales of octaves divided into 12 equal parts (ibid., p. 52)

of a monochord; because in that action, sound is considered as qualitative by the Musician, and not quantitative . . .][111]

One way or another, our Galilei senior brought the problem back to the practice of musicians, and the ear, which for him seemed to count more than the presumed Pythagorean-Ptolemaic truths. "... con più gusto è universalmente intesa la Quinta secondo la misura che gli dà Aristoxenus che dentro la sesquialtera [3:2] sua prima forma. Né da altro credo veramente ciò avvenga, che dall'haverci il mal uso corrotto il senso: imperoché la Quinta dentro la sesquialtera non solo pare nell'estrema acutezza che ella può andare, ma più tosto che ell'habbia un poco del duro per non dire (insieme con altri d'udito delicato) dell'aspro. Dove, nella maniera d'Aristoxenus, pare che quella poca scarsità gli dia gratia, & la faccia divenire, più secondo il gusto d'hoggi, molle e languida. Ne per altro credo io che ciò avvenga che dall'essere assuefatti udirle del continuo sotto tal forma o simile: dal che si trae un importante & efficace argumento, . . ., che si sia imparato di cantare questo modo dagli Strumenti di corde, & particolarmente da quelli che non hanno come il Liuto & la Viola i tasti." [. . . with greater *gusto*, the Fifth is universally heard in accordance with the measurement that Aristoxenus gives it, than in the sesquialtera [3:2], its original form. And I truly believe that this happens for no other reason than a bad use has corrupted our sense: however, it not only seems that the Fifth in the sesquialtera can go in an extreme acuteness, but rather that it reveals something hard, not to say (together with others whose hearing is delicate) rough. Where, in the

[111] Vincenzio Galilei 1581, p. 53.

manner of Aristoxenus, it seems that a limited scarcity gives grace to it, and makes it become, in line with the taste of today, soft and languid. Neither I believe that this happens except as a result of becoming accustomed to hearing it continuously in this, or a similar, form: we can draw an important, significant argument from this, . . . we have learnt to sing this mode from stringed instruments, and particularly from those that do not have frets, as the Lute and the Viola have.]

That "bad use", to Galilei father, seemed now to become even ". . . abuso, l'imperfettione della musica de nostri tempi; e di quanto l'universale si inganni, . . ., & quanta poca cognitione habbia della vera musica" [. . . abuse, the imperfection of the music of our times; and to what extent all are deceived, . . ., and how little they know the true music.] But wasn't he contradicting himself, therefore? If on the one hand he wanted to remove "slander from the back" of Aristoxenus, but then on the other, he criticised his contemporaries, who, without realising it, took their inspiration from this ancient Greek, thus corrupting true music. The ancient musicians, wrote our lute-player of the sixteenth century, would have exploited him (well) in a different way from the modern ones (who, on the contrary, exploited him badly). ". . . intese Aristoxenus & la più parte degli antichi musici in suprema eccellenza; oltre al non importar cosa del mondo la perfettione & imperfettione degli intervalli, al modo di cantare di quei tempi, per non servirsene (come intenderete) nella maniera che usiamo noi." [Aristoxenus and most ancient musicians of a supreme excellence understood; besides not caring in the slightest about the perfection or imperfection of intervals, or the way of singing of those times, so as not to exploit it (as you will understand) in the way that we use it.][112]

By exhuming Aristoxenus, here Vincenzio Galilei started to show the ground on which he was preparing the theoretical reshuffle. Together with the sacred vocal music sung in churches in a polyphonic style, to uplift the Christian faithful to God, an instrumental music was now being born, played in courts in a worldly monodic style for listeners. In some cases, it might accompany sonnets and poems, allowing the verses to be heard clearly, though: a "recitar cantando" [recital in song].

Among the various writers of treatises on music mentioned and variously criticised, like Didymus, Ptolemy, Boethius, Guido D'Arezzo, Franchino Gaffurio (1451–1522), Heinrich Glareanus (1488–1563), Lorenzo Valla (1447–1500), emerged even a renowned musician like Cyprien De Rore. He mentioned a couple of his compositions, and called him ". . . Musico in questa maniera di Contrapunto veramente singulare, . . .". [. . . a truly outstanding musician in this kind of counterpoint.] Better still, the general thesis of the book is that, in the comparison between ancients and moderns, the latter came off worse, and did not gain anything. Only those who did not know ancient music very well could have sustained the contrary, like Zarlino. ". . . gli antichi Greci & Latini vi [nella musica] dessero opera & studio più di quello che si fa hoggi, & superassero gli huomini de nostri tempi in ciascun'arte & scienza; [. . .] non si ode o pur vede hoggi un minimo segno di quelli che l'antica faceva; . . .". [. . . the ancient Greeks and Latins put

[112]Vincenzio Galilei 1581, p. 55.

their energy and study into it [music] more than is done today, and they surpassed the men of our times in every art and science; [. . .] today, it is impossible to hear or see the slightest sign of what ancient music did; . . .]. The "moderni prattici contrappuntisti" [". . . modern practical contrapuntists"], that is to say, executors or composers without any theory, ". . . tra le molte confuse regole loro . . ." [. . . among their many, confused rules . . .] prohibited ". . . per legge fatale . . ." [. . . by a fatal law . . .] that one perfect consonance should be followed by another one ". . . dell'istessa proportione & spezie . . ." [. . . of the same ratio and kind . . .]. While they made people sing ". . . le sillabe della medesima parola, nel cielo una, nella terra l'altra, & se più ve ne sono, nell'abisso, [. . .] strascinandone bene spesso una di esse sillabe sotto venti & più note diverse, imitando talhora in quel mentre il garrire degli uccelli, & altra volta il mugulare de cani." [. . . the syllables of the same word, one in heaven, the other on the earth, and if there are more of them, in the abyss, [. . .] often stretching one of these syllables well out over twenty or more notes, sometimes imitating in this way the chirping of birds, and other times the whining of dogs.][113]

Our Florentine lover of ancient glories condemned no less the aims of such modern music. It was not composed ". . . per fine alcuno virtuoso, ma per dare piacere a chi ode, & che questo piacere ancora vilmente si faccia: però affermiamo noi tali esercitij non essere da huomini liberi, ma da servili & meccanici artefici." [. . . for any virtuous end, but to give pleasure to listeners, and this pleasure is still unworthily offered: however we affirm that such exercises are not worthy of free men, but of servile, mechanical craftsmen.] "Imperoché quella spezie tanto reputata, la quale fu dalla Natura ordinata, usata nella sua semplicità, era grave, virile & costante, dove per l'opposito questa è per la sua incostanza, ridicola, effeminata & varia. Talmente che, di severa matrona che anticamente era, è divenuta la Musica una lasciva (per non dire sfacciata) Meretrice." [So, that highly esteemed kind, which was ordered by Nature, used in its simplicity, was grave, virile and constant, whereas on the contrary, this, for its inconstancy, is ridiculous, effeminate and fickle. To such a point that, from the severe Matron that she was, Music has become a lascivious (not to say impudent) Prostitute.][114]

Ancient music had maintained a respect for the poetic text, which modern music had now lost. "Imperoché il Musico allhora non era disgiunto dalla poesia, né il Poeta era separato dalla Musica." [Because then the Musician was not separated from the poetry, nor was the Poet separated from Music.] Because music ". . . è l'imitazione dei concetti che si trae dalle parole." [. . . is the imitation of ideas that are drawn from the words.] By harmony, the ancients had meant ". . . il bello & gratioso procedere dell'aria della Cantilena; le parole della quale s'intendevano tutte; & così il verso del Poeta & conseguentemente i concetti loro; senza essere interrotti da accidente alcuno che sviasse l'animo dalla virtù di quelli; l'opposito appunto che occorre alla musica & cantare d'hoggi . . .". [. . . the beautiful and gracious

[113]Vincenzio Galilei 1581, pp. 72 and 78–83.

[114]Vincenzio Galilei 1581, p. 83.

proceeding of the air of the *Cantilena*; the words of which were all understood; and thus the verse of the Poet, and consequently their ideas; without being interrupted by any mishap which distracted the mind from their virtue: quite the opposite of what happens to the music and singing of today ...] The tricks of the modern contrapuntists would have been "... di sommo impedimento a commuovere l'animo ad affettione alcuna: il quale occupato & quasi legato principalmente con questi lacci di così fatto piacere, non gli danno tempo d'intendere non che di considerare le mal profferite parole." [... a great hindrance to move the mind to any affection: to a mind, occupied and almost bound mainly by these ties of such pleasure, they do not give time to understand or consider words badly uttered.][115]

Consequently, the ancients would once have provoked the most surprising effects by means of music. Timotheus is said to have encouraged Alexander to achieve his great conquests; fish could be caught, or elephants calmed. Knowing the power of music, as the Swiss and the Germans subsequently did, the Spartans used it in war. For this reason, they had driven out the lyric singer, Timotheus, the son of Tersander, who instead chose the chromatic genre, more suitable for soft, effeminate types. He came from the island of Milos "... gli habitatori della quale erano (...) huomini lascivissimi & effeminati e tali (...) sono ancora hoggi." [... the inhabitants of which were (...) most lascivious, effeminate men, and such (...) they are still today.] Like the majority of Greeks, the Spartans preferred simplicity; "Il qual buon uso, mediante le lascivie & le delitie in progresso di tempo, s'abbandonò & si perdè interamente; trasferendosi poscia ai Latini la finta piutosto che la vera musica." [Which good use, due to lasciviousness and delights in the course of time, was abandoned, and completely lost; and later transferring to the Latins a false music, instead of the true one.] This Timotheus Milesius arrived at a cithara of eleven strings, and made many more holes on his *tibia* (or *aulos*, a kind of flute) because in that way it produced a more varied kind of music. But the Spartans got angry with him "... perché, rendendo la musica più varia, noceva agli animi de fanciulli & gli impediva dalla modestia della virtù: & l'harmonia che haveva ricevuta modesta rivolgeva nel genere Cromatico che è più molle." [... because, making his music more varied, he did harm to the spirits of young men, and turned them away from the modesty of virtue: and the modest harmony that he had received, he turned it into the chromatic genre, which is softer.][116]

Our Florentine musician distinguished the sciences from the arts: "Le scienze cercano il vero degli accidenti & proprietà tutte del loro subbietto & insieme le loro cagioni, havendo per fine la verità della cognitione senza più: & le arti hanno per fine l'operare, cosa diversa dall'intendere. L'Aritmetico cerca tutte le proprietà & accidenti de numeri ... L'Abbachista poi non si serve di cos'alcuna da queste, ma solo attende a moltiplicare, partire, trarre e raccorre i numeri ...". [The sciences seek the truth of accidents and the properties of their subject, and together their causes, having as their end the truth of knowledge, without anything else; and

[115] Vincenzio Galilei 1581, pp. 99, 88, 105, 87.

[116] Vincenzio Galilei 1581, pp. 86, 90, 100, 102, 106, 107.

the arts have as their end operating, which is different from understanding. The arithmetician seeks all the properties and the accidents of numbers . . . The abacus teacher, then, does not use any of these things, but attends only to multiplying, dividing, subtracting and adding numbers]. And yet he gave the differences between the principal schools as follows: "Alcuni de quali volevano principalmente seguire la ragione de numeri & questi furono i Pitagorici, Harmonici però detti. Altri che proponevano il senso dell'udito alla ragione di essi numeri erano detti Canonici & Canonisti, furono gli Aristossenici. Alcuni poi volevano per qualche via accordar questi & quelli insieme talmente che fra di loro non fussero in cosa alcuna discrepanti; e tali erano i Tolomaici." [Some of whom wanted mainly to follow the reason of numbers, and these were the Pythagoreans, however called Harmonics. Others, who put the sense of hearing before the reason of these numbers were called Canonicals and Canonists; these were the followers of Aristoxenus. Some, then, wanted in some way to reconcile the ones and the others, so that there would not be any discrepancies between them; and these were the Ptolemaics.][117]

However, in referring the divisions of the tetrachord (the interval of the fourth) operated by the different schools for the three genres (diatonic, chromatic and enharmonic), he fixed numbers also for those of Aristoxenus. Thus, for him, Galilei father imagined the interval of the fourth to be divided into 60 no-better-specified "particelle", [particles] perhaps obtained by ear on the instrument. The semitone, or two enharmonic quarters of a tone, contains 12, and each tone 24. Also whole numbers were made to correspond to the notes: $E[mi]120, F[fa]114, G[sol]102, a[la]90$. Thus the Florentine lutenist even put Aristoxenus into the Pythagorean fourth 120:90, that is to say, 4:3. Only the internal division abandoned the 'simple' Pythagorean ratios, for others which he gave without any justification. These appear to be approximations to those referring to the equable temperament. 120:114 $=$ 1.052 . . . [to be compared with the tempered semitone 1.059. . .]; 114:102 $=$ 1.117 . . . [to be compared with the tempered tone 1.122. . .]; and the other tone was 102:90 $=$ 1.133 . . . Thus they would be two different tones, and not equal as in Aristoxenus. Curious that, after what he had written before, he did not use continuous quantities, that is to say roots, for these divisions. Did something still stop him from doing so? For the other genres, the fourths attributed to Aristoxenus were divided with other numbers. For example, for the enharmonic, 120, 117, 114, 90.[118]

The father of Galileo Galilei made a critical reanalysis of the legend of Pythagoras, who was written to have invented the laws of harmony, thanks to the sounds produced by hammers of different weights. Plutarch (c.50–120) had sustained that the ratios of the weights, 6, 8, 9, 12 were the same as those of the length of strings to obtain the octave, the fifth, and the other intervals. But our musician of the sixteenth century must now have had doubts about it, because he

[117]Vincenzio Galilei 1581, pp. 105, 107.

[118]Vincenzio Galilei 1581, pp. 107–111. Earlier, on p. 41, he had written that 60 had been chosen because it is divisible by 2, 3, 4, 5, 6.

launched himself into considerations and comparisons with other cases. First he gave a description of sound in the air, which finally arrives in the ear as a stroke. "… l'intensione dell'aria che racchiusa nel mezzo di quelli strumenti che la percuotono schizza quasi del mezzo di loro fuora per forza; & con il suo empito tutta unita come l'è stata da quella ristretta insieme, urta in quella che l'è contigua all'intorno, spingendo sempre infino che la più vicina al sensorio sforzata da quel moto, quasi ferisce quelle cartilagini che ferite fanno il sentire; il qual colpo sentito è veramente il suono." [… the tension of the air, gathered inside those instruments that strike it, almost bursts out of them of necessity; and with its vehemence all compacted, as it has been by being squeezed together, crashes into the one that is closest around it, continually pushing until the closest one to the sensory organs, forced by that movement, almost injures the cartilages, which, with the injury, create the hearing; this stroke heard is really the sound.]

What if, instead of hammers, it had been weights? "Poteva Plutarco … considerare gli istessi numeri applicati a pesi attaccati a quattro corde uguali in lunghezza, grossezza & bontà; le quali percosse si udirebbe[ro] uscire da esse gli istessi musicali intervalli & per l'istesso ordine che si udivano ne quattro martelli: ma tanto più sonori & distinti in quelle che in questi, quanto le corde sono più atte (dopo l'esser tese e percosse) de quattro semplici pezzi di ferro, a rendere il suono intelligibile e rationale." [Plutarch could … consider the same numbers applied to weights attached to four strings equal in length, breadth and quality; when these are struck, the same musical intervals would heard coming from them, in the same order, as were heard in the four hammers: but all the more sonorous and distinct in the strings than in the hammers, seeing that the strings are more suitable (after being pulled taut and plucked) than four simple pieces of iron, to make the sound intelligible and rational.][119]

He thus set out along a different road, which in a few years was to lead him to quite different conclusions: he was to start listening with his own ears to the sound effects obtained by changing the weights in order to vary the tension of the strings, other conditions of length, breadth and quality being equal. Because in his *Discorso intorno all'opere di Messer Gioseffo Zarlino da Chioggia*, of 1589, he no longer indicated the usual Pythagorean 2:1, 3:2, 4:3, but rather, "… sendomi ultimamente accertato con il mezzo dell'esperienza delle cose maestra …" [having ascertained recently by means of experience, the teacher of things …], their squares 4:1, 9:4, 16:9. The ratios between the numbers of the weights were not the same as the lengths. A quadruple weight, and not a double one, was needed to generate the (higher) octave, produced also (at the lower level) by a string of double length.[120]

Galilei senior also considered other "resonant bodies" interesting. He described the sounds generated by glasses filled with different levels of water: the glass harmonica which was to become popular in the eighteenth century. He observed that the more water you put in, the more the sound lowered. He attributed the same

[119] Vincenzio Galilei 1581, pp. 132–133.

[120] Vincenzio Galilei 1589, pp. 102–104. Palisca 1989, pp. 170–172; Walker 1989c, p. 184.

ratios as the lengths of the vibrating strings, obtained on the monochord, also to organ pipes. Thus, *initially*, between a pipe of "dodici palmi" [twelve palms] and "... un'altra dell'istessa grossezza & del medesimo vano che sia lunga sei ..." [another of the same thickness and opening which is long six ...] the octave was formed "... facendo quella che conterrà le dodici parti il suono grave & l'acuto quella che ne conterrà sei." [... with the one containing twelve parts making the deep sound, and the one containing six the acute sound.]

Then he went on to examine also pipes of the same length, but with different diameters. "Circa poi alla proporzione delle canne della medesima lunghezza & di diversa larghezza, non ne ho mai trovato alcuna memoria;..." [Then, as regards the ratio of pipes of the same length and of different breadth, I have never found mention;...]. Thus he confirmed the absence of any Greek theory specifically for pipes. Here, in his *Dialogo della musica* of 1581, he declared that he was "certissimo" [absolutely certain] to obtain the interval of the Diapente (fifth) from two pipes of the length of "due braccia" [two ells], one "mezzo braccio" [half an ell] in circumference, and the other "... tre quarti ... rendendo questa il suono grave & quella l'acuto: & con i medesimi rispetti si potranno havere la più parte degli intervalli musici da canne d'uguale lunghezza & disuguale capacità;" [... three quarters ... with the latter making the deep sound, and the former the acute one: and in the same way, we may obtain most of the musical intervals from pipes of the same length, but of different capacities;] However, he added at once: "ma non però si havevano tutti di quella eccellenza & sonorità, dove la lunghezza ancora nella proporzion di esse per rata [calcolo] concorre." [but not all would be of the same excellence and sonority, seeing that the length still concurs in their proportion for the calculation.] Thus he sought the reason in the different increase in volume in the two cases: the first with a variable length and a fixed circumference, the second vice versa. To facilitate the calculation, he also presented some drawings.

Our composer for the lute was measuring by how much the area of the section of a pipe would increase if the diameter were enlarged (if circular) or the side (if square) to double, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$ and $\frac{6}{5}$. Thus he obtained, for the volume that would generate the lower octave, 4 times that of the acute one, for the Diapente (fifth) $\frac{9}{4}$, for the Diatessaron (fourth) $\frac{16}{9}$, for the major third $\frac{25}{16}$ and for the minor third $\frac{36}{25}$. These increases in volume were not equal to those obtained by lengthening the pipes, the section being equal. These still maintained the Pythagorean proportions, while the former were their squares. If, therefore, he attributed the lack of "eccellenza & sonorità" [excellence and sonority] to pipes tuned while changing the section, because the relative volumes did not respect the Pythagorean canons (except, perhaps, 4 for the octave), he would seem to have remained partly subject to their influence.[121]

We may wonder at this point how much experience with organs this lute-player had acquired. He must have carried out some experiments, if he asserted that the difference between an open pipe and a closed one of equal length and capacity was

[121] Vincenzio Galilei 1581, pp. 133–134.

the interval between "... Hypate & la Nete; facendo quella che sarà chiusa il suono grave & l'aperta l'acuto: ..." [Hypate and Nete; with the closed one making the deep sound, and the open one the acute sound:] From the following explanation, we can see that Galilei father considered that interval to be an octave rather than a seventh, because only Boethius used those Greek terms for the latter.[122] Very probably, he had thus listened with his own ears to the different notes produced by open and closed pipes. And yet some uncertainty in identifying that sound as an octave may have remained. Indeed, the "final effect" (which would require also tempering the diameters) must have made it approximate, and the change in the timbre of the two pipes (in the closed one the even harmonics disappear) might have made the comparison difficult.[123] Was it this lack of "eccellenza" for his ears, heard while tuning consonant intervals with organ pipes, that was to prompt him to make further investigations?

We can read of further novelties for pipes both in the *Discorso intorno ... Zarlino* of 1589, and in the "Discorso particolare intorno alla diversità delle forme del diapason" [Detailed discourse about the diversity of the forms of the diapason], extant in a hand-written form, and subsequent to the other *Discorso*, but maintained for us because it passed into the papers of Galileo Galilei. Here, the Florentine musician announced that in the organ, the lower octave was obtained by varying the volume of the pipe with the cube, and not the square, as he had written in the *Dialogo della musica* [Dialogue on music] in 1581. Thus the ratio in the volumes for two pipes tuned as an octave, had to be 8:1, unlike the Pythagorean 2:1. Two pipes will produce an octave if "... la lunghezza & il vacuo o vogliam dire il Diametro della grave sia dupla dell'acuta" [... the length and the void, or shall we say, the diameter, of the deep sound are twice that of the acute one.]. The same for the fifth: instead of 3:2, 27:8. "... di maniera che il vacuo de queste [canne] corrisponde al Cubo, i pesi sospesi alle corde, alle superficie [quadrato], e le corde semplicemente tese nello strumento alla linea." [... such that the opening of these [pipes] corresponds to the Cube, the weights suspended on the strings to the surface [squared], and the strings simply stretched in the instrument to the line.][124] The number 8 is easily obtained by multiplying the length by 2 and the diameter by 2, and thus the area of the section of the pipe increases by 4, and the volume by 2 times 4.

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[122] See above, the interval between $A[la]$ and $g[sol]$.

[123] Vincenzio Galilei 1581, p. 135. Resnick & Halliday 1961, pp. 432–434.

[124] Vincenzio Galilei 1589, p, 105. Vincenzio Galilei 15?? Mn. 50v. Palisca 1989, p. 172. Palisca 1989a, pp. 180–197. Walker 1989c, pp. 184–185. Unfortunately, Walker considers this cubic ratio a mistake made by Vincenzio Galilei. But modern theories on the 'end effect' confirm the Renaissance musician's sensibility of hearing, despite the Englishman's (scanty?) experience with organs. Any reader who wants to understand Walker's mistakes with organs, from which his mistaken judgement on Vincenzio Galilei derives, may consult Fletcher & Rossing 1991, pp. 474–477. We shall find, in Part II, Sect. 10.2, another Englishman, far more famous than Walker, who, in his attempt to reconcile physics with music, was likewise to make a curious mistake. And I hope I will not be taken for anti-British. With other arguments, also Palisca 1992 confuted Walker's affirmation about Vincenzio Galilei and his experience with pipes. Palisca 1989a, pp. 159 and 187.

In this way, Galilei senior had finally obtained “quella eccellenza & sonorità” [that excellence and sonority] in tuning, at the level of his lute-player’s ear. In order to succeed, he had now varied the length and diameter of pipes simultaneously, reaching a volume which increased to its cube. In ancient times, we have (in Chap. 3) seen only the Chinese tradition, which we know did not suffer from similar dogmas about presumed universal numbers, offer long since that explanation which harmonised the number for pipes with the ear. To my knowledge, Vincenzio Galilei is the first Western scholar who imagined a musical theory based on numerical ratios, also for pipes, with non-Pythagorean ratios.

