Environmental Fluid Mechanics

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To our wives, Elana Rubin and Nancy Atkinson

and our families

for their continued love and support

Preface

The purpose of this text is to provide the basis for an upper-level undergraduate or graduate course over one or two semesters, covering basic concepts and examples of fluid mechanics with particular applications in the natural environment. The book is designed to meet a dual purpose, providing an advanced fundamental background in the fluid mechanics of environmental systems and also applying fluid mechanics principles to a variety of environmental issues. Our basic motivation in preparing such a text is to share our experience gained by teaching courses in fluid mechanics, environmental fluid mechanics, and surface- and groundwater quality modeling and to provide a textbook that covers this particular collection of material.

This text presents a contemporary approach to teaching fluid mechanics in disciplines connected with environmental issues. There are many good fluid mechanics texts that overlap with various parts of this text, but they do not directly address themes and applications associated with the environment. On the other hand, there are also several texts that address water quality modeling, calculations of transport phenomena, and other issues of environmental engineering. Generally, such texts do not cover the fundamental topics of fluid mechanics that are relevant when describing fluid motions in the environment. Besides presenting contemporary environmental fluid mechanics topics, this text bridges the gap between those limited to fluid mechanics principles and those addressing the quality of the environment.

The term *environmental fluid mechanics* covers a broad spectrum of subjects. We have adopted the principle that this topic incorporates all issues of small-scale and global fluid flow and contaminant transport in our environment. We have chosen to consider these topics as divided into two general areas, one involving fundamental fluid mechanics principles relevant to the environment and the second concerning various types of applications of these principles to specific environmental flows and issues of water quality modeling. This division is reflected in the organization of the text into two main parts. The intent is to provide flexibility for instructors to choose material best suited for a particular curriculum. A full two-semester course could be developed by following the entire text. However, other options are possible. For example, a one-semester course could concentrate on the advanced fluid mechanics topics of the first part, with perhaps some chapters from the second part added to emphasize the environmental content. The second part by itself can be used in a course concentrating on environmental applications for students with appropriate fluid mechanics backgrounds. Although the book addresses principles of fluid mechanics relevant to the entire environment, the emphasis is mostly on water-related issues.

The material is designed for students who have already taken at least one undergraduate course in fluid mechanics and have an appropriate background in mathematics. Other courses in numerical modeling and environmental studies would be helpful but are not necessary. Because of the breadth of material that could be considered, some subjects have necessarily been omitted or treated only at an introductory level. These topics are left for continuing studies in the student's particular discipline, such as oceanography, meteorology, groundwater hydrology and contaminant transport, surface water quality modeling, etc. References are provided in each chapter so that students can easily get started in pursuing a particular subject in greater detail. Example problems and solutions are included wherever possible, and there is a set of homework problems at the end of each chapter.

We believe it is very important to introduce students to the proper use of physical and numerical models and computational approaches in the framework of analysis and calculation of environmental processes. Therefore, discussion and examples have been included that refer to scaling procedures and to various numerical methods that can be applied to obtain solutions for a given problem. A full discussion of numerical modeling approaches is included.

Both parts of the text are organized to provide (1) a review of introductory material and basic principles, (2) improvement and strengthening of basic knowledge, and (3) presentation of specific topics and applications in environmental fluid mechanics, along with problem-solving approaches. These topics have been chosen to introduce the student to the wide variety of issues addressed within the context of environmental fluid mechanics, regarding fluid motions on the earth's surface, underground, and in the oceans and atmosphere.

We believe that the wide scope of topics in environmental fluid mechanics covered in this text is consistent with present teaching needs in advanced undergraduate and graduate programs in fluid mechanics principles and topics related to the environment. These needs are subject to continuous growth and change due to our increasing interest in the fate of ecological systems and the need for understanding transport phenomena in our environment.

The authors are grateful to the US–Israel Fulbright Foundation for supporting a sabbatical leave for Joseph Atkinson at the Technion–Israel Institute of Technology, without which this text might not have been completed.

Finally, we are indebted to our own teachers, colleagues, and students, who have each made contributions to our understanding of this material and have helped in shaping the presentation of this text. We hope the book will contribute to this legacy.

Hillel Rubin
Joseph Atkinson

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Preliminary Concepts

1.1 INTRODUCTION

1.1.1 Historical Perspective

Fluid mechanics and hydraulics have long been major components of civil engineering works and were probably originally associated with problems of water supply in ancient civilizations. One of the first well-documented hydraulic engineers was Archimedes (ca. 287–212 B.C.). His discovery of the basic principles of buoyancy serves today as one of the fundamental building blocks in describing fluid behavior. He also designed simple pumps for agricultural applications, and some of his designs are still in use today. Other early engineers had to deal with moving water over large distances from sources to cities, as with the Roman aqueducts found in many parts of Europe and the Middle East (see Fig. 1.1). These designs needed to incorporate basic aspects of open channel flow, such as finding the proper slope to obtain a desired flow rate. Remains of water storage and conveyance systems have also been found from some of the earliest civilizations known, both in the Near East and in the Far East. Rouse (1957) provides an interesting history of the science and engineering of hydraulics, which is also summarized by Graf (1971), particularly as it relates to open channel flow. In a sense, these were the first kinds of problems that can be associated with the field of environmental fluid mechanics.

An equally important task for early engineers was to design procedures for disposing of wastewater. The simplest means of doing this, which was in use until the relatively recent past, consisted of systems of gutters and drainage ditches, usually with direct discharge into ponds or streams. Septic tanks, with associated leeching fields, are another example of a simple wastewater treatment system, though these can handle only relatively small flow rates. Within the last century the practice of wastewater collection and treatment has evolved considerably, to enable varying degrees of treatment of a waste stream before it is discharged back into the natural environment. This development has been

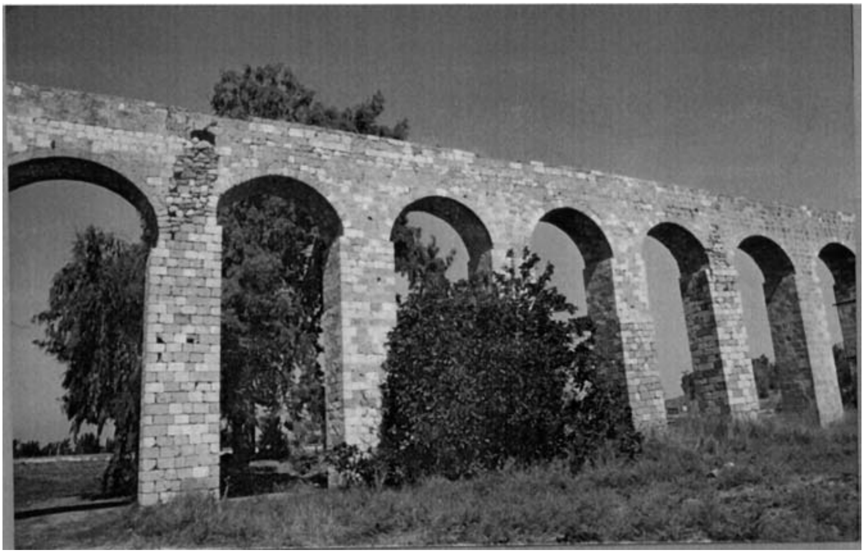

Figure 1.1 Remains of Roman aqueduct, built in northern Israel.

driven by increased demands (both in quantity and in quality) for treating municipal sewage, as well as increased needs for treating industrial wastes. Sanitary engineering, within the general profession of civil engineering, traditionally dealt with designing water and wastewater collection and treatment systems. This has evolved into the contemporary field of environmental engineering, which now encompasses the general area of water quality modeling, for both surface and groundwater systems. This has necessitated the incorporation of other fields of science, such as chemistry and biology, to address the wider range of problems now being faced in treating waste streams with a variety of characteristics and needs.

In addition to treating municipal or industrial wastewater, environmental engineers currently are involved in solving problems of chemical fate and transport in natural environmental systems, including subsurface (groundwater) and surface waters, sediment transport, and atmospheric systems. A knowledge and understanding of fluid flow and transport processes is necessary to describe the transport and dispersion of pollutants in the environment, and chemical and biological processes must be incorporated to describe source and sink terms for contaminants of interest. Typical kinds of problems might involve calculating the expected chemical contaminant concentration at a water supply intake due to an upstream spill, evaluating the spreading of waste heat discharged from power plant condensers, predicting lake or reservoir stratification and associated effects on nutrient and dissolved oxygen distributions, determining the relative importance of contaminated sediments as a continuing source of pollutants to a river or lake system, calculating the expected recovery time of a lake when contaminant loading is discontinued, or evaluating the effectiveness of different remediation options for a contaminated groundwater source. All of these kinds of problems require an understanding of fluid flow phenomena and of biochemical behavior of materials in the environment.