As hearing is an equally noble sense as sight, we encounter direct experiments conducted in order to understand phenomena, later called acoustic events. These were carried out by a musician, who was also father of his son, the famous Galileo Galilei. Let us continue, however, to leave to interested philosophers of science the pleasure of deciding how much experimental (Baconian?) method there was here already, among the notes of his father’s music, considering also that we shall come across less experiments than usual in the writings of his son, the natural philosopher.[125]

In the “Tavola della maggior parte delle cose che nell’opera si contengano” [“Table of the most parts of things that the works includes”], the Florentine musician underlined 56 “Errori del Zarlino” [“Mistakes made by Zarlino”]; 2 more were classified under “Gioseffo Zarlino musico prattico e teorico eccellentissimo” [“Gioseffo Zarlino, practical musician and most excellent theoretician.”] Only once were the “Laudi del Zarlino” [“Praises of Zarlino”] sung, together with those usually reserved for Franchino [Gaffurio] and Aristoxenus. The *Dialogue on music* thus sounded like a constant criticism of what the Venetian theoretician had sustained, even if he was recognized to be “. . . huomo essemplare di costumi, di vita & di dottrina . . .” [an exemplary man in his customs, life and doctrine.][126]

The work ended with the praise of musicians: “. . . molto più da essere reputati son quelli che c’insegnano le virtù, & maggiormente quanto più rare & eccellenti sono, che quelli che semplicemente (con le buffonerie loro) ci dilettano. [. . .] . . . perché qual si voglia semplice piacere del senso (per la sua inconstanza) ultimamente ci sazia & di sapere alcuno mai si trae sete: [. . .] . . . egli è impossibile di trovare un huomo che sia Musico veramente & che sia vitioso [. . .] sarà costui lontanissimo da ogni brutta & disonesta attione [. . .] sarà a se stesso & alla sua Republica di comodo & utile infinito: [. . .] osserverà del continouo il decoro, la modestia & la verecundia.” [. . . those who teach us virtues are much more to be regarded, and the more so, the more these are rare and excellent, than those who simply (with their buffoonery) amuse us. [. . .] . . . because any simple pleasure of the senses (due to its inconstancy) in the end satiates us, whereas nobody quenchs ever the thirst for knowledge:[. . .] . . .it is impossible to find a man that

[125] See Part II, Sect. 8.2. Incidentally, the Zarlino vs. Vincenzio Galilei controversy seems to continue today with the historians Walker vs. Palisca; Palisca 1989, Walker 1989c.

[126] Vincenzio Galilei 1581, p. 39.

is a true Musician and is depraved [. . .] he will stay far away from every bad and dishonest action [. . .] he will be infinitely helpful and useful to himself and to his Republic: [. . .] he will continually observe decorum, modesty and decency.] Galilei senior criticised those cantors and singers who limited themselves to exhibiting fine voices. "È ancora simile il sapere di questi alle caduche bellezze della Donna." [The knowledge of these people is still similar to the passing beauty of Woman.][127]

The end of the *Dialogo della musica*, which takes its inspiration from Plato's *Republic*, does not seem to be very consistent with the praise tributed to Aristoxenus at the beginning. There, Galilei the composer wanted to remove "slander from the back" of the Greek musician, and defended him for the tuning of his lute. Here, this book would now seem to be still looking at some traditional Pythagorean values. Did Aristoxenus above all have a polemical function against Zarlino? Did our Florentine try to drag even this ancient theoretician towards a certain orthodoxy, in order to construct a Greek world without any irreconcilable contrasts, to be globally enhanced and to be used *en bloc* against the modern contrapuntists that he disliked?

From the *Dialogo della musica*, however, a different background emerged compared with that of Pacioli, Maurolico or Zarlino. Pythagoras had refused, and had taken away ". . . dal mondo tutte le cose miste, impure, & varie; conoscendo egli che in esse l'incostanza & la temerità signoreggiava" [. . . from the world everything that was mixed, impure, and varied; for he knew that in them, inconstancy and recklessness dominated]; then, like a bee, he would extract only the nectar. This Galilei, on the contrary, did not disdain to make use of the ". . . purgato udito di colui che è bene esercitato, accompagnato appresso da naturale giudizio & da qualche buona regola [perché esso] non si inganna così di leggiero." [. . . purged hearing of one who is well-practised, accompanied nearby by natural judgement and some good rules [because he] will not be deceived so easily.] He did not despise any aspect of musical experience. In fact he referred to the ". . . autorità di un suo homiciattolo che dava vento alle canne [d'organo]" [. . . authority of little man of his, who blew wind into the organ pipes], who would have stopped working the bellows if "questi saputi" [teorici] [these know-alls [theoreticians] did not tune the instrument properly. Instead of seeking the origin of music only in the myths of Apollo and Mercury, or in the genealogies of the *Bible*, as others usually did, he wrote that the original way of singing, which had been corrupted in the Middle Ages, had, however, remained with ". . . i rustici agricoltori nel cultivare i campi, & i pastori per le selve & monti . . ." [the rustic farmers in cultivating the fields, and the shepherds in the forests and mountains . . .] [. . .] ". . . havere gli huomini apparata questa facultà nel cercare di imitare il canto degli uccelli." [. . . men had developed this faculty in trying to imitate the song of birds.][128]

He often made reference to "Nature". He invoked it to distinguish the Dorian, Phrygian and Lydian modes from one another, connecting them with the deep or

[127]Vincenzio Galilei 1581, pp. 148–149. Cf. Cardano, Sect. 6.5.

[128]Vincenzio Galilei 1581, pp. 32, 34, 36.

acute voices of the relative populations, as would happen, according to him, also among Lombards, Tuscans and Ligurians. However, he left it to natural philosophers to discuss whether this depended on the food, the water, the air or the climate. Few words he dedicated to the music of the spheres, commenting on Boethius and Ptolemy "… a guisa delle sfere celesti …" [… in the manner of the heavenly spheres …]; but here, too, he returned to earth: "… nella sfera del Mondo son termini al più lungo & al più breve giorno dell'anno." [… in the sphere of the World, there are limits to the longest and shortest days of the year.] Like other of his environment, he sometimes seemed to prefer poetry to the usual recurring myths. He attributed to the poetess Sappho the invention of the Mixolydian mode. This was one semitone more acute than the Lydian, not only because she was a woman, but "… dalla conformità che maggiormente havevano i concenti delle sue poesie con la proprietà & natura di quella si fatta harmonia." [… from the greater conformity that the harmonies of her poems had with the properties and nature of such harmony.][129]

Whether Galilei really liked him quite or not, Aristoxenus had reappeared on the scene in the *Dialogo della musica* [Dialogue on music], both as a theoretician to allow tones and octaves to be divided into equal parts, and as a practical musician, to inspire the relative tuning of instruments and to ease the first games of modulations. This testimony appears to be important: that at least some players and composers understood the advantages of Aristoxenus' tuning for their art, beyond the theoretical obstacles that other rival musicians might raise.

In the end, in the *Discorso intorno … Zarlino* and in the unedited manuscripts, the musician Galilei had to solve some doubts about the contrast perceived by his ears towards the Pythagorean series of numbers. If his preference to follow the former found confirmation in his writings about weights to tune strings, and in volumes to tune pipes, he must still have had a certain reluctance to abandon whole numbers completely. In the writings of his last years, sure, he alternated "some friends of mine who follow Aristoxenus" with other no-better-specified "nemici aristosseniani" [… enemies who are followers of Aristoxenus.][130]

In all this vacillation, sometimes not very consistent, between orthodox Pythagoreans, Ptolemaic variants, Aristoxenic scales, Zarlinian novelties, theoretical musicians, practical experts, voices, instruments, ears, numbers, words or notes, in the end Vincenzio Galilei even advanced a last opportunity to succeed in cutting the intricate Gordian knot of which he was a prisoner. He compared musical intervals to the words of a language, and thus, for him, they became various, acquired and handed down in different cultures, as well as subject to historical evolution. The idea, even if difficult to accept for some, is exactly the same one around which we are constructing this book.

[129] Vincenzio Galilei 1581, pp. 71, 66, 115, 70.

[130] Palisca 1989, p. 173; Walker 1989c, p. 180. The fact that, all things considered, Vincenzio was on the side of Aristoxenus, because he composed tablatures, and played the lute, is explained also by Brown 1992, p. 174.

In the pond of orthodoxy, which had remained motionless for too long, someone at last polemically started throwing stones. Of course, the mathematical sciences were still considered as those based, prevalently for music, on whole numbers only, and therefore many musicians continued to deal with them in a traditional way. And yet our lute-player, influenced by Aristotle and also by Aristoxenus, even if he was partly a friend, and partly an enemy, of various contemporary Aristoxenian musicians, thought of sound intervals as a *continuum* to be touched lightly with one's fingers on the strings: "… minime particelle, quasi simili agli Atomi …" [minimal particles, almost like Atoms …]. The numbers [of the Pythagoreans], instead, to him appeared to be distant, separate, and discrete.[131] The problem then became what relationship to hypothesise between the ones and the others, and thus whether to follow the classical *quadrivium*, or not. Not everyone would have done so, and not for ever. But it was necessary for the roots of numbers to be accepted and used like all the other numbers.

Francesco Maurolico sang during the ceremonies of his ecclesiastic profession. Michael Stifel struggled with the words and the musical notes in his "Johannes thüt uns schreiben", which followed the melody of a popular song in the tradition consolidated by Martin Luther. Girolamo Cardano composed polyphonic canons with up to 12 voices. Apart from his theory, Nicola Vicentino, who was a musician by profession, left us motets like "Musica prisca caput". Peter Pesic arrives at my same conclusions. "The comparison of these three [four] figures of theorists, composers, and mathematicians illuminates ways in which musical concerns, both practical and theoretical, influenced the acceptance of novel mathematical concepts, which in turn bore on musical matters."[132] Of all this, the building-block added by Vincenzio Galilei was the keystone which was to consolidate the building. Above all, it should be noted that musicians were the most willing to accept irrational numbers on the same plane as rationals, whereas natural philosophers still

[131]Vincenzio Galilei 15?? Mn. 54v. Palisca 1989, p. 173. Palisca 1989a, pp. 194–196; Palisca 2008. Chua 2001. In spite of some excesses in debatable philosophical speculations, which lead him in a different direction from the one taken here, also Luigi Borsacchini recognizes the important role of the *continuum* and of music played in the evolution of the ancient Greek mathematical sciences. This Italian scholar rightly asks: "why did the history of mathematics 'remove' the 'musical way'? In my opinion, the answer is given by the ancient prejudice concerning both *the presumably 'empirical', 'natural', 'phenomenological' character of the idea of "continuum" and the opposition discrete/continuous*. This is a common prejudice shared not only by historians and philologists, but even by almost all mathematicians." Unfortunately, misled by Needham, and by only partial readings of Graham (cf. here Chap. 3), Borsacchini does not grasp the Chinese *continuum*. Nor does he seem to understand that Aristoxenus heard his *continuum* in the strings of the lyre, which for a musician generate clearly audible sounds and notes. Thus, to understand the relationship of classical Greek sciences with their music, we have no need, either of anachronistic logical constructions, invented with the *Grundlagenstreit* between David Hilbert and Hermann Weyl, or, even less, of Chino-Indo-Arabic decimal numbers coming from other cultures. Everyone has the prejudices that he deserves. Borsacchini 2007, p. 287. Weyl 1985. Tonietti 1982a; 1985a; 1988; 1990.

[132]Tonietti 2006b. Pesic 2010, pp. 523–528.

shunned from declaredly doing so. Last, but not least, in the judgement of Cardano on musicians, and in the 'wager-trial' of Vicentino, the moralistic idea of music continued to have its weight, as part of the system to 'elevate' young people. In this way, the Europe of the sixteenth century again recalled Plato's Athens, with its inclinations to censure.[133]

[133]The anthology collected by Philippe Vendrix 2008 re-examines in its own way some of the problems dealt with here, with some omissions and different details. Among the most interesting, Brigitte van Wymeersch observes that translating the Arabic *asam* into the Latin *numerus surdus* [surd] should not be considered a misunderstanding of the sense of irrational number, without *logos*, like a corruption due to the passage between different languages. She points out rather the link with music, as sustained here, and as we shall find with Stevin in Chap. 8; Wymeersch 2008.

In Marsilio Ficino, Brenno Boccadoro discovers the problem of how it is possible to reconcile, or not, numbers for music with harmony and love; Boccadoro 2008. Guido Mambella analyses Zarlino's musical theory, underlining that in the end, even the Neopythagorean inventor of the *senarius* slid towards another point of view. Subject to the criticism of Vincenzio Galilei, and seeing what other musicians of his age loved, now Zarlino would have preferred to subject music to geometry, rather than to arithmetic. Unfortunately, he didn't do so; Mambella 2008. Daniele Sabaino recalls the theoretician Juan Caramuel Lobkowitz (1606–1682), overlooked here, who used logarithms base 2 to divide the octave temperately; Sabaino 2008.

For other details, the usual commonplaces, reproposals of old articles ignoring recent results, as well as different viewpoints, or unjustified convenient simplifications, see: Gozza 2000, Clark & Rehding 2001 and Christensen 2004. The common limit shown by many of these studies is always ignoring non-Western cultures.

Appendix A
[From the] **Suanfa tongzong** *[Compendium of Rules for Calculating] by Cheng Dawei*[1]

Dividing into three parts and decreasing by one, thus we also obtain the fraction $\frac{2}{3}$. Dividing into three parts and increasing by one, thus we also obtain the fraction $\frac{4}{3}$.[2] The rule says: the *lü*[3] Huang zhong is 9 cun long.[4] Take 9 cun and multiply it by itself to obtain 81, which serves for the note Gong. Again take 81 and multiply it by 2, obtaining 162 cun; divide this by 3, obtaining 54 cun. In other words, by decreasing 3 by one part [$\frac{2}{3}$], the note Zhi is generated: fire. Again take 54 and multiply by 4, obtaining a result of 216; divide by 3, obtaining 72 cun. That is to say, by increasing 3 by one part [$\frac{4}{3}$], the note Shang is generated: metal. Take 72 again, and multiply it by the fraction $\frac{2}{3}$, obtaining 48 cun, which generates the note Yu: water. Continue to multiply the 48 of Yu by $\frac{4}{3}$, thus obtaining 64, which generates the note Jiao: wood (Fig. A1).

This is the rule that generates the five notes from one another. The greatest [length] functions like an old man, i.e. it is turbid [deep], the shortest like a boy, i.e. it is similarly clear [acute].

Song of the lülü generated from one another.

> The tremor of the *lülü* made together is rare;
> As a basis, give the Huang zhong nine *cun*.
> The three decrease by one, from the *yang* generate the eighth *yin*;
> From the *yin lü* generate the *yang*, the odd increase by one.
> Of the Huang, Lin, Dacu the *cun* are all whole numbers;
> No more this the rest, change into fractions,
> Since the products, use all the nine fractions.
> The four seasons unite, appropriately, and the climate.[5]

(Fig. A2)

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[1]The author's translation from the Chinese text of Cheng 1592.

[2]The Chinese text erroneously indicates the fraction $\frac{1}{2}$.

[3]Pipe tuned in accordance with the rules.

[4]The "cun" was a unit of length of about 3.30 cm.

[5]The Chinese song is put into rhyming couplets.

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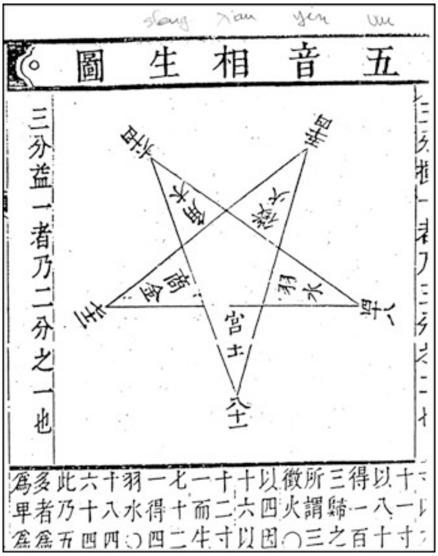

Fig. A1 Figure of the five notes generated from one another
[The tip at the bottom of the star bears the number "81", the character of the phase "earth" and the name of the note "Gong". From here, we move up the straight line to the right to the tip "54", "fire" and "Zhi", multiplying by the fraction $\frac{2}{3}$ indicated in the text. Then we go downwards to the left to the tip "72", "metal" and "Shang", multiplying by $\frac{4}{3}$. After this, we move horizontally, arriving at "48", "water" and "Yu", by means of the fraction $\frac{2}{3}$. From here, by multiplying once again by $\frac{4}{3}$, we move upwards to the left towards "64", "wood" and "Jiao". Though this last note is connected to the first one by another descending straight line, it is not possible to transform it by means of the preceding operations of "decreasing" and "increasing": the cycle stops at five notes, and is not closed]

The notes Huang zhong, Dacu, Guxi, Ruibin, Yize, Wushe, act as yang; Dalü, Jia zhong, Zhonglü, Lin zhong, Nanlü, Ying zhong act as yin. The yang *lü* generate the yin by dividing into three parts and then decreasing by one[6]; the yin *lü* generate the yang by dividing into 3 parts and increasing by one. Multiply by 2 and divide by 3; multiply by 4 and divide by 3: this is appropriate for the *lülü*. Only each *lü* of the Huang zhong, the Lin zhong and the Dacu is a whole number; all the rest have divisions with numbers that do not finish, because they use fractions.

The Huang zhong is of the yang type, its opening is 9 fen,[7] and the *lü* is 9 cun long; multiplying by 9 fen, we obtain the product 810 fen; its period is the beginning of winter [the solstice]. The yang *lü* generates the yin one, while with the rule from

[6] In the text, erroneously, "into 2 parts".

[7] Unit of measurement about one third of a cm; 1 cun = 10 fen.

Fig. A2 Figure of the *lülü* generated from one another
[The 12 tips are given the names of the *lülü* and the earthly branches shown in Table 3.1. The first note Huang zhong is at the bottom on the right]

9 cun, by multiplying by 2, we obtain 18 cun and dividing by 3, we obtain 6 cun; at a distance of 8 places,[8] the Lin zhong is generated.

The Lin zhong is of the yin type, its opening is 9 fen and the *lü* is 6 cun long; multiplying by 9 fen, we obtain the product 540 fen; its period is the "Great heat". The yin *lü* generates a yang one, while on the basis of the rule from 6 cun, by multiplying by 4, we obtain 24 cun, and dividing by 3 we arrive at a length of 8 cun; after 8 places, the Dacu is generated.

The Dacu is of the yang type, its opening is 9 fen and the *lü* is 8 cun long; multiplying by 9 fen, we obtain the product 720 fen; its period is "Water and rain". The yang *lü* generates a yin one, while following the rule from 8 cun, by multiplying by 2, we obtain 16 cun, and dividing by 3, we obtain $5\ \frac{1}{3}$ cun; after 8 places, the Nanlü is generated.[9]

Each of the 3 preceding *lü* obtains whole cun; the 9 following *lü* have not finished cun, and all use fractions.

The Nanlü is of the yin type, the length of the *lü* is $5\frac{1}{3}$ cun, but as the denominator of the fraction is 3, adding the 1, we obtain a total of 16 cun; multiplying by 9 fen and dividing by 3, we obtain the product 480 fen; its period is the autumn equinox. Multiplying by 4 the fraction with 16 as the numerator, we obtain 64 cun, and

[8]In the star of Fig. A2, count eight tips in a clockwise direction, starting from the one at the bottom on the right.

[9]In the text, erroneously, "Gong".

multiplying by 3 the 3 of the denominator, we obtain 9; with the rule of division, we obtain $7\frac{1}{9}$ cun; after 8 places, the Guxi is generated.

The Guxi is of the yang type, its *lü* is $7\frac{1}{9}$ cun long; thus we must add to the 7 cun multiplied by the 9 of the denominator the 1 of the numerator, obtaining a total of 64 cun; multiplying by the 9 fen of the opening, we obtain 5,760 fen, and dividing by the 9 of the denominator, we obtain 640 fen; its period is "Rain and wheat". Multiplying by 2 the 64 cun of the numerator, we obtain 128 cun, and multiplying by 3 the 9 of the denominator, we obtain 27; dividing in accordance with the rule, we obtain $4\frac{20}{27}$ cun; after 8 places, the Ying zhong is generated.

The Ying zhong is of the yin type, its *lü* is $4\frac{20}{27}$ cun long; thus we must add to the 4 cun multiplied by the 27[10] of the denominator the 20 of the numerator, obtaining a total of 128 cun; multiplying by the 9 fen of the opening, we obtain 11,520 fen. As the denominator is 27, by the rule of division, reducing the 18 of the numerator with the 9 of the denominator, we obtain the not integer quantity of $426\frac{2}{3}$ fen[11]; its period is "Light snow". By multiplying the 128 cun of the fraction by 4, we obtain 512 cun; by multiplying the 27 by 3, we obtain 81; by means of the rule of division, we arrive at $6\frac{26}{81}$ cun; after 8 places, the Ruibin is generated.

The Ruibin is of the yang type, its *lü* is $6\frac{26}{81}$ cun long; thus we must add to the 81 of the denominator multiplied by 6 the 26 of the numerator, obtaining a total of 512 cun; multiplying by the 9 fen of the opening, we obtain 46,080 fen. As 81 is the denominator, by means of the rule of division, reducing the 72 of the numerator with the 9 of the denominator, we obtain the not integer quantity of $568\frac{8}{9}$ fen[12]; its period is the summer solstice. Multiplying by 4 the 512 cun of the fraction, we obtain 2,048 cun, and multiplying by 3 the 81, we obtain 243; by means of the rule of division, we arrive at $8\frac{104}{243}$ cun; 8 places further on, the Dalü has its origin.

In accordance with the rule, the yang generates the yin, as mentioned above, but from the Ruibin we should use the 3 decreased by 1 [i.e. multiply by $\frac{2}{3}$], but then we again use the rule of 3 increased by 1 [multiply by $\frac{4}{3}$]. It is impossible to explain this rule, which again increases the 3 parts by 1. Or rather, may the reason be that at the summer solstice the Yin begins to increase? After that, the yin *lü* generates the yang by 3 which decreases by 1 and the yang *lü* generates the yin by the 3 that increases by 1.

The Dalü is of the yin type, its *lü* is $8\frac{104}{243}$ cun long; thus by adding to the numerator of the fraction 8 times the denominator, we obtain a total of 2,048 cun. Multiplying by 9, with 243 as the denominator, we use the rule of division, obtaining the not integer number with 126 as the numerator, and by reducing the fraction both in the

[10]Instead of the *shi* [10] in *ershiqi* [27], the text erroneously presents the character *cun*, which is almost the same.

[11]In the text, the *liu* of 6 is missing.

[12]The *ba* of the 8 is missing in the text.

numerator and in the denominator by 3, we obtain the sum of $758\frac{42}{81}$ fen[13]; its period is the "Great cold". In the fraction with 2,048, multiplying by 2 we obtain 4,096 as the numerator; multiplying by 3 the 243, we obtain 729; using the rule of division, we obtain $5\frac{451}{729}$ cun; after 8 places, the Yize is generated.

The Yize is of the yang type, its *lü* is $5\frac{451}{729}$ cun long; thus by adding to the numerator of the fraction 5 times the denominator, we obtain a total of 4,096 cun. Multiplying by the 9 fen of the opening, we obtain the sum of 368,640 fen; having 729 as the dividend, and using the rule of division, reducing the divisor 495[14] and the dividend by 9, we obtain the not integer quantity of $505\frac{55}{81}$ fen[15]; its period is the "Limit of the heat". Multiplying by 4 the 4,096 cun of the fraction, we obtain 16,384 cun; multiplying again by 3 the 729, we obtain 2,187; by means of the rule of division, we arrive at $7\frac{1{,}075}{2{,}187}$ cun; after 8 places, the Jia zhong is generated.

The Jia zhong is of the yin type, its *lü* is $7\frac{1{,}075}{2{,}187}$ cun long; thus, by adding to the numerator the denominator of the fraction multiplied 7 times, we obtain a total of 16,384 cun. Multiplying by the 9 fen of the opening, we obtain 1,474,560; dividing by the denominator 2,187, we obtain the not integer quantity $674\frac{58}{243}$ fen, after reducing by 9 both the numerator 522 and the denominator; its period is the spring equinox. Multiplying by 2 the 16,384 cun of the fraction, we obtain 32,768 cun as the dividend; multiplying 2,187 by 3, we obtain 6,561, and by using the rule of division, we obtain $4\frac{6{,}524}{6{,}561}$ cun; after 8 places, the Wushe is generated by means of a decrease.

The Wushe is of the yang type, its *lü* is $4\frac{6{,}524}{6{,}561}$ cun long; consequently, by adding to the numerator of the fraction 4 times the denominator, we obtain a total of 32,768 cun. Multiplying by the 9 fen of the opening, we obtain 2,949,120 fen; then, dividing by the denominator 6,561, we obtain the sum of 3,231, and from the proportion of the rule, we obtain the not integer quantity of $449\frac{3{,}231}{6{,}561}$ fen[16]; its period is the "Descent of frost". Multiplying by 4 in the fraction 32,768, we obtain 131,072 cun, and then multiplying the denominator 6,561 by 3, we obtain 19,683; from this, by following the rule of division, we obtain $6\frac{12{,}974}{19{,}683}$ cun; after 8 places, the Zhonglü is generated by means of an increase.

The Zhonglü is of the yin type, its *lü* is $6\ \frac{12{,}974}{19{,}683}$ cun long, and by adding to the numerator 6 times the denominator of the fraction, we obtain a total of 131,072 cun; multiplying by the 9 fen of the opening, we obtain 11,796,480 fen; using the denominator 19,683 in the rule of division, we obtain the sum of $599\frac{6{,}363}{19{,}683}$ fen[17]; its period is "Full of wheat".

[13]Here, the modern editor of the classical edition has inserted the wrong number 750, thinking that he was correcting the text, whereas the ancient version presents the right number 758.

[14]The text contains the wrong number 414.

[15]As a consequence of the preceding error, the text presents the wrong number $\frac{46}{581}$.

[16]The fraction has been simplified by a factor of 9 in Table 3.1.

[17]See Footnote 16.

Appendix B
Al-qawl ‘ala ajnas alladhi bi-al-arba‘a
[Discussion on the Genera Contained in a Fourth] by Umar al-Khayyam[1]

The ratio of the sample-unit plus one third [4:3][2] can be further subdivided into three ratios, which correspond to the three *ab‘ad* [intervals] limited to four sounds. For this reason, the ratios of the sample-unit plus one third was called a fourth [a tetrachord]. These three *ab‘ad* [intervals] either do not include an interval whose ratio is greater than the sum of the others, or else they include an interval whose ratio is equal to twice the remaining two. The first of the *ajnas* [among the species of the fourth] was called *qawi* or *tanin* [strong, diatonic], the second *mulawwan* [coloured, chromatic] or *mu‘tadil* [moderate, median], and the third *rikhw* [weak, enharmonic] or *ta’lif* [compound].[3]

The first of the strong species is the first, which doubles the interval; it is one whole unit plus one seventh [8:7], one whole unit plus one seventh [8:7], and one whole unit plus one of forty-eight parts of a whole [49:48]. It corresponds to the numbers 64, 56, 49, 48.[4] This would be an extremely strong, valid species, if it

[1]Translation from the Arabic manuscript by Michele Barontini, with technical help of the author; Barontini & Tonietti 2010.

[2]We use square brackets [ ] to indicate our additions which may help the reader to understand the text more easily.

[3]The Iranian editor, Humai, proposes *taniniyan*, rather than *taninan*, or even *tannan*, which would mean ‘sonorous’, seeing that the punctuation of the first *nun* and the *ya* are missing in the Manisa manuscript. The interpretation of Humai opts for a more modern reading, closer to current musical terminology, with the meaning of ‘diatonic’. This was a recovery of the Greek genera. The 4:3 tetrachord contemplated three of them. The diatonic one was formed by two tones and one semitone; the chromatic one included a minor third and two semitones. The enharmonic tetrachord consisted of a major third and two micro-intervals, similar to quarters of a tone. Curt Sachs, *The Rise of Music in the Ancient World. East and West*, (New York, 1943), pp. 206–207.

[4]The ratios of the four numbers are those indicated. For example, 56 can be obtained from 49 by dividing by 7 and multiplying by 8; 64 can be obtained from 56 in the same way. The three ratios united together, 8:7, 8:7, 49:48, give 4:3, which is thus divided into three parts. If the players of musical instruments were to follow a similar theory, they would tune them by placing the notes that divide the interval of the fourth on the basis of these ratios. For example, the re and the mi in the interval of the fourth $do - fa$, in accordance with modern notation. We will compare the

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were not for this interval, that is to say, one forty-eighth [49:48], which is a value very distant [from practice].

The second of the strong species is the second, which doubles [the interval]; it consists of one whole unit plus one eighth of a unit [9:8], with one whole unit plus one eighth of a unit [9:8], and one whole unit plus thirteen out of two hundred and forty-three parts [256:243], corresponding to the numbers 324, 288, 256, 243.[5] This is a species that is very close to normal practice, and as a result, it is the only one that is [commonly] used in most countries.[6]

The third species is the third, which doubles [the interval]; it consists of a whole unit plus one ninth [10:9], one whole unit plus one ninth [10:9] and one whole unit plus six parts out of seventy-five [81:75 = 27:25]. The numbers are 100, 90, 81, 75. It was discovered by Al-Farabi and, as far as I can judge, it was not among the most common.[7]

The fourth is the strong conjunct of the first species; it is made up of one whole unit plus one seventh of a unit [8:7], one whole unit plus one eighth of a unit [9:8], and one whole unit plus one twenty-seventh of a unit [28:27]. The numbers are 72 62[63] 56 54.[8] This is a very good calculation.

The fifth is the strong conjunct of the second species, and is made up of one whole unit plus one eighth of a unit [9:8], one whole unit plus one ninth of a unit [10:9], and one whole unit plus one fifteenth of a unit [16:15]; its numbers are 180 168[160] 144 165[135].[9] This species is the best of all, in my opinion.

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species, as al-Khayyam lists them, with those that are found in the works of Al-Farabi and Ibn Sina (Avicenne). This first way of dividing the fourth is also found in Ibn Sina, extrait du *Kitab al-Šifa [The Book of Healing]*, ‘Section des sciences éducatives, chapitre XII, La musique’, in *La Musique Arabe* 4 vol., ed. Rodolphe d’Erlanger, (Paris, 1930 and 1935), vol. II, p. 146. It is also described by Al-Farabi, *Kitab al-Musiqi al-kabir [Great Treatise on Music]*, in *La Musique Arabe*, vols. I–II, I, p. 109. See also Chap. 5.

[5] Al-Farabi, *Kitab al-Musiqi*, I, p. 109. Ibn Sina, *al-Šifa*, p. 149.

[6] This goes back to the Greek Pythagorean tradition. A. Barker, *Greek Musical Writings* (Cambridge, 1984 and 1989). Tito M. Tonietti, “The Mathematical Contributions of Francesco Maurolico to the Theory of Music of the 16th Century (The problems of a Manuscript)”, *Centaurus*, 48 (2006): 149–200. See also Chap. 2.

[7] Al-Farabi, *Kitab al-Musiqi*, I, p. 109.

[8] If we desire to maintain the same proportions as in the text, the number 62, which is found both in the Manisa manuscript and in Humai’s edition, should be corrected to 63, because this is the result of nine-eighths of 56. If, on the contrary, we desire to maintain 62 in the sequence, then the proportions should be changed in the text. Al-Farabi, *Kitab al-Musiqi*, I, p. 111. At this point, in his edition, Humai states: “I say this without being completely sure of the accuracy of the calculations for this subject and the following one. And Allah knows more about this.” This sentence authorises us to suspect that the editor of the printed edition did not know a lot about the Greek and Arabic theories of music.

[9] The numbers of the sequence do not correspond to the proportions in the text; 168 should be corrected to 160, and 165 to 135. In Humai’s edition, there is a sign of a horizontal parenthesis with a dot, placed next to 18, below 1/5. It probably derives from the zero 0 with a line over it in the Manisa manuscript. Zero was often written as an omicron with a line over it in manuscripts. Al-Farabi, *Kitab al-Musiqi*, I, p. 111.

Among the conjuncts of the third species, the sixth is made up of one whole unit plus one ninth [10:9], one whole unit plus one tenth [11:10], and one whole unit plus one eleventh [12:11]. Its numbers are 220 198 180 165.[10] This, too, is a good species.

The seventh, the first disjunct, is made up of one whole unit plus one seventh [8:7], one whole unit plus one ninth [10:9], and one whole unit plus one twentieth [21:20]. Its numbers are 80 60[70] 63 60.[11] This, too, is good and appropriate.

The eighth species, the second of the strong disjunct ones, is made up of one whole unit plus one eighth [9:8], one whole unit plus one tenth [11:10], and one whole unit plus twenty-three two hundred and ninety-sevenths [320:297]. Its numbers are 396 353[352] 320 297.[12] This species was discovered by Al-Farabi, but it is not appropriate, in spite of the fact that he identified the so-called *tanini* interval in it.[13]

Another species was discovered by the Imam of Saviours, Ibn Sina, and is made up of one whole unit plus one eighth [9:8], one whole unit plus one twelfth [13:12], [lacuna],[14] and he sustained that it is made up of one whole unit plus one seventh [8:7], one whole unit plus one thirteenth [14:13], and one whole unit plus one twelfth [13:12], the numbers of which are 16, 14, 13, 12. In my opinion, this species is far from being usable, on account of the small difference between its two similar intervals. Other examples can be found among these disjunct [species]; I have limited myself to these few disjuncts, because [by proceeding in this way, we find species which] are not used, and because they are far from any permitted *i’tilaf* [consonance].[15]

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[10]Al-Farabi, *Kitab al-Musiqi*, I, p. 111. On the contrary, Humai’s edition gives the partly mistaken numbers 225 198 18 165LA. This last symbol, LA, *lam alif* is probably due to the written form for zero in the Manisa manuscript.

[11]The error in the Manisa manuscript, 60 instead of 70, was aggravated in Humai’s edition by the numbers 85 65 63 6?, again probably due to the different notation for zero. Al-Farabi, *Kitab al-Musiqi*, I, pp. 112 and 114. Ibn Sina, *al-Šifa*, p. 147.

[12]Three hundred and fifty-three should be corrected to 352. Instead of the 0 of 320, Humai’s edition has the Arabic letters for LA.

[13]Al-Farabi, *Kitab al-Musiqi*, I, p. 112. For this interval, equivalent to the whole Pythagorean tone, 9:8, cf. also Carl Cowl, ‘The Risala fi hubr ta’lif al-’alhan of Al-Kindi’, *The Consort*, 23 (1966): 129–166.

[14]This lacuna was indicated by the copyist in the margin of the Manisa manuscript, but it was ignored in Humai’s edition. Anyway, the sequence should be completed by “one whole unit plus eleven one hundred and seventeenths” [128:117], which, together with the other ratios, gives 4:3. But then, it is necessary to start from 13:12, otherwise the whole units would not be correctly divisible. Consequently, the sequence of numbers becomes 156, 144, 128, 117. Ibn Sina wrote: “468($\frac{13}{12}$)432($\frac{9}{8}$)384 351”; Ibn Sina, *al-Šifa*, p. 150. It is interesting that the ratio 128:117 is also missing in the edition for Ibn Sina of Erlanger between the last two numbers, and that the numbers in the sequence are exactly three times our numbers. If this ratio were to be missing also in the most ancient Arabic editions, might this indicate a (perfectly plausible) direct contact between this text of al-Khayyam and that of Ibn Sina?

[15]Ibn Sina, *al-Šifa*, p. 148. With the ratios in a different order, 8:7, 13:12, 14:13, the species is found also in Al-Farabi, *Kitab al-Musiqi*, I, p. 114.

The first of the coloured species is one whole unit plus one fifth [6:5], one whole unit plus one nineteenth [20:19], and one whole unit plus one eighteenth [19:18]. The relative numbers are 24, 20, 19, 18.[16]

The second of the coloured species is made up of one whole unit plus one fifth [6:5], one whole unit plus one fourteenth [15:14], and one whole unit plus one twenty-seventh [28:27]. The numbers are 36, 30, 28, 27.[17]

The third of this species is one whole unit plus one fifth [6:5], one whole unit plus one thirty-ninth [40:39], and one whole unit plus one twelfth [13:12]. Its numbers are 48, 40, 39, 36. I believe that this was not found by Al-Farabi, and that the second and the third are far from practical use, in spite of the fact that [their ratios] are acceptable.[18]

The fourth of the species is one whole unit plus one fifth [6:5], one whole unit plus one twenty-fourth [25:24], and one part out of fifteen together with one whole unit [16:15]. The numbers are 60, 50, 8 45 [48, 45].[19] This species is close to common use.

The fifth is one whole unit plus one sixth [7:6], one whole unit plus one fourteenth [15:14], and one whole unit plus one fifteenth [16:15]. The numbers are 16 15 4[14] 12.[20] This species is good, in spite of the fact that we calculated the largest part at the end of the group, as we preferred to lighten it, but this does not create too much disturbance.

The sixth is one whole unit plus one sixth [7:6], one whole unit plus one eleventh [12:11], and one whole unit plus one twenty-first [22:21]. The numbers are 28 24 22 21, and this is good, too.[21]

The seventh is one whole unit plus one sixth [7:6], one whole unit plus one ninth [10:9], and one whole unit plus one thirty-fifth [36:35]. The numbers are 40 36 35 30.[22] Here, too, we have placed the largest interval at the end, but this set is far removed from practical use.

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[16] Al-Farabi, *Kitab al-Musiqi*, I, pp. 104 and 105.

[17] Humai’s edition has 35 instead of 30. Did copyists often confuse 5 with 0? In reality, there are various versions of zero designed by the Manisa manuscript copyists, which go from a kind of Greek gamma to a sort of Arabic *He*, or even a little circle like the modern-day five. There is no doubt, on the contrary, about the five: a sort of upside-down B in the most ancient forms of writing. Al-Farabi, *Kitab al-Musiqi*, I, p. 107. Ibn Sina, *al-Šifa*, p. 153.

[18] The third, sure enough, we did not find in Al-Farabi, *Kitab al-Musiqi*.

[19] 8 45 should be completed to read 48 45. Humai’s edition has *wa gu’zun min khamsata wa ‘asharin kullin* [and one part out of twenty-five of a whole unit]. On the contrary, the Manisa manuscript reads *wa gu’zun min khamst ‘ashar min kullin* [and one part out of fifteen of a whole unit], but it also contained two misleading dots under the letter *ra* of *’ashar*, thus indicating a dual, and creating confusion. Al-Farabi, *Kitab al-Musiqi*, I, p. 113. Ibn Sina, *al-Šifa*, p. 153.

[20] In the sequence, the 1 of 14 is missing, and the ratios are calculated in the opposite order. Al-Farabi, *Kitab al-Musiqi*, I, p. 104. Ibn Sina, *al-Šifa*, p. 152.

[21] Al-Farabi, *Kitab al-Musiqi*, I, p. 107. Ibn Sina, *al-Šifa*, p. 152.

[22] The ratios that link that four numbers in the proportion 4:3 are, on the contrary, in the order 10:9, 36:35 and 7:6. Ibn Sina, *al-Šifa*, p. 152.

As regards the first of the compound [enharmonic] species, it is one whole unit plus one quarter [5:4], one whole unit plus one thirty-first [32:31], and one whole unit plus one thirtieth [31:30]. The numbers are 40 32 31 30.[23]

The second one is one whole unit plus one quarter [5:4], one whole unit plus one thirty-ninth [40:39], and one whole unit plus one twenty-fifth [26:25]. The numbers are 100 80 78 75.[24] This is a good proportion.

The third one is one whole unit plus one quarter [5:4], one whole unit plus one thirty-fifth [36:35], and one whole unit plus one twenty-seventh [28:27]. The numbers are 140 112 1[0]8 105.[25] These two species are not quoted in the book of the ancient masters *Kutub al Qudama*, despite their validity. I am unable to suggest any reason for their absence, other than some form of negligence [in copying the texts].[26]

The fourth species is one whole unit plus one quarter [5:4], one whole unit plus one twenty-third [24:23], and one whole unit plus one forty-fifth [46:45]. The numbers are 60 48 47[46] 45.[27] This was discovered [by the ancient masters], but this species is not as consonant as the second and the third. Other examples of these species could be given, but they would not be usual.

The epistle was completed with praise to Allah and it was received under his good auspices.

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[23]Humai’s edition continued to write the letters LA instead of 0 in 40. We have already seen that this derives from the version of zero similar to an Arabic *lam-alif*. Al-Farabi, *Kitab al-Musiqi*, I, pp. 104–105. Ibn Sina, *al-Šifa*, p. 154.

[24]The second 0 of 100 is missing in Humai’s edition. In this sequence, we did not find the species either in Al-Farabi, *Kitab al-Musiqi*, or in Ibn Sina, *al-Šifa*. But Ibn Sina, on p. 154, gave a different order to the ratios, obtaining the numbers: “$80(\frac{40}{39})78(\frac{26}{25})75(\frac{5}{4})60$”.

[25]The 0 of 108 is missing in the manuscript; furthermore, the sequence of numbers fits in with a different order of ratios: 5:4, 28:27, 36:35.

[26] The third species, that is to say, the enharmonic one, composed in the order 5:4, 28:27, 36:35, is to be found in Al-Farabi, *Kitab al-Musiqi*, I, p. 113. In the form “$36(\frac{36}{35})35(\frac{5}{4})28(\frac{28}{27})27$”, it is also found in Ibn Sina, *al-Šifa*, p. 154.

[27]The number 47 does not fit in with the ratio 46:45, and should be substituted in the sequence by 46. Al-Farabi, *Kitab al-Musiqi*, I p. 107. The mistakes in the numbers written in Arabic figures, which are present in the sequences found both in the Manisa text and in the one in Teheran, are probably due to copyists and the editor. However, there are considerably fewer in the Manisa manuscript than in Humai’s edition. By contrast, the ratios expressed in words in the Manisa manuscript do not contain any.

Appendix C
Musica *[Music] by Francesco Maurolico*[1]

[Part I]

[I]

Having thus subtracted the greater semitone from the tone, what remains is the lesser semitone; consequently, if the tone is decreased by the lesser semitone, the greater semitone will remain, and as a result, when the greater and the lesser semitones are united, they will form the whole tone. Thus there are three of these systems and intervals by which all the kinds of melodies may be measured. Then there exist three genres of melodies, that is to say, chromatic, harmonic and diatonic. The harmonic genre abounds in very broad intervals, taking its name from the noble proportion of the composition: this genre is the most sought after and singable, only by the most excellent musicians; it includes all the following, and well arranged consonances; [we will say] more about them below.

The diatonic, indeed, is the genre that abounds in notes with legitimate intervals, in which after every two tones, a lesser semitone is inserted, so that more than two tones placed one next to another do not generate hardness. This genre, therefore, is natural, singable even by the most uneducated, and vigorous; healthy, respectful customs are guided by this.

Lastly, the chromatic is the genre which develops by semitones or by medium intervals; it is sweet, melancholy, and the most affectionate. In constructing for himself the seven-stringed lyre, therefore, Mercury chose the legitimate, natural genre, that is to say the diatonic, in such a way that the first string differed from the second one by one tone, the second from the third by a semitone, the third from the fourth by one tone, the fourth from the fifth by one tone, the fifth from the sixth by a semitone, and the sixth from the last one by one tone.

[1] Translation by the author from the Latin text in Maurolico 201?. In notes and in [ ] the additions of the translator. Sometimes, here, the order of the chapters is different from that of the critical edition.

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2187.	mi	Greater semitone
2048.	fa	

Chart 1 Greater semitone[2]

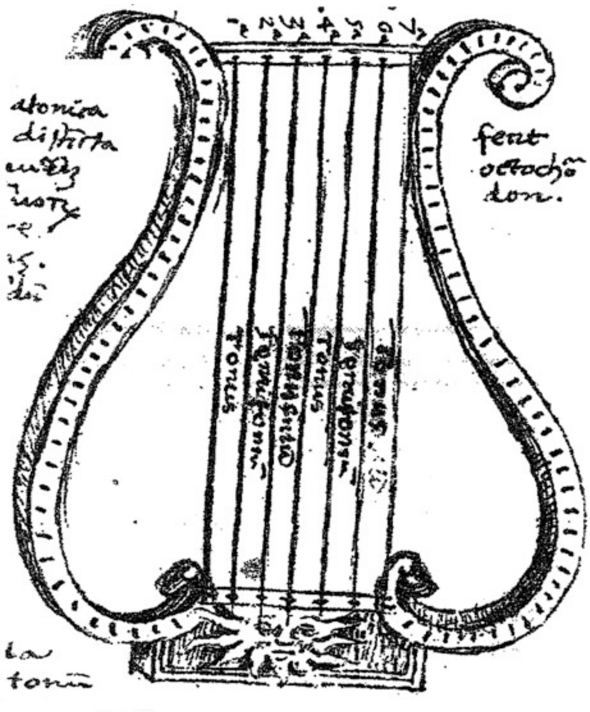


Fig. C1 Mercury's lyre, which Pythagoras designed with eight strings

Only then did Pythagoras add the eighth string, which differed from the seventh by one tone, so that the diapason [octave] would be completed, to honour the perfect consonance, from which before nothing was heard. In fact, the octave consists of five tones and two lesser semitones on the interval of the double ratio (Fig. C1).

Imitating them, our Guido [D'Arezzo] put together seven hexachords divided in the same diatonic extension. Perhaps considering the number of strings of the first lyre, we can now see that the hexachord was chosen for good reason. It contains within itself all the simple consonances: that is to say, the unison that derives from the same strings; the ditone, which includes two tones, taken naturally from the first to the third string; the semiditone, introduced through the tone and the semitone, in reality from the second to the fourth string; the diatessaron [fourth], containing a double tone and a lesser semitone, numbered from the first to the fourth string; the diapente [fifth], formed by three tones and a lesser semitone, from the first to

[2] The first folios of the manuscript are missing.