1.1.2 Objectives and Scope

The primary objective of this text is to provide a basis for teaching upper-level fluid mechanics and water quality modeling courses dealing with environmentally related issues and to give a compilation of applications of environmental fluid mechanics seen in contemporary problems. The text also is meant to serve as a reference for further study in the various subjects covered, so references are included for additional reading. It would be impossible to include an exhaustive discussion of all possible subjects in one text, and inclusion of these additional references should provide a good starting point for more in-depth study. Example problems are provided where appropriate, to amplify the discussion or help reinforce certain concepts, and unsolved problems are included at the back of each chapter, to provide exercises that might be included in a course.

Today, the area of environmental fluid mechanics spans a broad range of issues, including open channel hydraulics, sediment transport, stratified flow phenomena, transport and mixing processes, and various issues in water quality and atmospheric modeling. These topics are studied in a variety of ways, such as by theoretical analyses, physical model experiments, field studies, and numerical modeling. This text presents material that might traditionally be included in two separate courses, one in fluid mechanics and the other in water quality modeling. The emphasis here is on aqueous systems, both in surface and subsurface flows, though the basic principles are mostly applicable also for atmospheric studies. A major link between classic hydraulic engineering and water quality studies is in defining the advective and diffusive (or dispersive) transport terms of a water quality model, which are normally estimated from hydrodynamic calculations. Fluid mechanics deals with the study of fluid motion, or the response of a fluid to applied forces, and environmental fluid mechanics refers to the application of fluid mechanics principles to problems involving environmental flows, including purely physical applications (e.g., open channel flow, groundwater flow, sediment transport) and problems of water quality modeling. In the following chapters the analytical bases for the engineering evaluation and solution of these types of problems are developed. Governing equations for fluid motion are derived, as well as the equation expressing mass balance for a dissolved tracer, otherwise known as the advection–diffusion equation. Conservation equations for both mechanical and thermal energy also are developed, and these lead to descriptions of turbulent kinetic energy and temperature, respectively.

The text is divided into two parts. The first part is a discussion of theoretical principles used in describing fluid motion and includes the derivation of the basic mathematical equations governing fluid flow. Chapters 4 through 9 include discussions of potential flow theory, introductions to turbulence and boundary layer theory, groundwater flow, and large-scale motions where the rotation of the earth must be incorporated into the equations of motion. The second part of the text contains material more directly applied to environmental problems. Fundamental transport processes for contaminants are discussed, including advection, diffusion, and dispersion, and applications are described in modeling groundwater flow and contaminant transport, exchange processes between water surfaces and the atmosphere, stratified flows, jets and plumes, sediment transport, and remediation issues. Sections in various chapters are included that discuss associated numerical modeling issues, as we recognize the important role of numerical solutions in many of the problems faced in environmental fluid mechanics. Different solution approaches, boundary conditions, numerical dispersion and scaling considerations are addressed. The intent is that the material contained herein could serve as the basis for a two-semester upper level undergraduate or graduate course, with each part of the text providing a focus for each semester of instruction. Of course, single-semester courses can be developed, based on individual chapters.

The remainder of the present chapter is devoted to a review of fluid properties and mathematical preliminaries.

1.2 PROPERTIES OF FLUIDS

1.2.1 General

Most substances are categorized as existing in one of two states: solid or fluid. Solid elements have a rigid shape that can be modified as a result of stresses. This shape modification is termed *deformation* or *strain*. Different types of solids are identified by different relationships between the shear stress and the strain. A strained solid body is in a state of equilibrium with the stresses applied on that body. When applied stresses vanish, the solid body relaxes to its original shape.

Solid boundaries (i.e., a container) and interfaces with other fluids determine the shape of a fluid body. Unlike solids, even an infinitesimal shear force changes the shape of fluid elements. Differences between different types of fluid are identified by different relationships between the shear stress and the *rate of strain*. When applied stresses vanish, fluid elements do not return to their original shape. In addition, fluids usually do not support tensile stresses, though in many cases they strongly resist normal compressive stresses. In many cases they can be considered as *incompressible* materials or materials subject to *incompressible flow*, meaning that their density is not a function of pressure. In general, fluids may be divided into liquids, for which *compressibility* is generally negligible, and gases, which are *compressible* fluids. In other words, the volume of a liquid mass is almost constant, and it occupies the lowest portion of a container in which it is held. It also has a horizontal free surface in a stationary container. A gas always expands and occupies the entire volume of any container. However, gases like air are usually well described in the atmosphere using *incompressible* flow theory.