Chart 2 Guido's hexachord

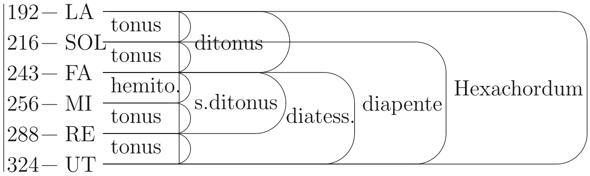
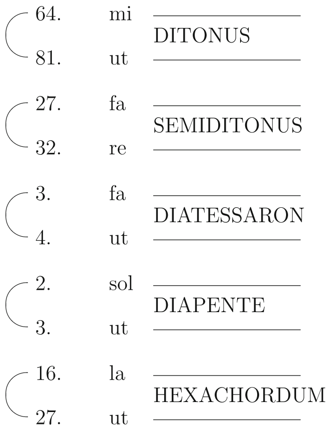


Chart 3 Ratios of musical intervals



the fifth string; lastly the hexachordal consonance containing four tones and a lesser semitone, which completes the whole hexachord.

The ditone thus includes three strings, as does also the semiditone. The diatessaron has four, the diapente five, and the hexachordal consonance six. Few intervals are in unity with all the strings and the consonances. On the other hand, the concord of two voices is that harmonious union that delights the hearing with a certain sweetness, and is called in Latin *consonantia*.

In order to measure and arrange the sounds of the hexachord, on the other hand Guido found six syllables to be pronounced with the human mouth, which are *.ut.re.mi.fa.sol.la.* and which contain all the simple consonances as sustained above; this is clearly evident in this description.

From the syllable *ut* to *re* there is one tone, from *re* to *mi* one tone, from *mi* to *fa* one semitone certainly, from *fa* to *sol* one tone, and from *sol* to *la* one tone. Actually, from *ut* to *mi*, as from *fa* to *la*, there is a ditone, or a major third, with an interval which exceeds 64 parts [of a unit] by 17. Thus the terminal notes of the ditone, which are *ut* and *mi*, or *fa* and *la*, stand in a ratio of 81:64; a ratio composed of those of two tones, from which the ditone derives.[3]

From *re* to *fa*, or from *mi* to *sol*, there is a semiditone, a ditone decreased by a greater semitone, also called a minor third, with the ratio that exceeds 27 parts [of a unit] by 5. Thus, among the terminal notes of the semiditone, *re* stands to *fa*, or *mi*

[3]The ratio of the Pythagorean tone is 9:8; combining this with itself, we obtain 81:64, because $\frac{9}{8} \times \frac{9}{8} = \frac{81}{64}$. See below, notes 8 and 12.

stands to *sol*, respectively in the ratio 32:27. This ratio is composed of the two ratios of the tone and the semitone, from which the semiditone derives.

From *ut* to *fa*, or from *re* to *sol* or from *mi* to *la*, we count the diatessaron, with a sesquithird interval [$1 + \frac{1}{3}$]. Thus *ut* stands to *fa*, or *re* to *sol*, or *mi* to *la*, in the ratio 4:3. This ratio is composed of the two ratios of the semiditone and the tone, or the ditone and the semitone, from which the diatessaron is obtained.

From *ut* to *sol*, or from *re* to *la*, we measure the diapente, with a sesquialtera interval [$1 + \frac{1}{2}$]. Between the terminal notes of this consonance, *ut* stands to *sol*, or *re* stands to *la*, in the ratio 3:2, because this ratio is composed of the two ratios of the ditone and the semiditone, or the diatessaron and the tone, from which the diapente derives.

Lastly, from *ut* to *la*, the hexachordal consonance proceeds through the interval that exceeds 16 parts [of a unit] by 11. Thus *ut* stands to *la* in the ratio 27:16, seeing that the ratio is developed from the two ratios of the diatessaron and the ditone, or the diapente and the tone, from which the whole hexachord is composed.

Thus all the simple consonances, listed in the preceding ratios, are contained in the hexachord, but in order to form others from these, it was necessary to connect several hexachords. Guido himself did so, fixing the first string of the second hexachord over the fourth string of the first one, i.e. *fa*. Likewise, the third was fixed over the fourth string of the second one. He indeed put the fourth hexachord over the second string of the third one, that is to say, over the *re*. In this way, lastly, the third string of this fourth hexachord, i.e. *mi*, was at the distance of a greater semitone from the fourth string of the third one, i.e. *fa*.

Continuing, the three remaining hexachords are places similarly to the three that follow the first one. Thus the greater semitone is placed in two positions, i.e. between the fourth string of the third hexachord and the third string of the fourth one, as well as between the fourth string of the sixth hexachord and the third string of the seventh one. Thus we have two semitones at the distance of the diapason consonance, separated by the interval of the double ratio, as is indicated by the letter ♭ used in the two places. Hence, all the strings of hexachords are indicated in accordance with the cycle of seven letters, so that from one letter to the next same letter, there should be a perfect consonance of one octave. The figure inserted [Chart 4] shows all these elements more clearly than the above points.

From this representation, it can also be seen more clearly how from any one simple consonance the others can be composed. If you assume, above unison, the octave, you will form the first consonance composed of the octave. If instead you go up above the ditone by that same octave, you will create the consonance of a major tenth. If, instead, you pass over the semiditone by an octave, you will form the consonance of a minor tenth.

Of course, no consonance derives from the diatessaron (because in itself, it is not a consonance). Naturally, if you add the same octave to the diapente, you will clearly form the 12th. However, the diapente may be diminished in the greater semitone, as from *E la mi* to *B fa* and sometimes in the lesser one, as from *F fa ut* to ♭ *mi*. However, then these, and all those that derive from it, are melodies that do not agree. Lastly, if you add the often-mentioned octave to the hexachord, you will create the thirteenth.

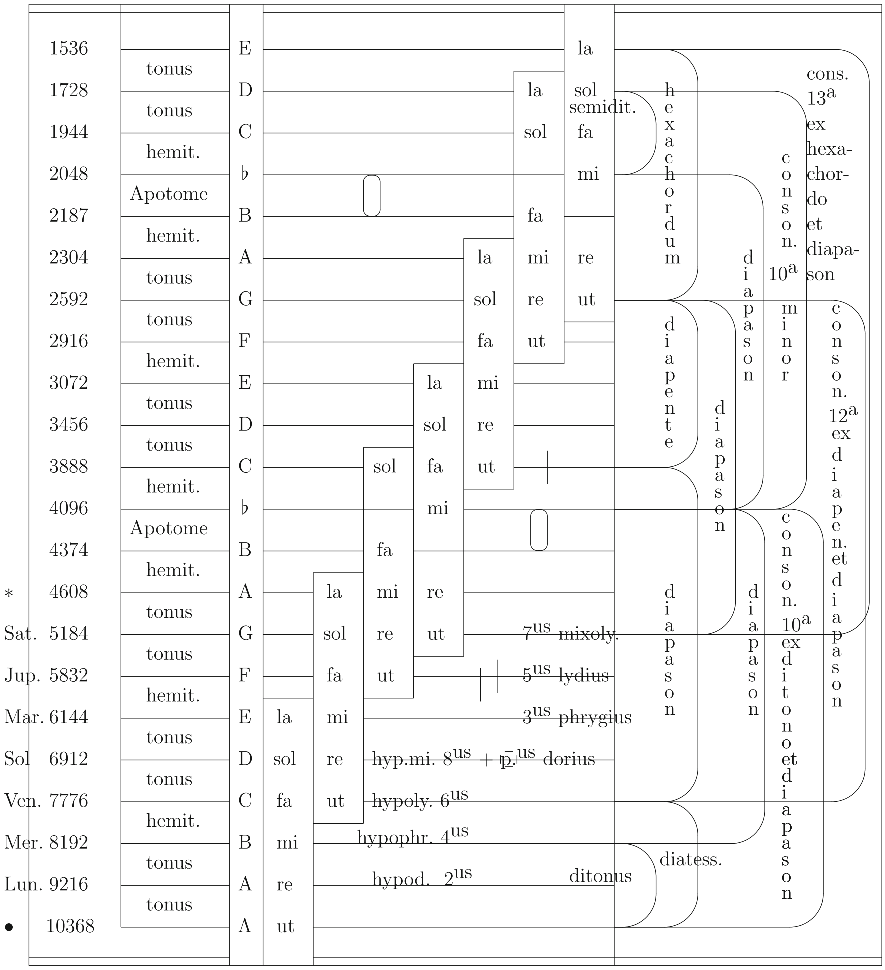


Chart 4 Guido's icosichord

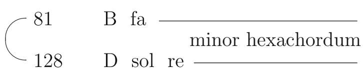


Chart 5 Hexachord minor

The hexachord may also lack the greater semitone, as happens in passing from *D sol re* to *B fa*, by means of a ratio that exceeds 81 parts [of a unit] by 47, because these two notes stand in a ratio 128:81.

Thus the hexachord, on the contrary, will be called minor, or minor sixth, and combined with the octave, it will likewise form a minor thirteenth. And for this reason, they do not agree with the melodies. From these compound consonances, by adding the same octave, yet others may be derived. If you unite to the octave another octave, you will form the bisdiapason, that is to say, a perfect 15th. If you combine the same octave with a major tenth, you will generate a major 17th; if you do the same with a minor tenth, you will obtain a minor 17th; if you do so with a 12th, you will obtain a 19th; with a major 13th, you will obtain a 20th, which includes all seven hexachords.[4]

For this reason, then, Guido created these seven hexachords so interlaced, so that they would include all the consonances compounded together once or twice. Therefore, from these consonances joined together for the second time by means of the continuous union of the octave, if you wished to deduce others combined four times, more than seven hexachords would be necessary. If then you add the octave to the minor 13th, which goes from the deeper *D sol re* to the most acute *B fa* (having previously added, as is right, two hexachords to the acute), you would create the minor 20th, that is to say, diminished by the greater semitone.

Furthermore, I believe that this should not be forgotten at all, that unison, and naturally the other consonances derived from it, the octave or diapason, the bisdiapason or 15th, and the others, are perfect. Whereas all the others, whether simple of compound, are said to be imperfect. Wherever they have been fixed, the former maintain their ratio invariable, whereas the latter (as has been said) are subject to decreases in various places.[5]

Thus the octave is the consonance that sounds in the same way as unison; it is composed of the diatessaron and the diapente, growing with the mean term in common. With its double ratio, it completes five tones and two lesser semitones, as in the case from Γ *ut* to *G sol re ut* first; it is a ratio that is composed of the other two, i.e. the diapente and the diatessaron, from which the octave itself is created. Hence, two notes that differ by an octave stand in the ratio 2:1.

The major tenth is the consonance that sounds the same as the ditone or the major third; produced by the same ditone and the octave, it includes seven tones and two lesser semitones in the ratio that exceeds twice 32 parts [of a unit] by 17, going from Γ*ut* to ♭ *mi*, which stand in the ratio 81:32. This ratio is composed by uniting the ratio of the ditone together with that of the octave, on which the tenth itself is based.

The minor tenth is the consonance that sounds the same as the semiditone, or minor third; it is a compound that combines the semiditone and the octave. It contains six tones and three lesser semitones, in a ratio that exceeds twice 27 parts by 10, as in the case from *A re* to *C sol fa ut*, which stand in the ratio 64:27. This ratio is composed by putting together the ratio of the semiditone with that of the octave, from which this tenth derives. Furthermore, this consonance is diminished by a greater semitone, compared with its major, like the semiditone.

The 12th is the consonance that sounds the same as the diapente, which goes to create it together with the octave. It includes eight tones and three lesser semitones

[4] Compare with Chart 5.

[5] They are not invariant by translation. On these kinds of scales in the Pythagorean tradition, if we calculate fourths and fifths from the various degrees, we do not find the same ratios, 2:3 and 3:4. From this problem and other musical requirements, in time, the equable temperament gained ground.

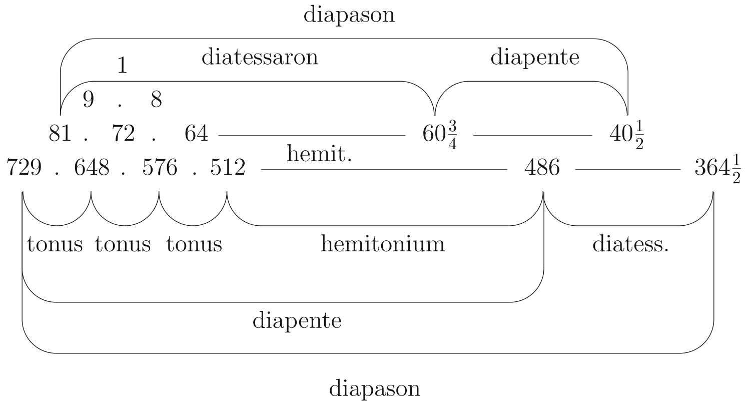


Chart 6 Division of the octave

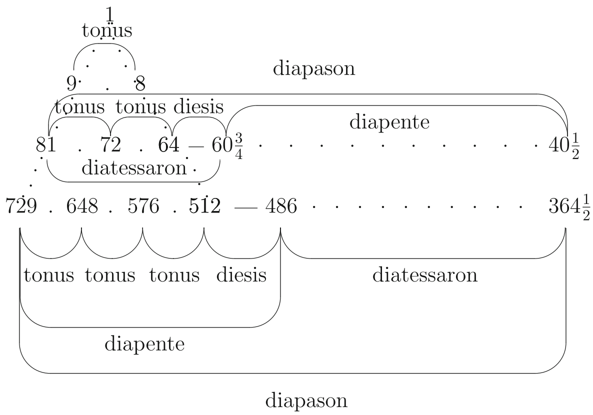


Chart 7 Division of the octave

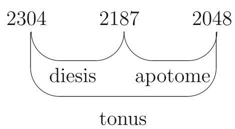


Chart 8 Division of the tone

in a triple ratio, as in the case of Γ *ut* and the more acute *D sol re*, which stand each other in the ratio 3:1. This ratio comes from the ratio of the diapente and the octave, from which the 12th itself derives.

The major 13th is the consonance which sounds the same as the hexachord, or major sixth; it is composed of the sixth itself and the octave. It contains in itself nine tones and three semitones, in a ratio that exceeds three times 8 by 3, as in the case of Γ*ut* and the most acute *E la mi*, which stand in the ratio 27:8. This ratio derives from that of the hexachord and that of the octave, on which the 13th itself is based.

Then the minor 13th is the consonance that sounds the same as the minor sixth; it is formed by the latter together with the octave. It comprehends eight tones and four lesser semitones, in a ratio that exceeds three times 81 parts by 13, as in the case of *D sol re* and the more acute *B fa*, which stand each other in the ratio 256:81. This ratio is formed by the ratio of the minor sixth and the octave, on which this 13th is based.

The lesser semitone is called diesis.
The greater semitone is the apotome.
Their difference is called a comma.
The schism is said to be half the comma.
The diaschism is half of the diesis, that is to say, half of the lesser semitone.[6]

[II]

C.1 Rules to Compose Consonant Music

The first rule states that, at the beginning, every melody should start with perfect consonances, in other words, with unison or its compounds, or with a fifth or related intervals, which is included among the perfect ones in view of its sweetness. Actually, this rule is arbitrary rather than necessary, and in many cases melodies begin with consonances that are not perfect.

The second rule states that two perfect consonances of the same species cannot be places one immediately after the other, either ascending or descending.

The third rule states that between two perfect consonances of the same kind, raised or lowered in different motion or similarly, another imperfect one should be formed, like a third or a sixth.

The fourth rule states that several perfect and dissimilar, ascending or descending consonances can be formed: like a fifth after unison, or the octave after a fifth.

The fifth rule states that two similar perfect consonances can be placed one immediately after another, provided that they proceed by contrary motion: thus, if one octave rises in the acute, then on the contrary, the other octave will go down towards the deep; the same goes for the fifth.

[6]On March 17th 1569, at this point in the manuscript, Maurolico noted down the project of the compendium in the following points: "Theory and ratios of music 30. Eight-stringed lyre with its preamble. Account of Guido's Icosichord. Properties of modes. Inventors of instruments. Precepts of melodies. Calculation of systems. Ratios of the values of notes". There follow calculations and charts still at the drafting stage, of which we repeat here only those that are most significant. The numbers of the charts are not arranged in the same order as in the critical edition. Maurolico 201?.

The sixth rule states that the *tenor* and the deep *cantus*[7] should proceed by motion contrary to each other: in this way, if the *cantus* rises, the *tenor* goes down, and vice versa. This rule is rather arbitrary, and often they rise or descend together.

The seventh rule states that, moving from the sixth to the octave, as from the third to the fifth, or from unison to the third, or vice versa, the contrary movement of the *cantus* and the *tenor* sounds particularly sweet. The same is true when passing from the minor sixth to the fifth, one part remaining fixed, while the other moves. The same you see of their combinations.

The eighth rule states that the melody should finish on a perfect consonance, seeing that the end of everything is perfection.

Dissonance cannot be formed on notes that are breves and semibreves, or on greater values. Minimal notes and undoubtedly those of a lesser value may admit some dissonances, because they pass more rapidly. In the same way, dissonance may be admitted on passing from an imperfect consonance to a perfect one by means of contrary motion, as in the seventh rule. In this case, the passage is not only through a minim, but also a semibreve.

Three voices harmonise best, when the extremes play the octave together, and the middle one is a diapente away from the deeper, and therefore a fourth close to the higher. Thus the fourth, the diatessaron, does not agree, unless it is placed in this position. The same is said of the consonances that they form.

The elapsing of thirds or tenths, which rise or fall for notes whose duration is similar, is likewise pleasant.

[Part II]

[III]

Consonances of musical melodies lie not only in the commensurable ratios, but also in particular numbers. For incommensurable ratios (seeing that they are irrational and unknowable) always produce dissonance: because notes composed of such ratios, due to their incommensurability, respond to one another by means of strokes that are not well-ordered, but always different (this difference generates disharmony).

Then, the particular numbers are 1, 2, 3, 4. First of all, 1:2 is that ratio of the diapason [octave] consonance. The diapente [fifth] is based on the ratio 2:3. The diatessaron [fourth] on the ratio 3:4. These two together form the ratio 2:4, which is the diapason. From these are derived the intervals of tones and semitones, from which we obtain the natural and diatonic levels of ascending or descending melodies. Actually, the tone is the interval of the sesquieighth ratio [8:9], which means that it is the difference[8] between the diapente and the diatessaron; therefore it

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[7] *Tenor* was used to indicate the main voice that guided the composition. *Cantus* meant the melody.

[8] In calculations, it should be remembered that subtraction between intervals corresponds to a division in the relative ratios. If, anachronistically, fractions were used, we would thus obtain

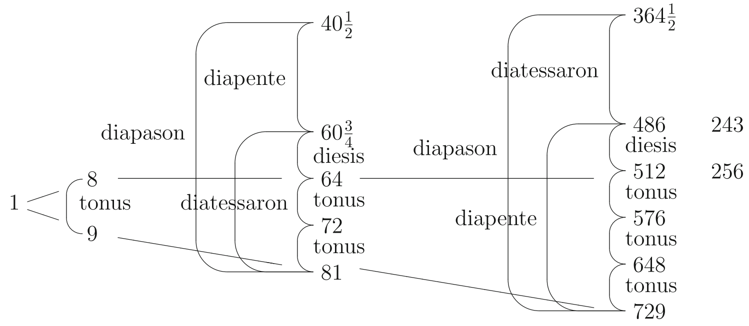


Chart 9 Division of the octave

is the difference between the sesquialtera [2:3] and the sesquithird [3:4],[9] which we find in the numbers 9, 8, 6. Instead, the tone taken away twice from the diatessaron leaves the semitone lesser or diesis.

Consequently, in a natural melody, we go up by one tone, another tone and a diesis; again by one tone, another tone and a diesis. Two single diatessaron intervals are thus completed, and if yet another tone is added to these, the diapente is clearly completed, which consists of three tones and a semitone. In this way, also the octave of five tones and two [minor] semitones is also completed, as is clear from the numbers indicated above.

And seeing that, after completing the octave, the same order must be maintained (so that the octave consonance will always finish at the eighth place, with the above-said five tones and two [minor] semitones), rising further for this reason, we again proceed by tone, tone and diesis, and so on, repeating the succession of the seven previously described intervals one, twice, three times and *ad infinitum*.

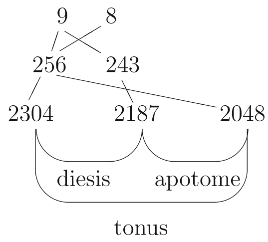
Hence, in the progression of any octave, we are forced to admit three tones next to one another. The third of these is usually divided by musicians into a greater and a lesser semitone, or in other words an apotome and a diesis, in order avoid the hardness of the tritone. And for this reason that melody which completes the tritone, i.e. $\natural mi$, takes its name from the same square hard $\natural$, seeing that it tires the singer and strains the voice, as a result of its excessive hard rise. On the contrary, the interposed melody which divides the third tone of the preceding tritone, i.e. $\flat fa$, is given the name of the same soft round $\flat$, in view of its sweetness, ease and gentleness of performance.

The preceding hexachord, which does not include the tritone or make the above-said division of the third tone into semitones, takes its name instead from Nature, as the intermediary between hardness and sweetness. Furthermore, the hexachord was

$\frac{2}{3} : \frac{3}{4} = \frac{8}{9}$. In the following chapters, Maurolico will be found to calculate the ratios, without using fractions, by means of a different procedure. See the following note and Chap. IV.

[9] Here, Maurolico made a mistake, writing erroneously sesquithird and sesquifourth [4:5].

Chart 10 Division of the tone



chosen because, as we have said elsewhere,[10] it includes all the simple consonances, that is to say, unison, the third, the fourth, the fifth, the sixth, or better, unison, the ditone, the diatessaron, the diapente and the hexachord itself. If indeed the octave is added to these single intervals, the same number of secondary consonances will be created for composition. Adding the octave, a consonance of the same quality is always produced. And thus the interval of a 13th will contain consonances of the second order, that is to say, the octave, the 10th, the 11th, the 12th and the 13th.

The addition of another octave to the 13th will create the 20th, which finishes all the icosichord of Guido.[11] This includes systems of the third order, that is to say, the double octave, the 17th, the 18th, the 19th and the 20th. And continuing like this, one could proceed *ad infinitum*.

Hence the origin is clear of these notes that make up the hexachord, that is to say, *ut, re, mi, fa, sol, la*. Instead, eight letters are fixed, $\Gamma.a.b.c.d.e.f.g$, so that any octave of these, repeated also in the eighth place, will always indicate the consonance of the double ratio. And the numbers arranged in the single strings of the icosichord display this, like all the other consonances and the other intervals.

As it is voiced, the letter *g* starts the $\natural$ hexachord of the square, and hard genre.

As it is intermediate between the aspirated and the voiced, *c* starts the hexachord of Nature in the diatonic genre.

As it has an aspirated, soft sound, *f* starts the $\flat$ hexachord of the chromatic, and soft genre.

Thus it is clear in numbers how tones proceed in the sesquieighth ratio [8:9], and how the ratios of the sesquialtera [2:3] and sesquithird [3:4] are formed, that is to say, the diapente and diatessaron components of the octave, which leave the interval of the diesis between them.

From these, it can be verified that the ratio of the diesis lies in the numbers 256:243. Now we must see the ratio of the greater semitone, the apotome. As the ratio 9:8 produces the tone, multiply the 256 and also the 243 by 9, obtaining the two numbers 2304 and 2187, whose ratio is the same as 256:243, i.e. again, the diesis. Multiply 256 also by 8, obtaining 2048. 2304:2048 will thus be like 9:8. Consequently, the ratio 2304:2048 composes the tone, and if the ratio 2304:2187

[10] Cf. Chap. I.

[11] See below, Chap. VII.

Chart 11 Calculation of tone and diesis

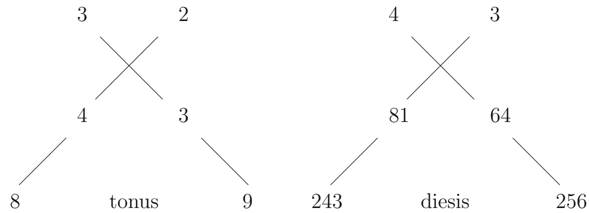


Chart 12 Calculation of apotome

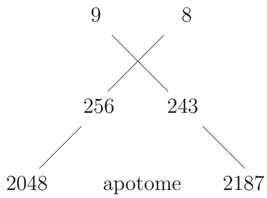
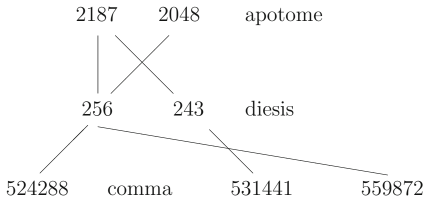


Chart 13 Calculation of comma



forms the diesis, or lesser semitone, then the ratio 2187:2048 will be left for the greater semitone, or apotome.

Fifth calculation that obtains the apotome by the difference between the tone and the diesis.[12]

Furthermore, the difference between the apotome and the diesis is called the comma, which is obtained by subtracting one ratio from the other, as is clear above.

It will also be verified that the diesis is more than three commas, but less than four.

Whereas the apotome is more than four commas, but less than five.

Hence also the tone exceeds eight commas, but is less than nine; these are all things that can be verified by means of a long calculation, with many figures. See Boethius and Faber in the Elements of music.[13]

Distinguishing the eight strings with the tones and the diesis.

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[12]In the Graeco-Western Pythagorean tradition, the numbers corresponding to the notes are distributed in a geometrical progression. Thus, if calculations are performed by translating the proportions into ratios, and these into numerical fractions, as we were taught at school, it is necessary to divide the relative fractions in order to obtain the difference between intervals, and to add them, they must be multiplied. See Chap. 5 below, where Maurolico gave the general rules, and performed all the necessary calculations in an orderly manner.

[13]Boethius 1867. Tonietti 2006b. See above, Part I, Chap. 6.

*									(*toni*)
Sat.	....	$4\frac{1}{2}$	8	—— 18	*Nete* — 54 —	81		....	..7^{us}
	tonus								
Jup.	....	$5\frac{1}{16}$	9	*Para* $20\frac{1}{4}$	*nete* — $60\frac{3}{4}$ —	$91\frac{1}{8}$	*diate.*	....	..5^{us}
	diesis								
Mar.	....	$5\frac{1}{3}$	$9\frac{13}{27}$	*Para* $21\frac{1}{3}$	*mese* — 64 —	96		....	..3^{us}
	tonus							*diapason*	
Sol	....	6	$10\frac{2}{3}$	—— 24	*Mese* — 72 —	108		....	8^{us} 1^{us}
	tonus								
Ven.	....	$6\frac{3}{4}$	12	—— 27	*Licha* — 81 *nos*	$121\frac{1}{2}$	*diapente*	....	6^{us} .
	diesis								
Mer.	....	$7\frac{1}{9}$	$12\frac{52}{81}$	*Parhy* $28\frac{4}{9}$	*pate* — $85\frac{1}{3}$ —	128		....	4^{us} .
	tonus								
Luna	....	8	$14\frac{2}{9}$	—— 32	*Hyp.* — 96 —	144		....	2^{us} .
	tonus								
•	....	9	16	*Proslam* 36	*banome* 108 *nos*	162		.....	
	14								

Chart 14 Correlation between notes and planets[14]

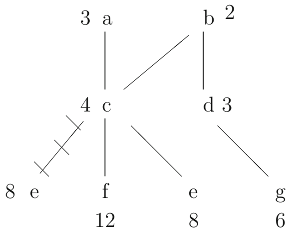


Chart 15 Rule of unification

[IV]

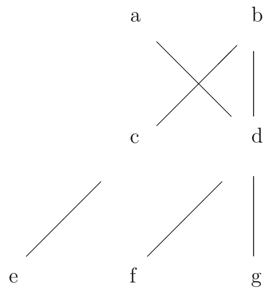
C.2 Rule of Unification

Arranging two ratios one immediately after the other.

Let two ratios be given, $a : b$ and $c : d$, which it is necessary to arrange one after the other. Produce b to c, obtaining e. Produce a to c, obtaining f. It will be

[14]The chart bears the date of Monday, December 30th, 1566. The manuscript contains calculations shown in Maurolico 201?. I have substituted the symbols of stars in the manuscript with their names.

Chart 16 Rule of taking away



true that $a : b = f : e$[15] seeing that products multiplied by the same number are proportional.

Produce the same b to d obtaining g. It will likewise be true that $c : d = e : g$. But the ratio $f : g$ is composed of the ratios $f : e$ and $e : g$. Thus the same ratio $f : g$ is composed of the ratios $a : b$ and $c : d$, which are arranged one immediately after the other under the terms f and g. This is generally valid.

C.3 Rule of Taking Away

Given two ratios, take one of the two from the other one.

It is proposed to take away the ratio $a : b$ from the ratio $c : d$. Produce b to c, obtaining e. Also a to d, obtaining f. And again, b to d, obtaining g. And it will be true, as products remain proportional after multiplication, that $c : d = e : g$, and likewise $a : b = f : g$. But $e : f$ is the ratio that remains of the ratios $e : g$ and $f : g$. In other words, taking away the ratio $f : g$ from the ratio $e : g$, what remains is the ratio $e : f$.

Thus, also taking away the same ratio $a : b$ from the ratio $c : d$, what remains is the ratio $e : f$; and this is the desired remainder. Actually the three numbers position, one immediately after the other, the two ratios that compose the ratio contained in the extreme terms. Of these, one or the other is the excess of the compound ratio with respect to the other.

First calculation. The compounding of the sesquialtera [3:2] and sesquithird [4:3] generates the double ratio [2:1]. Hence, if the sesquialtera is taken away from the double ratio, what is left is the sesquithird, and vice versa. This derives from the multiplication of the terms 3.2 and 4.3, from which we obtain 12.8.6. Thus the diapente forms the diapason [octave] with the diatessaron, and taking either of the two from the octave, the other one is left over.

[15]Initially, Maurolico made a mistake, putting e on the left of f instead of on the right.

Chart 17 Unification of fifth and fourth

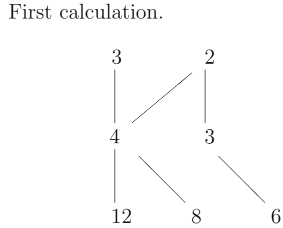


Chart 18 Taking away the fourth from the fifth

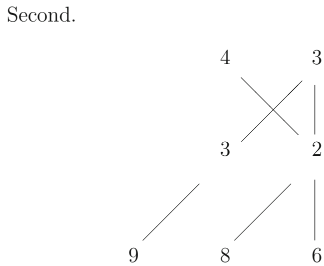
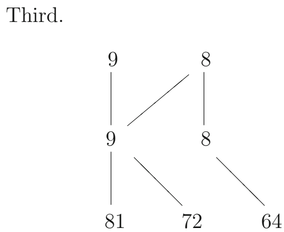


Chart 19 Unification of two tones



Second. If the sesquithird [4:3] is taken away from the sesquialtera [3:2], the sesquieighth [9:8] will be left over. Hence, the sesquithird forms, together with the sesquieighth, the sesquialtera, as can be verified by multiplying in this order. The following numbers are thus obtained: 9.8.6. Thus, if the diatessaron is taken away from the diapente, the tone is left over. And together with the diatessaron, the tone forms the diapente.

Third. Compounding two sesquieighths [9:8], we generate the ratio which exceeds 64 parts of a unit by 17, i.e. 81:64. In this way the three numbers 81.72.64 are produced, which contain two sesquieighths. Thus two tones placed next to each other form the ditone.

Fourth. Taking away the ratio 81:64 from the sesquithird [4:3] in this way, the ratio 256:243 is left over. Thus, if the ditone is taken away from the diatessaron, what is left over is the diesis, that is to say, the lesser semitone. Placed immediately after the ditone, this forms the diatessaron.

Fifth. If the ratio 256:243 is taken away from the sesquieighth [9:8], what is left is the ratio 2187:2048, multiplying in this order. Thus, taking away the diesis from the tone, the apotome is left over. And together with the diesis, this forms the same tone again.

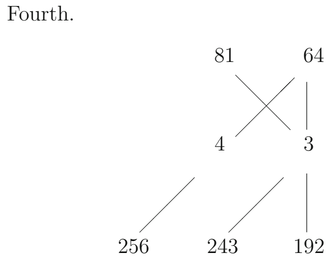


Chart 20 Taking away two tones from the fourth

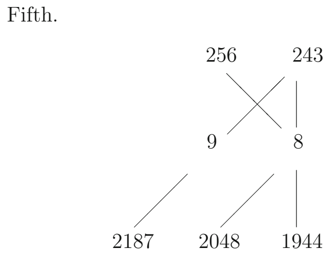


Chart 21 Taking away the diesis from the tone

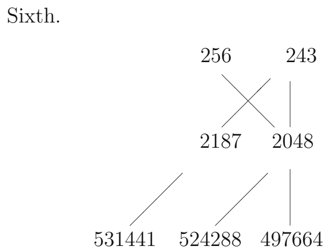


Chart 22 Taking away the diesis from the apotome

As the mean proportional number between 2187 and 1944 is higher than 2061, consequently the ratio 2048:1944 of the lesser semitone is less than half of a tone. And the ratio 2187:2048 of the greater semitone is more than half of a tone.[16]

Sixth. Taking in this way the ratio 256:243 from the ratio 2187:2048, what if left is the ratio 531441:524288. Thus, taking away the diesis from the apotome, that is to say the lesser from the greater semitone, what is left is the comma, which together with the diesis forms the apotome again.

Seventh. Here follows the seventh calculation of Boethius, in which he concluded that the tone was more than eight commas, but indeed less than nine. Similarly, the diesis is more than three, and less than four. Also the apotome is more than four and less than five. We have reported this calculation, demonstrating furthermore that the diesis is more than three and a half commas, and that the apotome is more than four and a half commas. This is clear below.[17]

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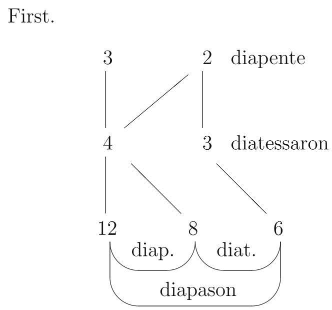
[16] To verify this, Maurolico left a series of calculations in the manuscript; Maurolico 201?.

[17] See Chap. V.

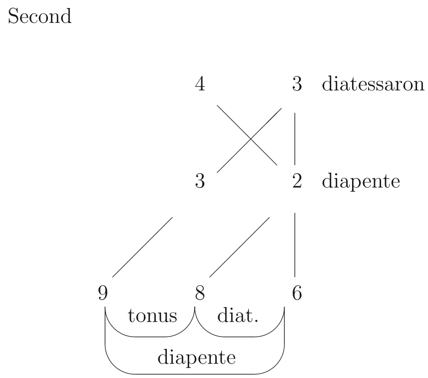
First.

Chart 23 Calculation and division of octave



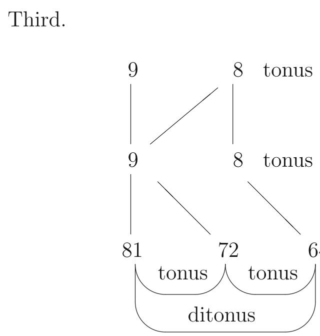
Second.

Chart 24 Calculation and division of fifth



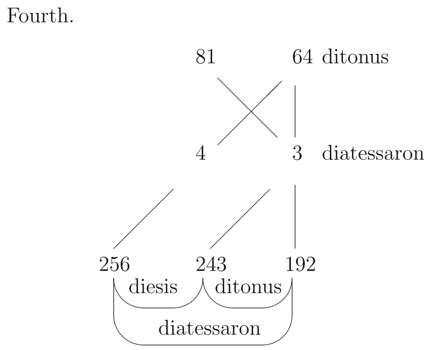
Third.

Chart 25 Calculation and division of ditone



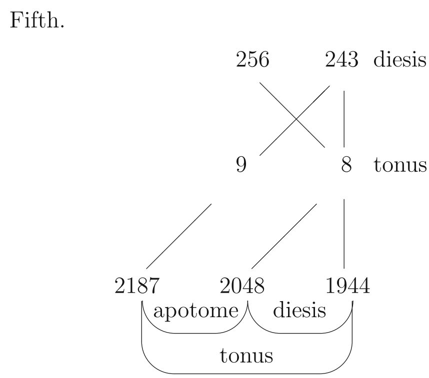
Fourth.

Chart 26 Calculation and division of the fourth



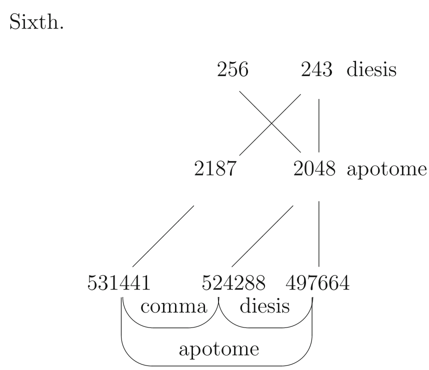
Fifth.

Chart 27 Calculation and division of the tone



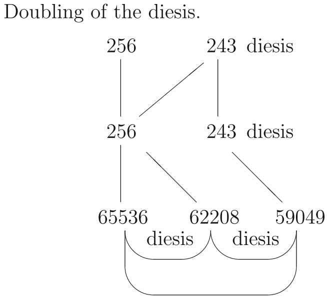
Sixth.

Chart 28 Calculation and division of the apotome



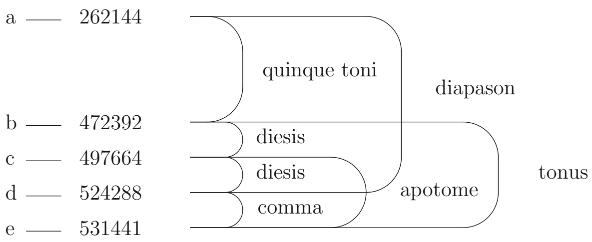
Doubling of the diesis.

Chart 29 Doubling of the diesis



C.4 The Calculation of Boethius for the Comparison of Intervals

Chart 30 Boethius' calculation



f —	59049	differentia toni
g —	25272	differentia diesis
h —	33777	differentia apotomes
k —	7153	differentia commatis
l —	21459	triplum differentiae commatis
m —	28612	quadruplum
n —	35765	quincuplum
p —	57224	octuplum
q —	64377	nonuplum

This is the calculation of Boethius in Book III of his *Musica*, in which he concluded that the tone was undoubtedly less than nine commas, and more than eight. Analogously, the diesis was less than four commas and more than three. Thus the apotome was less than five commas, and more than four.

However, he made a mistake in this calculation, because he used the same differences in the division of the ratio. Whereas proportional ones should have been used: this is how ratios are multiplied, and not by means of equal differences. Effectively, in this way he avoided the effort of performing the multiplications, and yet he arrived at the correct result, and thus we may excuse him.[18]

[V]

C.5 Comment on the Calculation of Boethius

The numbers 64, 65, 66, 67, 68, 69, 70, 71, 72 are listed. Let the ratio of the tone be 72:64; then it is clear that the ratio 72:71 is less than one eighth part of the tone. But 531441:524288, the ratio of the comma, is like $72 : 71\frac{16425}{531441}$, that is to say, less than 72:71. Thus, a fortiori, it will be less than one eighth part of the tone. Hence, eight commas are less than one tone. Therefore, Boethius was right to say that the tone exceeded eight commas.

Consequently, the diesis, together with the apotome, also exceed eight commas. But the diesis exceeds three commas and the apotome four, whereas their difference is equal to the comma. Thus the diesis necessarily exceeds three and a half commas, while the apotome exceeds four and a half commas. This thing was left out by Boethius. The excesses over these half-tones must also necessarily be equal. Otherwise the difference between the diesis and the apotome would not be equal to the comma, but more than a comma.

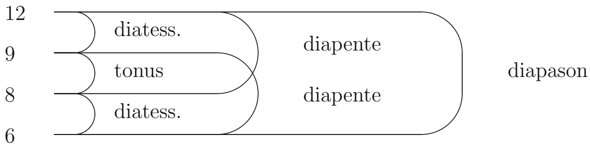
In the same way, based on the arguments of Boethius, it is certain that the diesis is less than four commas, and the apotome less than five; likewise, the tone is less than nine. But the diesis cannot have three commas and the apotome four, because otherwise the whole tone would include only seven: on the contrary, it has been established that the tone is more than eight commas. However, it is not possible, either, that the diesis contains exactly three and a half commas, or that the apotome contains exactly four and a half, because the whole tone would then contain exactly eight commas, when it has been calculated to be more than eight commas. Thus the diesis must necessarily exceed three and a half commas, and the apotome must exceed four and a half. Therefore the whole tone will exceed eight commas, as we have shown.[19]

---

[18]The correction to Boethius bears the date of January 31st, 1567.

[19]Maurolico's proof, triggered by the mistake of Boethius, bears the date of February 6th, 1567. It is based on the remark that the arithmetic mean is always more than the geometrical mean. Antiquity has not handed down any others to us. The elegance of avoiding long, impossible calculations should be noted; these would have involved multiplying six-figure numbers nine times among themselves. On this subject, see Tonietti 2006b.

Chart 31 Ratios of intervals



[VI][20]

Arithmetic mean, in which the differences are equal: 1, 2, 3.

Geometrical mean, in which the differences are proportional: 1, 2, 4.

Harmonic mean, in which the diatessaron, together with the diapente, forms the octave, that is to say, the sesquithird [4:3], together with the sesquialtera [3:2], forms the double ratio [4:2].

These four numbers contain the main harmonic systems, from which the remaining intervals may be derived. Among them, we find the arithmetic mean, 6, 9, 12, where the differences between the terms remain the same. There is the same difference in terms for the initial, central and final terms 1, 2, 3.

Among them, we also find the geometrical mean, because 6:8 = 9:12, and consequently the product of the extremes, 72, equals the product of the terms in the middle. There is the same difference for the terms in the middle, and first and last terms 1, 2, 4, or 4, 6, 9, or 2, 4, 8.

Among them, we find the harmonic mean, in which the sesquithird ratio, together with the sesquialtera, forms the double ratio, as in 6, 8, 12.

Among these numbers, we can find the number of surfaces of a cube, six, of course. And also the number of solid angles of a cube, i.e. 8. And also the number of edges at the sides of the surfaces, that is to say, 12.

We see that Boethius mentions all these in Book II of *Arithmetica* on 54th, where he asserted that these single numbers 6, 8, 9, 12 were solid. Why should we be surprised? Or what is noteworthy? After all, every number can be linear, planar or solid.

This is the main subject that Faber would like to introduce into Chap. 3 of his *Musica* on 33th, but in such an obscure manner that I cannot understand. However, in the last proposition in Book II of his *Musica*, he says that the tone is more than seven commas, not realising that it is more than eight, as Boethius shows in his *Musica* on 15th of Book III. He abounds in obscure intricacies and lacks the things that are necessary.

[20]Written on June 21st, 1567, following the request of Prince Pietro Barresi.

e.	4	...	...	e	...	...	...	...	...	...	la	...	e ...	8		24	e	8
												to.	...		tonus			
d.	$4\frac{1}{2}$	...	...	d	...	...	...	...	...	la	sol	...	d ...	9		27	d	9
												to.	...		to.			
c.	$5\frac{1}{16}$	...	...	c	...	...	...	...	...	sol	fa	...	c ...	$10\frac{1}{8}$		$30\frac{1}{4}\frac{1}{8}$	c	$10\frac{1}{8}$
												sem.	...		di.			
♮.	$5\frac{1}{3}$	...	...	♮	...	...	...	...	...	...	mi	...	♮ ...	$10\frac{2}{3}$		32	♭	$10\frac{1}{2}\frac{1}{6}$
			maius															
	♭	$5\frac{89}{128}$		♭						fa		to.	♭ tonus	$11\frac{25}{64}$	to.+	$34\frac{11}{64}$		
			minus															
a.	6	...	...	a	...	...	...	...	la	mi	re	...	a ...	12		36	a	12
												to.	...		to.			
g.	$6\frac{3}{4}$	...	...	g	...	...	...	...	sol	re	ut	...	g ...	$13\frac{1}{2}$		$40\frac{1}{2}$	g	$13\frac{1}{2}$
												to.	...		to.			
f.	$7\frac{19}{32}$	...	...	f	...	...	...	...	fa	ut	...	...	f ...	$15\frac{3}{16}$		$45\frac{1}{2}\frac{1}{16}$	f	$15\frac{1}{8}\frac{1}{16}$
												sem.	...		di.			
e.	8	...	...	e	...	...	...	la	mi	...	...	...	e ...	16		48	e	16
												to.	...		to.			
d.	9	...	...	2 d	...	...	la	sol	re	...	...	...	d ...	18		54	d	18
												to.	...		to.			
c.	$10\frac{1}{8}$	...	...	c	...	...	sol	fa	ut	♮	...	...	c ...	$20\frac{1}{4}$		$60\frac{1}{2}\frac{1}{4}$	c	$20\frac{1}{4}$
												sem.	...		di.			
♮	$10\frac{2}{3}$	...	...	♮	...	...	...	mi	...	...	...	...	♮ ...	$21\frac{1}{3}$		64	♭	$21\frac{1}{3}$
			maius															
	♭	$11\frac{25}{64}$		♭			fa					to.	♭ tonus	$22\frac{25}{32}$	to.+	$68\frac{11}{32}$		
			minus															
a.	12	...	...	a	...	la	mi	re	...	...	...	...	a ...	24		72	a	24
												to.	...		to.			
g.	$13\frac{1}{2}$	...	...	3 g	...	sol	re	ut	...	...	...	...	g ...	27		81	g	27
												to.	...		to.			
f.	$15\frac{3}{16}$	...	...	f	...	fa	ut	...	...	...	♮	...	f ...	$30\frac{3}{8}$		$91\frac{1}{8}$	f	$30\frac{1}{4}\frac{1}{8}$
												sem.	...		di.			
e.	16	...	...	e	la	mi	...	...	...	...	...	...	e ...	32		96	e	32
												to.	...		to.			
d.	18	...	...	4 d	sol	re	...	...	...	...	...	...	d ...	36		108	d	36
												to.	...		to.			
c.	$20\frac{1}{4}$	...	...	c	fa	ut	...	...	...	...	...	...	c ...	$40\frac{1}{2}$		$121\frac{1}{2}$	c	$40\frac{1}{2}$
												sem.	...		di.			
♮.	$21\frac{1}{3}$	...	...	♮	mi	...	...	...	...	...	...	...	♮ ...	$42\frac{2}{3}$		128	♭	$42\frac{1}{2}\frac{1}{6}$
			maius															
	♭.	$22\frac{25}{32}$		♭								to.	♭ tonus	$45[\frac{9}{16}]$	to.	$136\frac{11}{16}$		
			minus															
a.	24	...	...	a	re	...	...	...	...	...	...	...	a ...	48		144	a	48
												to.	...		to.			
g.	27	...	...	Γ	ut	...	...	...	...	...	...	...	Γ ...	54		162	Γ	54

Chart 32 Guido's icosichord[21]

[VII]

C.6 Guido's Icosichord

Guido's icosichord, with the tones and greater and lesser semitones, together with the intervals of the consonances represented by means of the smallest numbers possible. Where it is indicated that the harmonic genre is represented by the square ♮, in which naturally the interval of the tritone is admitted in order to maintain the consonance of the octave. The chromatic genre is represented by the soft round ♭, where to sweeten [the tritone] a division is made into the greater and

[21]The manuscript also contains another chart that has been cancelled; this is shown in Maurolico 201?.

$5\frac{1}{3}$	a		*Anthypate*	4	$*$	*firmamentum*
		tonus				
6 ..	g		*Nete*	$4\frac{1}{2}$	*Sat.*	*Mixolydius* 7^{us}
		tonus				
$6\frac{3}{4}$	f		*Paranete*	$5\frac{1}{16}$	*Jup.*	*Lydius* 5^{us}
		diesis				
$7\frac{1}{9}$	e		*Paramese*	$5\frac{1}{3}$	*Mar.*	*Phrygius* 3^{us}
		tonus				
8 ..	d		*Mese*	6	*Sol*	*Dorius* 1^{us} *Hypomixolydius* 8
		tonus				
9 ..	c		*Lichanos*	$6\frac{3}{4}$	*Ven.*	.... *Hypolydius* 6
		diesis				
$9\frac{13}{27}$	♮		*Parhypate*	$7\frac{1}{9}$	*Mer.*	.... *Hypophrygius* 4
		tonus				
$10\frac{2}{3}$	a		*Hypate*	8	*Luna*	.... *Hypodorius* 2
		tonus				
12 ..	Γ		*Proslambanomenos*	9	$\bullet$	*Centrum*

Chart 33 The arrangement of the tones in the seven-stringed lyre[23]

lesser semitones. The natural and diatonic genre is intermediate between these, and proceeds by two tones and semitones.

The theory of ratios, in consonant intervals of voices, and in the performance of songs, is presented here. This is Guido's icosichord, arranged in accordance with the diatonic genre, in which then the tritone was admitted in the eighth position. That is to say in this, the ratio of the octave consonance will be composed of five tones and two diesis, as in each voice of its following octaves. Actually, in order to avoid the hardness of the difficult tritone, the third of these tones is divided into the greater and lesser semitones, i.e. into the apotome and the diesis. This division may be performed by musicians and by well-prepared singers in accordance with the chromatic genre in any tone. As happens in citharas, organs and instruments with many strings. These ratios cannot be represented without fractions, or without very large numbers.[22] In this connection, if it were desired to transform the fractions back to whole numbers, the single terms should be multiplied by 384; actually this number contains all the fractions of the terms.

It should be underlined that the pyramid contains four [solid] angles, like the units of the highest number. The octahedron has six angles, the units of the fifth number. The cube has eight angles, the units of the eighth number. The icosahedron has 12 angles, the units of the 12th number. The main consonances are contained in these. The dodecahedron has 20 angles, and the icosahedron has the same number of surfaces, and all the strings of Guido's icosichord add up to the same number. This was not only nice to know, but also admirable and worthy of being noted.

[22]We read in the manuscript the date of November 20th, 1567.

[23]The manuscript also includes calculations, as in Maurolico 201?.

[VIII]

In this seven-stringed lyre, there are seven strings which contain eight modes of singing: four of these are authentic, the main guiding ones, called Dorian, Phrygian, Lydian and Mixolydian; and the other four are plagal, subordinate and secondary, which are called Hypodorian, Hypophrygian, Hypolydian, and Hypomixolydian, because they stand one diatessaron below their respective authentic modes.[24] By singers, the former are indicated, respectively, as first, third, fifth and seventh, and the latter as second, fourth, sixth [and eighth]. Thus, with this seven-stringed lyre of Mercury, it is necessary for the middle string to assume two of these modes, i.e. the first one, called Dorian, and the eighth one, Hypomixolydian, which will have to be positioned at the interval of a diatessaron from its authentic Mixolydian.

Furthermore, each string is either between two tones, or between a tone and a diesis. In the diatonic genre, if it is between two tones, either these are two tones by themselves, or they are the two of the tritone. If they are two by themselves, then the string assumes the first Dorian mode at the letter *d*, because the middle string is most suitable and equally distant from the higher and the lower tritone. If, on the contrary, the two tones, between which the string stands, come from the tritone, either they are the higher two or the lower two. If they are the higher, then the string assumes the second Hypodorian mode at the letter *a*. If they are the lower two, then the string assumes the seventh Mixolydian mode at the letter *g*; one diatessaron below this stands the eighth Hypomixolydian mode at the letter *d*, where the first mode is also positioned.

If indeed the string stands between the tone and the diesis placed above it, then it assumes the third Phrygian mode at the letter *e*; one diatessaron below this stands the fourth Hypophrygian mode at the letter ♮ in a similar position. Now if the string stands between the tone and the diesis placed below it, then it assumes the fifth Lydian mode, and below this at the above-said interval stands the sixth Hypolydian mode in a similar position. From all this, it is clear that each authentic mode occupies a position similar to its plagal equivalent.

Actually, both the Dorian and the Hypodorian stand between two tones: one between the tones of the ditone, the other, between those of the tritone. Similarly, the Hypomixolydian and the Mixolydian; thus the Phrygian and the Hypophrygian have the diesis above them and the tone below them. Lastly, the Lydian and the Hypolydian have the tone above them and the diesis below them.

In this way, my distinguished reader, you have strings distributed in seven parts, and these places assume the aforesaid eight modes of singing. Furthermore, the arrangements of the strings that follow one another towards the acute are similar to

[24]There were seven Greek modes, whereas there were eight ecclesiastic ones in Gregorian chant: four authentic and four plagal.

the previous ones, and like those correlated with them, when the similar position and the variety of tones and diesis are repeated *ad infinitum* in the icosichord of Guido.

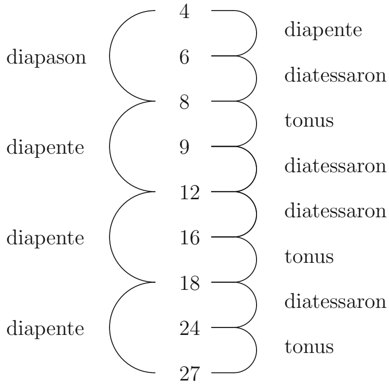
Hence let it be manifest that as the Sun is the centre of the planets, so the middle string of the lyre assumes the Dorian, the first of the modes. Consequently, the three highest strings are assigned to the three remaining authentic modes, and to the three higher planets: the seventh Mixolydian to Saturn, the fifth Lydian to Jupiter, the third Phrygian to Mars. On the other hand, the three lowest strings correspond to the three plagal modes and to the three lower planets, that is to say, the sixth Hypolydian to Venus, the fourth Hypophrygian to Mercury and the second Hypodorian to the Moon. The Hypomixolydian, which stands together with the Dorian, is assigned to the eighth heaven, as it is the eighth.

Ptolemy and Aristoxenus also add, above the others, four acute modes: Hyperdorian, Hyperphrygian, Hyperlydian and Hypermixolydian. Of these, the Hyperdorian coincides with the Mixolydian, and then the remaining three follow, which correspond to a certain extent in their arrangement to the Hypodorian, Hypophrygian and Hypolydian.

Likewise, consider that the numbers of the modes correspond to the number of the days in a week, which are ruled by the planets from which they take their names. Thus the first mode, Dorian, is attributed to the Sun, which governs the first day of the week. The second mode, Hypodorian, to the Moon, which governs the second day of the week. The third mode, Phrygian, to Mars, which governs in the third day. The fourth, Hypophrygian, to Mercury, which dominates in the fourth day. The fifth mode, Lydian, to Jupiter, which governs in the fifth day. The sixth, Hypolydian, to Venus, which governs over the sixth day. The seventh mode, Mixolydian, to Saturn, which dominates the seventh day. The eighth mode, Hypomixolydian, to the Sun, which follows the Saturday, or the eighth heaven. Furthermore, lastly, the four authentic modes exceed their relative underlying modes by the interval of a diatessaron. And thus each of the higher planets is a fourth, with respect to its lower equivalent, calculating the Sun twice, as lower than Saturn and higher than the Moon. This is the order of the modes observed most recently by people.

Take note, my reader, that Boethius, guided by Ptolemy, did not place in *A la mi re* the Hypomixolydian, but the Hypermixolydian corresponding to the eighth sphere, which follows over Saturn. Thus the two modes do not coincide in the Sun. On the contrary, more up-to-date people (as mentioned above) do not posit only one acute mode, like Ptolemy, but four. However, it is not necessary to procure other modes, beyond the number eight of the heavenly spheres, seeing that, distributing the modes in the spheres, we can transfer to the seven relative higher letters the same arrangement and nomenclature of the systems.

Chart 34 Numbers of musical intervals



[IX]

The difference of the seven letters corresponds to the variety of the seven voices. The arrangement of the planets, from which the properties of the modes in the seven-stringed lyre derive. The diversity of the seven strings corresponds to the eight types of modes which we call tones.

It is to be underlined that the centre of the world is on the Earth. Seven planets, the eighth sphere and the upper heaven represent 11 strings, with 10 intervals, in which we distinguish eight kinds of modulations through ascending and descending tones and diesis; as appears here below, and as Cicero mentioned in his book "Scipio's dream".[25]

Thus the modes are placed in accordance with the order of the planets in the sky, but they choose the number with which to refer to them in accordance with the property of their succession in the week. Thus it is necessary that the eighth mode should be attributed to the eighth heaven, which follows Saturn: or that is should coincide with the Sun, whose property takes the place of that of Saturn. Likewise, the Earth is placed ninth, motionless, in the *proslambanomenos* string, as Tullius [Cicero] says. Above the eighth is placed the first mobile. Under the Earth, the centre, so that there are 11 strings and ten intervals.

[Part III]

[X]

Mese, because it comes from the mean of the seven ancient strings, or because it is placed on the middle finger and is the middle of the two tetrachords.

Lichanos, because it is placed on the following index finger.

[25] At the end of his *Republic*; Cicero 1992.

					supremum coelum	
$3\frac{5}{9}$ ...	♮					
	♭	*tonus*				
4	*a*		*	*Anthypate*		
		tonus				
$4\frac{1}{2}$ ...	7^{us} *g*		*Sat.*	*Nete*	*Inferior in tritono*	*Mixolydius*
		tonus				
$5\frac{1}{16}$ ...	5^{us} *f*		*Jup.*	*Paranete*	*Supra diesim altam*	*Lydius*
		diesis				
$5\frac{1}{3}$ ...	3^{us} *e*		*Mar.*	*Paramese*	*Infra diesim altam*	*Phrygius*
		tonus				
[6] *modi*	8^{us} p^{us} *D*		*Sol*	*Mese*	*Media in ditono*	*Dorius Hypomixolydius*
		tonus				
$6\frac{3}{4}$ ...	6^{us} *c*		*Ven.*	*Lichanos*	*Supra diesim imam*	*Hypolydius*
		diesis				
$7\frac{1}{9}$ ...	4^{us} ♮		*Mer.*	*Parhypate*	*Infra diesim imam*	*Hypophrygius*
	♭	*tonus*				
8 ...	2^{us}*a*		*Luna*	*Hypate*	*Superior in tritono*	*Hypodorius*
		tonus				
9	*g*		•	*Proslambanomenos*		
		tonus				
$10\frac{1}{8}$	*f*					
					centrum	

Chart 35 Correlation of ratios, planets, tones, and modes

			*				
g . . .	7		*Sat.*	$4\frac{1}{2}$		*Neate vel Neta*	*Acutisona*
					tonus		
f . . .	5		*Jup.*	$5\frac{1}{16}$		*Paranete*	*Iuxta acutam*
					diesis		
e . . .	3		*Mar.*	$5\frac{1}{3}$		*Paramese vel Trete*	*Iuxta mediam*
					tonus		
d . .	8 1	*loci tonorum*	*Sol*	6		*Mese*	*Media*
					tonus		
c . .	6		*Ven.*	$6\frac{3}{4}$		*Lichanos*	*Index*
					diesis		
♭ . .	4		*Mer.*	$7\frac{1}{9}$		*Parhypate*	*Iuxta principalem*
					tonus		
a . .	2		*Luna*	8		*Hypate*	*Principalis*
					tonus		
γ . . .			•	9		*Proslambanomenos*	*Adiuncta vel Coassumpta*

Chart 36 Correlation of planets and tones with their Greek and Latin names

The etymology of the following strings is self-evident.

Mercury's lyre had been a tetrachord. Orpheus transformed it into a heptachord. Pythagoras added the eighth *proslambanomenos* string, and completed the octave.

3^a minor		3^a minor aucta					
a	d	a	d	dimi.			
	d		a	non potest			
				esse			
3^a maior		3^a maior dimi.					
a	d	a	d	aucta			
a	d	d	d	non potest			
				esse			
4^a		4^a aucta					
a	d	a	a	diminuta			
a	d	a	d	non potest			
	d		d	esse			
5^a		5^a diminuta			5^a minor		
a	d	a	d		a	d	
a	d	a	d	aucta	a	d	5^a
a	d	d	d	non potest		d	minor
	d		d	esse		d	
6^a minor		6^a minor aucta					
a	d	a	d	6^a minor			
a	d	a	d	diminuta			
a	d	a	d	non potest			
	d		d	esse			
	d		a				
6^a maior		6^a maior diminuta					
a	d	a	d				
a	d	a	d	aucta			
a	d	a	d	non potest			
a	d	d	d	esse			
	d		d				

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~~a~~

Chart 37 Decomposition of intervals in apotome and diesis

[XI]

It is narrated that the Muses added the middle string to Mercury's lyre (with which Apollo had a certain familiarity), Linus added the *lichanos*, Orpheus the *hypate*, Thamyris the *parhypate*, as Diodorus wrote. In fact Mercury's testudo had been a tetrachord, Pythagoras added the eighth string to complete the octave.

[XII][26]

[XIII]

1. Sound derives from the movement of the body, which makes the air vibrate: with a collision, with a blow, with a fracture.
2. The denser body vibrates more quickly, as the bronze string more than the [ox] sinew, and more than relaxing the tension.
3. The smaller body vibrates more quickly, like the thinnest or shortest sinew and the smallest reed or tibia, due to the faster movement of the air.
4. The faster vibration makes the sound more acute, and consequently it is necessary that:
   Corollary, both the denser body, like the bronze string compared with the sinew, and the smaller one, like the thinnest or shortest sinew or and the smallest reed-pipe, sound more acute: thus the bronze pipe, more than the lead one, and the smallest tube sound more acute.
5. And thus, when the qualities of voices are distinguished into acute and deep, these derive also from the qualities and the quantities of the bodies that make the air vibrate.
6. Furthermore, if the densities of bodies stand in a direct proportion[27] to their magnitudes, it happens that the sounds are produced in unison.
7. Furthermore, understand all this: making comparisons between bodies. The diversity of form diversifies the sound (although the bodies may be of the same material and quantity)
   Hence it is clear that:
8. The quality of the sound is distinguished by the quality of the material, by the magnitude of the body and by the form of the instrument.
9. The air is made to vibrate by the string and vice versa, the string by the air, which vibrates with the same tenor; the string vibrates with the same tenor as the air. Thus it happens that the string of the lyre that is not touched vibrates in accordance with the tenor of unison.
10. Unison is the beginning of consonances, as the unit is the beginning of numbers, equality for ratios, the base for steps. Likewise, unison is the most perfect of consonances, thanks to the correspondence of the same number of strokes.
11. Consonances consist of commensurable ratios: actually it is impossible to let incommensurable sounds agree together. As it is impossible to make the vibrations of incommensurable speeds correspond, seeing that harmony, like consonance, derives from the correspondence of strokes.

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[26]In Maurolico 201? there is a small figure with the temporal values of the notes.

[27]Forgetting his previous Proposition 2, Maurolico made a mistake, writing, on the contrary, "a reciprocal proportion".

Chart 38 Musical ratios

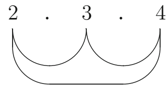
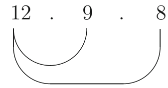


Chart 39 Musical ratios



12. Particular numbers generate the finest consonances. Hence, after unison, which rests on a monadic basis, the double ratio, which is indicated by the particular unit of numbers and its double, produces an excellent consonance. Those that derive from this produce perfect consonances, in view of the correspondence of strokes.
13. After these, indicated by the numbers two and three, the sesquialtera ratio produces the diapente, which does not possess the same perfection, because to make [the strokes] correspond, the whole is divided, requiring a unit of the slower and a unit and a half of the faster.[28]
14. Thus the sesquithird, which consists of the numbers 3 and 4, produces the diatessaron.
    Hence it is clear:
    Corollary, that multiplicity produces a more perfect consonance than superparticularity[29] and that these particular numbers are better than those that follow. Because where the correspondence of strokes is clearer, there a sweeter consonance arises.
15. The diapente and the diatessaron, placed one after the other, together form the octave, in this way:
16. Their difference is indeed the tone. Thus:
    Corollary. The octave will be composed of five tones and two diesis.
17. Taking away the tone from the diatessaron once and then again, what remains is the diesis, i.e. less than half of the tone
18. Thus taking away the tone also from the diapente three times, what remains is the same diesis.
    Hence it is necessary:
    Corollary, that natural melody proceeds, not through the intricacies of ratios, that is to say, not through unknown ratios, but through intervals derived from the main numbers. Thus, rising by a tone, a tone and a diesis, we find the diatessaron, and after another tone we establish the diapente. Going on, through a tone, a diesis and a tone, we reach the interval of the octave. This order

[28]In order to understand the subject better, the reader should do what Euler did: he represented every ‘stroke’ with a dot in a drawing. In unison, all the dots correspond to one another; in the octave, one every two; in the fifth, one every three. See below, Part II, Sect. 11.1, Fig. 11.2.

[29]Multiplicity: ratio between numbers of the type n:1, like the octave 2:1. Superparticularity: ratio of the type n + 1:n, like the fifth 3:2 and the fourth 4:3.

of melodies will thus be called legitimate, and established by nature, as is postulated and laid down by the relationship, called diatonic, which proceeds by natural tones and semitones. Repeating it several times ad infinitum, this process thus admits double tones and single diesis, and as a result, tripling the tone with its repetitions, with the eighth string it will also generate the octave, inserting the remaining intervals.

19. However, admitting three tones, one immediately after the other, though necessary in order to complete the octave interval everywhere, became difficult for singers. Therefore, in order to mitigate this hardness, artists divided the third of these tones into semitones. And thus, taking away the diesis from the third tone, that is to say, the legitimate semitone (which is less than half a tone), we are left with the apotome (which is more than half a tone), the greater semitone. In this way, three semitones, one after the other, are tolerated.
20. Knowing these things, that natural melody proceeds through double tones and single diesis, and so does tritonic melody through tritones, and the chromatic melody through[30] semitones more sweetly. These are the three genres of melodies.
21. The hexachord contains the simple consonances, i.e. unison, the ditone, the diatessaron, the diapente and the hexachord. Or in other words, unison, the third, the fourth, the fifth and the sixth. Thus it is a connected relationship of six syllables.
22. Indeed, adding the octave to these single consonances, we generate the composed second-order consonances, which are of the same quality, that is to say, the octave, the 10th, the 11th, the 12th and the 13th.
23. Again, continuing to add the octave to these single consonances, we form the third-order consonances, that is to say, the double octave, the 17th, the 18th, the 19th, and the 20th, so called from the number of strings. These are all included in the same icosichord of the most ingenious Guido, who distinguishes them by means of the joints of the left hand.
24. Following the same process, and continuing with the same octave, we prepare the consonances of the fourth, the fifth and the following orders, as may continue ad infinitum in the larger instruments.
25. But now, seven hexachords complete the whole icosichord of Guido, each expressed by means of six syllables, i.e. *ut, re, mi, fa, sol, la*, with seven letters that are repeated (so that the same letter, repeated in every eighth place, will indicate the octave). Furthermore, as the first, the fourth and the seventh of the hexachords admit the tritone, the name of square and hard ♮ is chosen in view of this hardness. On the contrary, as the second and the fifth proceed through two tones and a single diesis, they are spontaneously called diatonic. And the third and the sixth, in which the tritone is divided into diesis and apotome in order to mitigate the hardness, take the name of soft, from the round ♭. This division

[30]As a few sheets are missing from the manuscript, we have integrated this part with the "Musicae traditiones" that derive from it; Maurolico 1575.

may occur not only here, but also in the single tones, as the most expert singers do, and as appears in instruments.

26. The square ♮ hexachord, which is hard and voiced, starts rightfully from the voiced letter *g*. Indeed the diatonic, which is intermediate and natural, rightly starts from the letter *c*, which is intermediate between the voiced and the aspirated. Clearly, the chromatic, which is soft and round ♭, starts from *f*, which sounds like the aspirated Φ, soft and lingering.
27. Likewise, the same letter *B* assumes the *fa* of the soft hexachord, and also the *mi* of the hard hexachord; in order that this passage may be avoided by singers, there is the apotome interval. However, the letter changes its sign with *fa* round ♭, to denote the chromatic ease, and with square *mi* ♮, to indicate the hardness of the tritonic genre. In this way, with the variety of signs, it indicates the diversity.
28. The excess of the apotome over the diesis is called the comma. That is to say, fragment.
29. The diesis exceeds three and a half commas, but is less than four.
30. The apotome is more than four commas, but undoubtedly less than five.
    Thus it is clear that:
    Corollary, the tone is more than eight commas, and less than nine. This is obtained from the calculation of Boethius.[31]
31. No attention therefore should be given either to Aristoxenus, who divided the tone into equal parts, or to Philolaus, who divided it in another manner.
32. Tones establish the modes of singing in accordance with the customs of nations, i.e. Dorian, Hypodorian, Phrygian, Hypophrygian, Lydian, Hypolydian, Mixolydian and Hypomixolydian. Their properties and positions are illustrated below.

[XIV]

Order of the compendium.[32]

[Part IV]

[XV]

From the calculation of Boethius.

The lesser semitone is more than three and a half commas, and as a result, the greater one is more than four and a half, consequently the tone exceeds eight commas, but it is less than nine.

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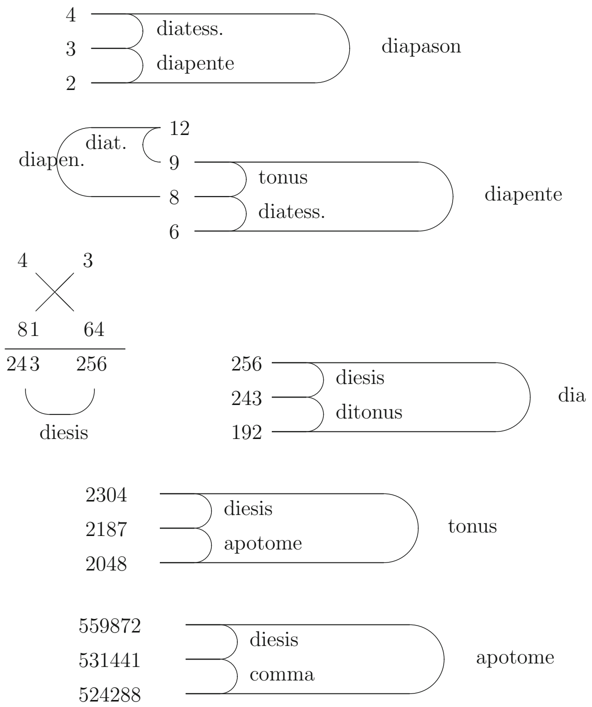
[31] See above, Chaps. IV and V.

[32] See note 6 above; Maurolico 201?.

Chart 40 Ratios of musical intervals



[XVI]

The Dorian drives away sleepiness: pretty and playful.
The Hypodorian induces sleep: moving, unpretentious.
The Phrygian incites and exasperates: severe, irascible.
The Hypophrygian soothes and exalts: conciliatory and persuasive.
The Lydian praises and consoles: cheerful and insolent.
The Hypolydian sympathises and gladdens: compassionate and tearful.
The Mixolydian is similar to the third, the fourth and the fifth: varied tuneful and impudent.
The Hypomixolydian extols and invokes heavenly things, and despises [those of the earth].

Through the octave, the authentic modes rise and fall; the plagal modes, from the place of their authentic modes, rise by a diapente, fall by an octave, and rising up by a diatessaron, return to their original position.

	firmamentum	∗ .	4 .	*a . la*	8 *modi canendi*	*A anthypate*
				tonus		
diapason 2 . 1 / *diate. 4 . 3*	*Sat.* .		$4\frac{1}{2}$	*g . sol*	7^{us} *mixolydius*	*g nete*
				tonus		
	Jup. .		$5\frac{1}{16}$	*f . fa*	5^{us} *lydius*	*f paranete*
				semit.		
	Mar. .		$5\frac{1}{3}$	*e . la*	3^{us} *phrygius*	*e paramese*
				tonus		
	Sol .		6 .	*d . sol*	*primus dorius* 8^{us} *hypomixolydius*	*d Mese*
diapente 3 . 2				*tonus*		
	Ven. .		$6\frac{3}{4}$	*c . fa*	6^{us} *hypolydius*	*c lichanos*
				semit.		
	Mer. .		$7\frac{1}{9}$	♮ . *mi*	4^{us} *hypophrygius*	♮ *parhypate*
				tonus		
	Luna .		8 .	*a . re*	2^{us} *hypodorius*	*A hypate*
				tonus		
	• .		9 .	Γ . *ut*	. *Centrum*	Γ *proslambanomenos*

Chart 41 Octochord of music

[XVII][33]

[Part V]

[XVIII]

C.7 MUSIC[34]

The air is struck by the blows of bodies; this is where sound and voice are born: as in the case of a stone that strikes water. The larger bodies vibrate more slowly, and make a sound deeper. The smaller ones vibrate more quickly and sound more acute. Consequently, the thicker and longer sinew and the larger tibia make a deeper sound. The thinner or shorter ones make a more acute sound. Similarly, the sparser material

[33]In the manuscript, the chapter contains only a few names of the notes, placed in scales, as in Guido's Icosichord; Maurolico 201?.

[34]In this part of the manuscript, Maurolico had started to arrange his ideas on music in order, and in a fair copy, even giving them a fine title in capital letters. But he was already very old, and death prevented him from completing the work that he had begun.

makes a deeper sound, and the denser a more acute one. For this reason, lead makes a deeper sound, bronze a more acute one, and iron more acute than bronze. The more relaxed string sounds deeper, the tighter one more acute. Furthermore, the form of the resounding body changes the sound. As is clear in the opened or closed mouthpiece of the pipe, or of a bell.

Thus the ratio of sounds in depth and acuteness follows the ratio of the magnitudes, the densities and the forms of the resounding bodies. Whereas in the same way as the string that had been made to vibrate, or the collision of the air, makes the air vibrate: so, reciprocally, the air that vibrates causes vibrations in the string, or at the sides of the reed-pipes: and thus the sound is propagated by the reciprocal collision.

Here it happens that the air caused to vibrate by the plucked string of the nearby lyre, in turn provokes vibrations in the vicinity, at least in the same string of a lyre that has not been touched. There is still other concerning the generation and the ratios of sounds and natural melodies.

Art then harmonises the strings, the pipes and the instruments with the measurements of music and of the consonances. These measurements follow, and imitate, particular numbers: in which commensurability is manifest. For incommensurability cannot produce consonance. Here, the consonance of sounds and melodies is the sweet concord of hearing. Likewise, the proceeding of sounds produces sweetness, as in melodies in unison, or in the alternate correspondence of strokes with the remaining consonances. For I do not agree with B[oethius], who says that consonance is made of different melodies. As equality is the first of the ratios, so unison is the first of the consonances. Thus, the first ratios of musical melodies are taken from the first few numbers. That is to say, from the monad, the dyad, the triad and the tetrad. Thus the double ratio produces the diapason [octave] consonance, the sesquialtera produces the diapente [fifth], the sesquithird produces the diatessaron [fourth], which are two that, joined together, form the octave. Furthermore, the excess of the diapente over the diatessaron is the tone. Whereas the excess of the diatessaron over the ditone is the diesis, that is to say, the lesser semitone. Subtracting this from the tone, what remains is the apotome, the greater semitone. Lastly, the difference between the apotome and the diesis is called the comma.

Furthermore, on the basis of the calculation of Boethius, the diesis exceeds three and a half commas, which means that the apotome will be more than four and a half commas. And likewise, the tone will exceed eight commas, without reaching nine.

Here the rise and fall of harmonies are described and put in order. At the end, the human voice, rising like an instrument from the deeper level of two tones, is led to the ditone; from there, through the diesis, to the diatessaron; one tone further on, to the diapente and up another tone to the hexachord. Lastly, rising by another tone and a diesis, it reaches the octave. And so on *ad infinitum*, through these systems and consonances.

The singer, then, is allowed to divide the single tones into diesis and apotome. Because the chromatic genre of singing, seeing that it is more modulated, is more pleasant than the diatonic. Such a division is usually admitted in the third tone of

a . 4	diat.	d . 9	tonus	g . 256	k . 2304	n . 559872
b . 3	diap.	e . 8	diat.	h . 243	l . 2187	o . 531441
c . 2		f . 6		i . 192	m . 2048	p . 524288

ab . diatessaron	gh . diesis	no . diesis
bc . diapente	hi . ditonus	op . comma
ca . diapason	ig . diatessaron	pn . apotome

de . tonus	kl . diesis
ef . diatessaron	lm . apotome
fd . diapente	mk . tonus

Chart 42 Symbols of musical intervals

the tritone; where it proves to be more necessary, in order to temper the hardness of the tritone.

And, not without reason, some consider the hexachord to be composed of six degrees *ut, re, mi, fa, sol, la* interwoven here with 13 strings, and then with 20, to complete the same number of joints the left hand.[35] For the hexachord [contains] all the simple consonances; 13 strings contain those compounded with the first, 20 those compounded with the second. This compounding takes place by adding the octave.

The number seven of letters is chosen: so that the same letter, repeated also in the eighth place, may indicate the consonance of octave. The hexachord of the above-mentioned notes is interwoven seven times. And for the one among them that begins with the voiced letter *g*, the name of square ♮ is chosen, in view of the sonority generated by the tritone produced in the ♮ *mi*. Instead, the one that begins with the aspirated letter *f* is named after the round, soft ♭: due to the division that it produces in the third tone of the tritone, through the insertion of the letter and of the note ♭ *fa*. Lastly, the hexachord that begins with the delicate, temperate letter *c*, which stands in the middle between the two previous hexachords of the square ♮ and the round ♭, as it is moderate and diatonic, takes the name of natural. This regards the reason for the letters in the icosichord, and the properties of hexachords.

Indeed, different nations choose different way of playing; in order to calm or excite the spirits, and because it is convenient to pay attention to customs. While there are eight modes distributed over the single notes and letters of the octave: and they are attributed to the planets, whose nature is imitated. And arranged in this way, as is indicated in the margin.

Of these, the first, the third, the fifth and the seventh are said to be authentic, the second, the fourth, the sixth and the eighth, plagal, which means that they are a diatessaron interval lower than their relative authentic equivalents. To these

[35] See above the icosichord of Guido D'Arezzo in Chap. VIII, together with Charts 4 and 32.

		anthyp.	∗ .	4 .	*A* .	——	*firmamentum* ——
						tonus	
diapason	diate.	*nete*	*Sat.* .	$4\frac{1}{2}$	*g* .	——	7^{us} *mixolydius* ——
						tonus	
		paranete	*Jup.* .	$5\frac{1}{16}$	*f* .	——	5^{us} *lydius* ——
						semit.	
		paramese	*Mar.* .	$5\frac{1}{3}$	*e* .	——	3^{us} *phrygius* ——
						tonus	
	diapente	*mese*	*Sol* .	6 .	*d* .	——	*primus dorius* 8^{us} *hypomix.*
						tonus	
		lichanos	*Ven.* .	$6\frac{3}{4}$	*c* .	——	6^{us} *hypolyd.* ——
						semit.	
		parhyp.	*Mer.* .	$7\frac{1}{9}$	♮ .	——	4^{us} *hypophr.* ——
						tonus	
		hypate	*Luna* .	8 .	*A*	——	2^{us} *hypodo.* ——
						tonus	
		proslambanomenos	• .	9 .	Γ	——	*Centrum* ——

Chart 43 Correlation of tones, planets, and modes

correspond the properties of the planets, in the order in which the weeks are divided into the various days.

While the authentic modes rise and fall in the octave, the plagal mode, on the contrary, rises by a diapente from its authentic equivalent, and falls by an octave, rising again by a diatessaron to the above-said definite place.

The Dorian mode is pretty and playful, it drives away sleepiness. The Hypodorian, moving and unpretentious, induces sleep. The Phrygian, severe and irascible, incites and exasperates. The Hypophrygian, more tender, exhorts more, attracts and renders unrestrained. The Lydian, cheerful and insolent, praises and consoles. The Hypolydian, compassionate and tearful, gladdens and sympathises. The Mixolydian, varied, tuneful and impudent, has the qualities of the third, the fourth and the fifth. The Hypomixolydian exalts, invokes, takes an interest in heaven, despising the [earthly] things. These first eight strings are also called *Proslambanomenos, Hypate, Parhypate, Lichanos, Mese, Paramese, Paranete, Nete.* Their etymology is manifest easily.[36]

And it is clear that *mese* is the proportional mean both between *hypate* and *nete*, and between *proslambanomenos* and the [string] following *nete*. That is to say, the Sun is the middle term in the proportion both between the Moon and Saturn, and between the centre [the Earth] and the firmament. Because, that is, we move away one diatessaron on both sides, but here one diapente on both sides. Thus $9, 8, 6, 4\frac{1}{2}, 4$, as was clear in the enneachord described above. Or better still, the string *mese* produced is the proportional mean between all the couples that are equally distant from it, seeing that the same proportion is maintained in equal

[36]The Greek terms mean, in the order: added, main, next to the main, index, middle finger, next to the middle finger, next to the acute, acute. See Chart 36.

Chart 44 Correlation between planets and modes

—	$*$	4 .	*a* .	*firmamentum*
T.				
—	*Sat.*	$4\frac{1}{2}$	*g* .	7^{us} *mixolydius*
T.				
—	*Jup.*	$5\frac{1}{16}$	*f* .	5^{us} *lydius*
S.				
—	*Mar.*	$5\frac{1}{3}$	*e* .	3^{us} *phrygius*
T.				
—	*Sol*	6 .	*d* .	*primus dorius hypomix.*
T.				
—	*Ven.*	$6\frac{3}{4}$	*c* .	6^{us} *hypolyd.*
S.				
—	*Mer.*	$7\frac{1}{9}$	♭ .	4^{us} *hypophr.*
T.				
—	*Luna*	8 .	*a* .	2^{us} *hypodo.*
T.				
—	•	9 .	Γ .	*Centrum* .

3 . 206
diatess.
4 . 200
diapente
6 . •
diapason

systems. That is to say, between *parhypate* and *paranete*, moving away on both sides by one tone and one diesis, as between *lichanos* and *paramese*, at the distance of one tone on both sides. The value of this middle string rests on this: it is the seat both of the first mode and of the Sun, the prince of the planets.

Appendix D
The Chinese Characters

道生一, 一生二, 二生三，三生萬物
Dao sheng yi yi sheng er er sheng san san sheng wan wu.
[“The tao generates one, one generates two, two generates three, three generates ten thousand things.”]
萬物負陰而抱陽。沖氣以為和。
Wan wu fu Yin er bao Yang. Chong qi yi wei he.
[“The ten thousand things bring the Yin and embrace the Yang. Thanks to the qi, they then become harmony.”]
道德經
Daodejing
yuejing [classic for music] 樂經
yinyue [music] 音樂
jing [classic] 經
shujing [classic for books] 書經
shijing [classic for odes] 詩經
yijing [classic for changes] 易經
liji [memories of rites] 禮記
chunqiu [Springs autumns annals] 春秋
zhong [bell] 鐘
ju [set square] 矩
taiyang [sun] 太陽
yue [moon] 月
lunyu [dialogues, Analects] 論語
liuyi [six arts] 六藝
he [harmony] 和
heping [peace] 和平
se [large lute] 瑟
zhan [war] 戰
wanshi [ten thousand generations] 萬世
shiji [memoires of a historian] 史記

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hanshu [books of the Han] 漢書
yuzhou [universe] 宇宙
wannian [ten thousand years, for ever] 萬年
QIAN HANSHU [Books of the early Han] 前漢書
Lulizhi [Annals of pipes and the calendar] 律歷志
lulu 律吕
Yi yue: can tianliangdi er yi shu
易曰:参天兩地而倚數
["Yi[jing] says: 'refer both to the sky and the earth, and trust yourself to numbers"']
Tian zhishu shi yuyi, zhongyu ershiyouwu. qiyi jizhi yisan, gu zhiyi
天之數始於一,終於二十有五。其義紀之以三,故置一
desan, you ershiwufen zhiliu, fan ershiwu zhi, zhongtian zhishu,
得三,又二十五分之六,凡二十五置,終天之數,
deba shiyi, yi tiandi wu weizhi he zhong yushizhe chengzhi, wei
得八十一,以天地五位之合終於十者乘之,為
babai yishi fen, ying litong, qianwubai sanshijiu suizhi zhangshu.
八百一十分,應曆統,千五百三十九歲之章數
["The numbers of the sky start from 1, and all together they add up to 25.
Take 1 in order to obtain 3; write down 25, the number of the whole sky,
for each of the three, and add six more, obtaining 81. Take the number 5 of the sky and the earth, and add it, arriving at 10, and multiply the above result by 10. This makes 810 fen. The calendar is based on tong, the number of zhang in 1539 years."]
shijiusui wei yizhang, yitong fan bashiyi zhang.
十九歲為一章,一統凡八十一章。
["19 years make one zhang, every 81 zhang one tong is completed."]
Huangzhong zhi shiye. . . . . cizhiyi, qi shier lu[lu] zhi zhoujing.
黃鐘之實也。...此之義,起十二律〔呂〕之周徑。
["It [810] is also the solid of the Huangzhong. This means
constructing the diameter for the circumference of the 12 pipes."]
huangzhong [yellow bell] 黃鐘
Lu kongjing sanfen, can tianzhishu ye; wei jiufen, zhongtian zhishuye.
律孔徑三分,參天之數也;圍九分,終天之數也。
["3 fen is the diameter for the opening of the pipe, and
it also refers to the number of the sky; the circumference is
9 fen, also the number of the whole sky."]
Dizhishu shi yuer, zhongyu sanshi. Qiyi jizhi yiliang, guzhi yideer,
地之數始於二,終於三十。其義紀之以兩,故置一得二
fan sanshizhi zhongdi zhishu, de liushi, yidi zhongshu liu chengzhi,
凡三十置終地之數,得六十,以地中數六乘之
wei sanbai liushifen dangqi zhiri, linzhong zhishi.
為三百六十分當期之日,林鐘之實。
["The numbers of the earth start from 2, and all together they add up to 30.

Write down 30, the overall number of the earth, for every two, obtaining 60.
Multiply this by 6, which is the intermediate number of the earth: the result is
360 fen. This number is equal to the number of days in the periods.
This is the solid of the Linzhong."]

...Linzhong chang liucun, wei liufen. Yi wei cheng chang, deji sanbailiu shifenye.

林鐘長六寸,圍六分。以圍乘長得積三百六十分也。

["The length of the Linzhong is 6 cun, the circumference
6. Multiply the length by the circumference, obtaining 360 fen."]

Qiyin ji. wei shier yue wei yiqiye.

期音基。謂十二月為一期也。

["Periods and sounds are the bases. This means that 12 months make up the periods."]

Renzheji tian shundi, xuqi chengwu.

人者繼天順地,序氣成物。

["That [the number] of man follows the [numbers] of the sky
and of the earth, in that order; the qi realises their matter"].

tong bagua, diao bafeng, li bazheng, zheng bajie, xie bayin,
統八卦,調八風,理八政,正八節,諧八音,
wu bayou, jian bafang, bei bahuang, yizhong tiandizhi gong, gu baba
舞八侑,監八方,被八荒,以終天地之功,故八八
liushisi. qiyiji tiandizhi bian, yi tiandi wuwei zhihe zhongyu shizhe
六十四。其義極天地之變,以天地五位之合終於十者
chengzhi, wei liubai sishifen, yiying liushisi gua, dazu zhishiye.
乘之,為六百四十分, 以應六十四卦,大族之實也。

["It connects the 8 gua [trigrams], moves the 8 winds, manages the 8
transactions, adjusts the 8 knots, harmonises the 8 sounds, causes the
8 stimuli to dance, observes the 8 directions, satisfies the 8
shortages, and takes care of all the services in the sky and on
earth, therefore 8 times 8 equals 64. This means changing the sky
and the earth completely. Add 5 for the sky and 5 for the earth,
and multiply the total by 10; that makes 640 fen, because
it deals with the 64 gua [hexagrams]. This is also the solid
of the Dazu [big group]."]

Dazu chang bacun, wei bafen, weiji liubai sishifenye.

大族長八寸，圍八分為積六百四十分也。

["The length of the Dazu is 8 cun, the circumference 8 fen; their product is 640 fen."]

Shu[jing] yue: tiangong renqi daizhi.

書[經] 曰:天功人其代之

["The Classic for books says: in the service of the sky, man takes his place."]

...yansheng renbing tianzao huazhi gongdai er xingzhi.

言聖人稟天造化之功代而行之。

["......the sage says that man receives the sky as a gift, with
the task of producing transformations; let him act on its behalf."]

Tian jiandi, ren zetian, guyi wuwei zhihe chengyan,
天兼地,人則天,故以五位之合乘焉,
‘weitian weida, weiYao zezhi’ zhixiangye.
唯天為大,唯堯則之之象也。
[“The sky in accord with the earth; let man imitate the sky.
In this, therefore, let him add 5 and multiply. Again, let him
take as his model ‘Only the sky becomes great’ and ‘Follow only Yao.”]
ze, faye. ‘lunyu’ cheng Kongzi yue: ‘dazai Yao zhiwei junye, weitian
則,法也。論語稱孔子曰:大哉堯之為君也,唯天
weida, weiYao zezhi’; mei diYao neng fatian er xinghua.
為大,唯堯則之;美帝堯能法天而行化。
[“ ‘Follow’, also take as your model. In the Lunyu [Dialogues],
Kongzi [Confucius] gives advice, and says: ‘Also the jun
[gentleman] acts like the true great Yao and follows the one and only
Yao of ‘Only the sky becomes great’. The good emperor Yao was capable
of modelling himself on the sky, and effecting transformations’.”]
Diyi zhongshu chengzhe, yindao libing, ….. santong xiangtong, gu
地以中數乘者,陰道理丙...三統相通,故
Huangzhong, Linzhong, Taizu luchang jie quancun erwang yufenye.
黃鐘,林鐘,太族律長皆全寸而亡餘分也。
[“Multiply that central number of the earth [6], the third of the
Yin principles, … link all three with one another; thus the Huangzhong,
Linzhong and Tai[Da]zu pipes each have the length of
a whole number of cun [9, 6, 8] and lose the remaining fen.”]
qing [chime-stones] 磬
zhuo [turbid] 濁
qing [clear] 清
hundun [chaos] 渾沌
五音相生
wu yin xiang sheng
[“Five notes generated from one another.”]
gong [palace] 宮
zhi 徵
shang 商
yu [feather] 羽
jiao [horn] 角
sunyi [decrease increase] 損益
xing [phases] 行
土，火，金，水，木
tu huo jin shui mu
earth, fire, metal, water, wood
律呂相生
lulu xiang sheng
[“tuned pipes, generated from one another”.]
huangzhong [yellow bell] 黃鐘,

dacu [big cluster] 大簇,
guxi [purification of women] 姑洗,
ruibin 蕤賓,
yize [sure rule] 夷則,
wushe [without discharge] 無射,
dalu [big pipe] 大呂,
jiazhong [compressed bell] 夾鐘,
zhonglu [central pipe] 仲呂,
linzhong [bell of the woods] 林鐘,
nanlu [Southern pipe] 南呂,
yingzhong [bell that answers] 應鐘。
hou [climatic season] 候
jieqi [solar terms] 節氣:
dongzhi [winter solstice] 冬至, dashu [great heat] 大暑
dizhi [earthly branches] 地支: zi [son] 子zishi [time of the son] 子時
乃三分益一之法此又不可曉者。
nai sanfen yiyi zhi faci you buke xiao zhe.
抑夏至一陰始生之故歟
yi xia zhi yiyin shisheng zhi guyu.
["It is impossible to explain this rule, which again increases
the 3 parts by 1. Or rather, may the reason
be perhaps that at the summer solstice, the Yin starts to
increase?"]
gongwei [circumference] 空圍
qi 氣
律以統氣類物
Lu yi tong qi lei wu
["The lu use an interconnected system whose substance
is similar to that of the qi"].
周髀算經
Zhoubi suanjing
[Classic for calculating the gnomon of the Zhou]
xiantu [figure of the string] 弦圖
ziran [naturally] 自然
qiqiaoban [tangram] 七巧板
ju [set-square rectangle] 矩
jing [straight line] 徑
Fuxi 伏犧
令出入相補
lingchu ru xiangbu
["[One] is made to go out, and [the other] to come in; they
compensate for each other."]
九章算術
Jiuzhang suanshu

[The art of calculating in nine chapters]
fangcheng [measuring according to the square] 方程
li [texture explanation] 理
・・・解體用圖・・・
. . . jieti yongtu . . .
[“. . . disassembling the bodies, using figures . . . ”].
斯膠柱調瑟之類
. . . sijiao zhudiao se zhilei.
[“. . . similar in that case to tuning the se [the ancient lute] with the pegs glued up.”]
wuwei [not doing anything] 無為
he [harmony] 和
wansui [cheers!] 萬歲
Zhonguo [country in the center, China] 中國
Yijing [book of changes] 易經
Lulu chengshu [complete book of the lulu] 律呂成書
dayan [big development] 大衍
kaifangchu [divide by opening the square] 開方除
bujin [no termination] 不盡
lu [ratio] 率
daodejing [classic of the way and of virtue] 道德經
不得一為無，故自古無未有之時而常存也。
Bude yi weiwu, gu zigu wuwei you zhi shi er chang cun ye
[“It is not possible for the same [something] to become nothing, and so from ancient times, there has never been an [initial] moment of existence, but in fact it exists continually”].
li [half a kilometre, a third of a mile] 里
九數出於句股，句股出於河圖，故河圖為數之源。
jiushu chuyu gougu, gougu chuyu Hetu, gu Hetu wei shu zhi yuan
[“The nine numbers come from the gougu [right triangle], the gougu comes from the Hetu [Figure of the Yellow River], thus the Hetu is the source of numbers.”]
論其理設為幾何之形而明所以立算之故
lun qili shewei Jihe zhi xing er ming suoyi li suan zhi gu
[“To discuss its li [reasons], to establish [what] serves as the xing [form] for the Jihe [how much it is, geometry], to make clear and thus establish the reasons for calculating”].
使理與數協
shi li yu shu xie.
[“to apply the li [reasons] with the shu [numbers] in a harmonious manner”].
論其理設為幾何之形而明所以立算之故。
lun qili shewei Jihe zhi xing er ming suoyi li suan zhi gu
[“To discuss its li [reasons], to establish [what] serves as the xing [form] for the Jihe [how much it is,

geometry], to make clear and thus establish the reasons for calculating"].
使理與數協
shi li yu shu xie.
["to apply the li [reasons] with the shu [numbers] in a harmonious manner"].
xin [heart-mind] 心
sheng [life] 生
天人同道
Tianren tongdao
["Heaven and Man, the same Tao."]
水凝為冰，氣凝為人
Shuining wei bing, qining wei ren
["As water turns into ice, so the qi crystallises to form the human body."]
崑崙山有水，水氣上蒸為霞
Kunlunshan youshui, shuiqi shang zheng weixia
["In the Kunlun mountains, there is water; the qi of the water rises, evaporates and becomes clouds."]
xiangcheng [ride on one another] 相乘
qizhen [is shaken] 氣振
xiangya [mutual grinding] 相軋
sheng [mouth-organ] 笙
jing [essence] 精
shen [demonic spirit] 神
buce [unpredictable] 不測
Yueji [Memoires of music] 樂記
知此則，樂者天地之和也
zhicize, yuezhe tiandi zhihe ye.
["Music thus [realises] the harmony of heaven and earth, relating this to us."]
律者所以通氣
luzhe suoyi tongqi
["Those lulu therefore canalise qi"].
ying [echos, causes a sympathetic reaction] 應
houqi [watching for the qi] 候氣
氣有潛通，數亦冥會，物之相感。
qi you qiantong, shuoyi minghui, wuzhi xianggan.
["The qi has an invisible penetratingness, rapidly effecting a mysterious contact, according to the mutual responses of material things."]
fengshui [wind-water] 風水
ge [song] 歌
qise [vapour-colour of qi] 氣色
qihaoran [great justice] 氣浩然

知不知上，不知知病。
Zhi buzhi shang buzhi zhi bing
[“Knowing that you don’t know is superior, not knowing that you know is a mistake.”]
zhengming [to rectify the names] 正名
you [there is, exists, to have] 有
wu [material things] 物
shi [right, yes, this] 是
wei [to do, to act, to become] 為
dongxi [east-west, thing] 東西
shanshui [mountains-waters, picturesque panoramas] 山水
yuzhou [space-time, universe] 宇宙
互文見義
huwen jianyi
[“the ones with the others the writings, [and] you will see [their] correctness.”]
shensuo [lengthen-shorten] 伸縮
fa [rules] 法
li [reasons] 理
taiji [the greatest limit] 太極
yin 陰yang 陽
xin [heart-mind] 心
zhen [real, the true] 真zhenli [real reasons] 真理
cuowu [the false] 錯誤
雖有文辭，斯亂道破義。
Suiyou wenci, si luandao poyi.
[“Although he writes in a classical style, [Zhang Heng] confuses the Dao [the way, the procedure] and damages what is right.”]
不可兩不可也。
buke liang buke ye.
[“It is not possible that both [are] impossible.”]
bian [to dispute] 辯
bian [to differentiate] 辨
有無相生
youwu xiangsheng
[“nothing and existence are born from one another.”]
wuwei [non-doing] 無為
ren [benevolence pietas] 仁
de [virtue] 德
道可道，非常道；名可名，非常名。
dao ke dao, fei chang dao; ming ke ming, fei chang ming.
無名天地之始；有名萬物之母。
wu ming tian di zhi shi; you ming wan wu zhi mu.
[“The way, the right way [is] not the unalterable way;

the description, the right description [is] not the unalterable description; the beginning of heaven and earth is undescribable; the mother of the 10,000 things is describable.”]
有定不能測無定
youding buneng ce wuding
[“. . . [it] remains fixed, it cannot measure that which is not fixed.”]
fajia [school or family of the rules, legalists] 法家
suanfa [rules for calculation] 算法
然世傅此法
ran shifu cifa
[“However, for generations, this rule has been taught,”]
方程法
fangcheng fa
[“. . . fangcheng rule . . . ”]
shu [art] 術
xinfa [new rules] 新法
wuli [reasons for material things, physics] 物理
lidongxi [putting things in order] 理東西
xiang [images, models] 像
qixiang [image of the qi, atmosphere] 氣像
yigeren [a person] 一個人
yizhimao [a cat] 一隻貓
yibenshu [a book] 一本書
yipima [a horse] 一匹馬
yibeishui [a glass of water] 一杯水
you [to have, there is] 有
wu [nothing] 無
qi 氣
dao 道
li [reason] 理
ziran [at ease, natural] 自然
zi [by itself] 自
lulu [tuned pipes] 律呂
qing [clear] 清
zhuo [turbid] 濁
dizhi [earthly branches] 地支
qihou [seasonal terms] 氣候
yuzhou [universe, space-time] 宇宙
cai [materials] 材
xing [phases] 行
聖人以無為待有德
Shengren yi wuwei dai youde
[“Following ‘non-doing’, the sage waits to have the virtue [capacity]”].
造化鍾神秀
Zaohua zhong shen xiu

陰陽，割，昏曉
YinYang ge hun xiao
[Magic and beautiful feels the nature with love
YinYang, she gathers, dusk and dawn.]
杜甫
Du Fu
算法統宗校釋
Suanfa tongzong jiaoshi
〔九七七〕五音相生圖
[jiuqiqi] Wuyin xiangsheng tu
三分損一者乃三分之二也
sanfen sun yizhenai sanfen zhierye
三分益一者乃二分之一也
sanfen yi yizhenai erfen zhiyiye
法曰黃鍾[1]之管長九寸以九寸自乘得八十一寸為宮
fayue huangzhong zhiguanchang jiucunyi jiucunzi chengde bashiyicun weigong
音。que[2]以八十一以二因之得一百六十二寸以三歸之得五十四寸所謂
yin. queyi bashiyi yier yinzhide yibailiushiercun yisan guizhide wushisicun-suowei
三分損一而生徵火。que[2]以五十四以四因之得二百一十六以三歸
sanfensunyi ershengzhihuo. queyi wushisi yisi yinzhide erbaiyishiliu yisangui
之得七十二寸所謂三分益一而生商金。que[2]以七十二以二因
zhide qishiercun suowei sanfen yiyi er shengshangjin. queyi qishier yieryin
三而一得四十八寸而生羽水。復以羽數四十八四因三而一得
saner yide sishibacun ershengyushui. fuyi yushu sishiba siyin saner yide
六十四而生角木。此乃五音相生之法多者為尊為濁
liushisi ersheng jiaomu. cinai wuyin xiangsheng zhifa duozhe weizun weizuo
少者為男為清也。
shaozhe weinan weiqingye.

[1] For zhong-bell, Cheng Dawei wrote the character 鍾, to which others gave a different meaning, while they painted for zhong-bell 鐘.

[2] In my software, the character of this ’que’ [yet, while] is missing; a similar is 卻.

〔九七八〕律呂相生圖
[jiuqiba] Luluxiangshengtu
律呂相生歌
luluxiangshengge
律呂相生戰者稀
lulu xiangsheng zhanzhexi
黃鍾九寸是根基
huangzhong jiucunshi genji
陽八生陰三損一
yang basheng yin sansunyi
陰律生陽益一奇

yinlu shengyang yiyiji
黃，林，大簇皆全寸
huang, lin, dacu jiequancun
餘者通之更不是
yuzhe tongzhi gengbushi
俱用九分乘見積
juyong jiufen chengjianji
四時氣候配攸宜
sishi qihou peiyouyi
黃鍾，大簇，姑洗，蕤賓，夷則，無射為陽；大呂，夾鍾，仲呂，
huangzhong, dacu, guxi, ruibin, yize, wushe wei yang; dalu, jiazhong, zhonglu,
林鍾，南呂，應鍾為陰。陽呂生陰二[3]分損一，陰呂生陽
linzhong, nanlu, yingzhong wei yin. yanglu shengyin erfensunyi, yinlu shengyang
三分益一，二因三除為損，四因三歸為益。律呂之中惟黃鍾
sanfen yiyi, eryin sanchu weisun, siyin sangui weiyi. lulu zhizhong wei huangzhong
林鍾，大簇之律皆得全寸。餘者皆有崎零不盡之，數以法通之。
linzhong, dacu zhilu jiede quancun. yuzhe jieyou qiling bujinzhi, shuyi fatongzhi.
[3] mistake for 三san・
〔九七九〕
[jiuqijiu]
黃鍾屬陽空圍九分，律長九寸以九分因之得積八百一十
Huangzhong shuyang kongwei jiufen, luchang jiucun yijiufen yinzhi deji babaiyishi
分其候冬至。陽律生陰，之法que[2]以九寸二因之得一十八寸，
fen qihou tongzhi. yanglu sheng yin, zhifa queyi jiucun eryin zhide yishibacun,
三歸之得六寸，隔八下生林鍾。
sangui zhide liucun, geba xiasheng linzhong.
林鍾屬陰空圍九分律長
Linzhong shuyin, kongwei jiufen luchang
六寸以九分因之得積五百四十分，其候大暑。陰律生陽之法
liucun yijiufen yinzhi deji wubaisishifen, qihou dashu. yinlu shengyang zhifa
que[2]以六寸四因之得二十四寸三歸之得長八寸隔八下生大簇。
queyi liucun siyin zhide ershisicun sangui zhidechang bacun geba xiasheng dacu.
大簇屬陽空圍九分律長八寸以九分因之得積七百二十分其候
dacu shuyang kongwei jiufen luchang bacun yijiufen yinzhi deji qibaiershifen qihou
雨水。陽律生陰之法que[2]以八寸二因之得一十六寸三歸之得
yushui. yanglu shengyin zhifa queyi bacun eryin zhide yishiliucun sangui zhide
五寸三分之一隔八下生宮[4]。以上三律皆得全寸自此以下
wucun sanfen zhiyi geba xiasheng gong [4]. yishang sanlu jiede quancun zici yixia
九律不盡之寸俱用通之。
jiulu bujin zhicun juyong tongzhi.

〔九八0〕南吕屬陰，律長五寸三之分一

[jiubaling] Nanlu shuyin, luchang wucun sanzhi fenyi

[4] mistake for nanlu 南呂.

que[2]以分母三通五寸加分子之一共得一十六寸以九分因之以三

queyi fenmu santong wucun jiafen zizhi yigongde yishiliucun yijiufen yinzhi yisan

歸之得積四百八十分，其候秋分。que[2]以通寸一十六以四因之得六十四

guizhi deji sibaibashifen, qihou qiufen. queyi tongcun yishiliu yisi yinzhide liushisi

寸。另以三因分母三得九為法歸之得七寸九分之一寸隔八

cun. lingyi sanyin fenmu sande jiuwei fagui zhide qicun jiufen zhiyicun geba

下生姑洗。

xiasheng guxi.

姑洗屬陽，律長七寸九分之一寸que[2]以分母九通

Guxi shuyang, luchang qicun jiufen zhiyicun queyi fenmu jiutong

七寸加分子之一共得六十四寸，以空圍九分因之得五千七百六十

qicun jiafen zizhi yigongde liushisicun, yikongwei jiufen yinzhide wuqianqibailiushi

分以分母九歸之得積六百四十分，其候穀雨。que[2]以通寸六十四以二因

fenyi fenmu jiugui zhideji liubaisishifen, qihou guyu. queyi tongcun liushisi yieryin

之得一百二十八寸。另以三因分母九得二十七為法除之得四寸

zhide yibaiershibacun. lingyi sanyin fenmu jiude ershiqi weifa chuzhide sicun

二十七分寸之二十隔八下生應鍾。

ershiqifen cunzhi ershi geba xiasheng yingzhong.

應鍾屬陰，律長四寸

Yingzhong shuyin, luchang sicun

二十七分寸之二十que[2]以分母二寸[5]七通四寸加分子〔九八一〕二十共

ershiqifen cunzhi ershi queyi fenmu ercunqi tong sicun jiafenzi [jiubayi] ershi gong

得一百二十八寸以空圍九分因之得一萬一千五百二十分以分母

de yibaiershibacun yi kongwei jiufen yinzhide yiwanyiqianwubaiershifen yi fenmu

二十七除之不盡一十八分法實皆九約之得積四百二十〔六〕[6]分三分寸

ershiqi chuzhi bujin yishibafen fashi jiejiu yuezhi deji sibaiershi[liu] fen sanfen cun

[5] mistake for shi十.

[6] liu六is missing.

之二，其候小雪。que[2]以通寸一百二十八以四因之得五百一十二寸。

zhi er, qihou xiaoxue. queyi tongcun yibaiershiba yisiyin zhide wubaiyishiercun.

另以三因二十七得八十一為法除之得六寸八十一分寸之二十六

lingyi sanyin ershiqi de bashiyi weifa chuzhide liucun bashiyi fencunzhi ershiliu

隔八下生蕤賓。

geba xiasheng ruibin.

蕤賓屬陽，律長六寸八十一分寸之二十六que[2]以分母八十一通

Ruibin shuyang, luchangliucun bashiyifen cunzhi ershiliu queyi fenmu bashiyi tong

六寸加分子二十六共得五百一十二寸以空圍九分因之得

liucun jiafenzi ershiliu gongde wubaiyishier cunyi kongwei jiufen yinzhide

四万六千令八十分以分母八十一為法除之不盡七十二分法實皆以

siwanliuqianlingbashi fenyi fenmu bashiyi weifa chuzhi bujin qishierfen fashi jieyi

九約之得積五百六十〔八〕[7]分九分寸之八其候夏至。que[2]以通寸

jiuyue zhideji wubailiushi[ba] [7] fen jiufen cunzhiba qihou xiazhi. queyi tongcun

五百一十二以四因之得二千令四十八寸。另以三因八十一得

wubaiyishier yi siyin zhide erqianlingsishiba cun. lingyi sanyin bashiyi de

二百四十三為法除之得八〔九八二〕寸二百四十三分寸之一百令四

erbaisishisan weifa chuzhide ba [jiubaer] cun erbaisishisan fencunzhi yibailingsi

隔八上生大呂。

geba shangsheng dalu.

按蕤賓陽律生陰之法當用三分損一如上所云乃三分

Anruibin yanglu shengyin zhifa dangyong sanfen sunyi rushangsuo yunnai sanfen

益一之法此又不可曉者抑夏至一陰始生之故歟自此以後陰律

yiyi zhifa ciyou buke xiaozheyi xiazhi yiyin shisheng zhigu yuzi ciyihou yinlu

生陽三分損一陽律生陰三分益一。

shengyang sanfen sunyi yanglu shengyin sanfen yiyi.

[7] ba 八is missing.

大呂屬陰律長八寸二百四十三分寸之一百令四que[2]以分母

Dalu shuyin luchang bacun erbaisishisan fencunzhi yibailingsi queyi fenmu

通八寸加分子共得二千令四十八寸以九分因之以分母

tongba cunjia fenzi gongde erqianlingsishiba cunyi jiufen yinzhiyi fenmu

二百四十三為法除之不盡一百二十六分法實皆三約之得積

erbaisishisan weifa chuzhi bujin yibaiershiliu fenfa shijie sanyue zhide ji

七百五十八分八十一寸[8]寸之四十二其候大寒。que[2]以通寸

qibaiwushibafen bashiyi cun[8]cunzhi sishier qihou dahan. queyi tongcun

二千令四十八寸以二因之得四千令九十六寸為實。另以三因

erqianlingsishiba cunyi eryin zhide siqianlingjiushiliucun weishi. lingyi sanyin

二百四十三得七百二十九為法除之得五寸七百二十九分寸之

erbaisishisan de qibaiershijiu weifa chuzhide wucun qibaiershijiu fencunzhi

四百五十一隔八下生夷則。

sibaiwushiyi geba xiasheng yize.

〔九八三〕夷則屬陽律長五寸七百二十九分寸之四百五十一。

[jiubasan] Yize shuyang luchang wucun qibaiershijiufen cunzhi sibaiwushiyi.

que[2]以分母通五寸加分子共得四千令九十六寸以空圍

queyi fenmu tongwu cunjia fenzi gongde siqianlingjiushiliu cunyi kongwei

九分因之得三十六萬八千六百四十分為實以七百二十九為法

jiufen yinzhide sanshiliuwanbaqianliubaisishi fenwei shiyi qibaiershijiu weifa
除之不盡四百一十四[9]分法實皆九約之得積五百八十一[9]分寸之
chuzhi bujin sibaiyishisi[9]fenfa shijie jiuyue zhideji wubaibashiyi[9] fencunzhi
[8] better 分fen.
[9] mistake for 四百九十五sibaijiushiwu; mistake for 八十一bashiyi.
四十六[10]其候處暑。que[2]以通寸四千令九十六以四因之得
sishiliu[10] qihou chushu. queyi tongcun siqianlingjiushiliu yisi yinzhide
一万六千三百八十四寸另以三因七百二十九得二千一百八十七
yiwanliuqiansanbaibashisi cunling yisanyin qibaiershijiu de erqianyibaibashiqi
為法除之得七寸二千一百八十七分寸之一千令七十五隔八
weifa chuzide qicun erqianyibaibashiqi fencunzhi yiqianlingqishiwu geba
上生夾鍾。
shangsheng jiazhong.
夾鍾屬陰律長七寸二千一百八十七分寸之一千令七十五。
Jiazhong shuyin luchangqicun erqianyibaibashiqi fencunzhi yiqianlingqishiwu
que[2]以分母通七寸加分子共得一萬六千三百八十四寸以
queyi fenmu tongqi cunjia fenzi gongde yiwanliuqiansanbaibashisi cunyi
空為九分因之得一百四十七万四千五百六十分以分母
kongwei jiufen yinzhide yibaisishiqiwansiqianwubailiushi fenyi fenmu
二千一百八十七除之不盡五百二十二分法實皆九約之得積
erqianyibaibashiqi chuzhi bujin wubaiershier fenfa shijie jiuyue zhideji
六百七十四分二百四十三分寸之五十八其候春分。que[2]〔九八四〕
liubaiqishisifen erbaisishisan fencunzhi wushiba qihou chunfen. que [jiubasi]
以通寸一萬六千三百八十四寸以二因之得
yi tong cun yiwanliuqiansanbaibashisi cun yier yin zhide
三萬二千七百六十八寸為實另以三因二千一百八十七得
sanwanerqianqibailiushiba cunweishi lingyi sanyin erqianyibaibashiqi de
[10] mistake for 五十五wushiwu.
六千五百六十一為法除之得四寸六千五百六十一分寸之
liuqianwubailiushiyi weifa chuzhi desicun liuqianwubailiushiyi fencunzhi
六千五百二十四隔八下生無射。
liuqianwubaiershisi geba xiasheng wushe.
無射屬陽律長四寸六千五百六十一分寸之
Wushe shuyang luchang sicun liuqianwubailiushiyi fencunzhi
六千五百二十四que[2]以分母通四寸加分子共得
liuqianwubaiershisi queyi fenmu tongsi cunjia fenzi gongde
三萬二千七百六十八寸以空圍九分因之得
sanwanerqianqibailiushiba cunyi kongwei jiufen yinzhide
二百九十四万九千一百二十分que[2]以分母六千五百六十一分為法
erbaijiushisiwanjiuqianyibaiershi fenque yifenmu liuqianwubailiushiyi fenweifa
除之不盡三千二百三十一分以法命之得積四百四十九分
chuzhi bujin sanqianerbaisanshiyi fenyi faming zhideji sibaisishijiufen
六千五百六十一分寸之三千二百三十一其候霜降。que[2]以
liuqianwubailiushiyi fencunzhi sanqianerbaisanshiyi qihou shuangjiang. queyi
通寸三万二千七百六十八寸以四因之得

tongcun sanwanerqianqibailiushiba cunyi siyin zhide
一十三萬一千令七十二寸另以三因分母六千五百六十一得
yishisanwanyiqianlingqishier cunling yisanyin fenmu liuqianwubailiushiyi de
一万九千六百八十三為法除之得六寸一萬九千六百八十三
yiwanjiuqianliubaibashisan weifa chuzhi deliucun yiwanjiuqianliubaibashisan
〔分〕[11]寸之一萬二千九百七十四隔八上生仲〔九八五〕呂。
[fen][11]cunzhi yiwanerqianjiubaiqishisi geba shangsheng zhong [jiubawu]lu.
仲呂屬陰律長六寸一萬九千六百八十三分寸之
Zhonglu shuyin luchang liucun yiwanjiuqianliubaibashisan fencunzhi
一萬二千九百七十四que[2]以分母通六寸加分子共得
yiwanerqianjiubaiqishisi queyi fenmu tongliu cunjia fenzi gongde
一十三萬一千零七十二寸以空圍九分因之得
yishisanwanyiqianlingqishier cunyi kongwei jiufen yinzhide
一千一百七十九萬六千四百八十分以分母
yiqianyibaiqishijiuwanliuqiansibaibashi fenyi fenmu
一萬九千六百八十三為法除之得積五百九十九分
yiwanjiuqianliubaibashisan weifa chuzhi deji wubaijiushijiu fen
一萬九千六百八十三分分[12]之六千三百六十三其候
yiwanjiuqianliubaibashisan fenfen[12]zhi liuqiansanbailiushisan qihou
小滿。
xiaoman.
[11] 分fen is missing.
[12] mistake for 分寸fencun.
律學新說
Luxue xinshuo
新製律準
xinzhi luzhun
創立新法。置一尺為實以密率除之凡十二遍所求律呂真數.
chuangli xinfa. zhiyi chiwei shiyi milu chuzhifan shier biansuoqiu lulu zhenshu.
tuibu [stage in deducing] 推步
yixue [foreign doctrine] 異學
yangwu [foreign affaires] 洋務
bi [gnomon] 髀
suicha [difference of the year, precession of the equinoxes] 歲差
lifa [rules of the calendar] 曆法
wangyuanjing [lenses to see far away, telescope] 望遠鏡
li [half a kilometer] 里

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