1.2.2 Continuum Assumptions

All materials are composed of individual molecules subject to relative movement. However, in the framework of fluid mechanics we consider the fluid as a continuum. We are generally interested in the *macroscopic* behavior of a fluid material, so that the smallest fluid mass of interest usually consists of a *fluid particle* that is much larger than the mean free path of a single molecule. It is therefore possible to ignore the discrete molecular structure of the matter and to refer to it as a *continuum*. The continuum approach is valid if the characteristic length, or size of the flow system (e.g., the diameter of a solid sphere submerged in a flowing fluid) is much larger than the mean free path of the molecules. For example, in a standard atmosphere the molecular free path is of the order of 10^{-8} m, but in the upper altitudes of the atmosphere the molecule mean free path is of the order of 1 m. Therefore, in order to study the dynamics of a rarefied gas in such heights a kinetic theory approach would be necessary, rather than the continuum approach.

1.2.3 Review of Fluid Properties

The *density* ρ of a fluid is a measure of the concentration of matter and is expressed in terms of mass per unit volume. The volume and mass of fluid considered for the calculation of the fluid density should be small, but not so small that variations on a molecular level would become important. Therefore, we define

$$\rho = \lim_{\delta V \to \delta V'} \frac{\delta m}{\delta V} \tag{1.2.1}$$

where δm is an amount of mass contained in a small volume δV, and $\delta V'$ is the volume of the smallest fluid particle that is still much larger than the mean free molecular path. The *specific weight* γ is the force of gravity on the mass contained in a unit volume of the substance,

$$\gamma = \rho g \tag{1.2.2}$$

The density of water is 1000 kg/m^3 (at 4°C) and the acceleration of gravity $g = 9.81$ m/s^2. Therefore, the nominal specific weight of water is

$$\gamma = (1000 \text{ kg/m}^3)(9.81 \text{ m/s}^2) = 9810 \text{ N/m}^3 \tag{1.2.3}$$

The diffusive flux of a dissolved constituent in a fluid is expressed by *Fick's law*, which states that the flux is proportional to the constituent concentration gradient (see also Chap. 10). In a one-dimensional domain this law is expressed as

$$q_{\mathrm{m}} = -k_{\mathrm{m}} \frac{\partial C}{\partial x} \tag{1.2.4}$$

where q_{m} is the mass flux (kg m^{-2} s^{-1}) of the constituent in the x direction, C is the constituent concentration (kg m^{-3}), and k_{m} is the mass diffusivity (m^2 s^{-1}), whose value depends on the fluid and on the constituent. The relationship represented by Eq. (1.2.4) is based on empirical evidence and is called a *phenomenological law*. A similar phenomenological law is *Fourier's law of heat diffusion*, which in a one-dimensional domain can be written as

$$q = -k \frac{\partial T}{\partial x} \tag{1.2.5}$$

where q is the heat flux (J m^{-2} s^{-1}), T is the temperature (°C), and k is the thermal conductivity (J m^{-1} s^{-1} °C^{-1}), whose value depends on the fluid.

Another phenomenological law is the law of Newton, expressing proportionality between the strain rate and the shear stress in so-called *Newtonian fluids*. In a one-directional flow with velocity u in the x direction and with the velocity a function of y, the shear stress τ that develops between fluid layers is expressed as

$$\tau = \mu \frac{\partial u}{\partial y} \tag{1.2.6}$$

Here the constant of proportionality μ (Pa s) is the *dynamic viscosity*, whose value depends on the fluid and on temperature. The ratio of dynamic viscosity to density appears often in the equations describing fluid motion and is called the *kinematic viscosity* ν (m^2 s^{-1}),

$$\nu = \frac{\mu}{\rho} \tag{1.2.7}$$

There is some similarity between Eqs. (1.2.4), (1.2.5), and (1.2.6). However, the mass flux given by Eq. (1.2.4) and heat flux given by Eq. (1.2.5) are components of flux vectors, whereas the shear stress given by Eq. (1.2.6) is a component of a tensor. These issues are described further in the following sections of this chapter.

The interface between two immiscible fluids behaves like a stretched membrane, in which tension originates from intermolecular attractive (cohesive) forces. Near an interface, say between the fluid and another fluid or between the fluid and the solid walls of a boundary or container, all the fluid molecules are trying to pull the molecules on the interface inward. The magnitude of the tensile force per unit length of a line on the interface is called *surface tension* σ (N m^{-1}), whose value depends on the pair of fluids and the temperature. If p_1 and p_2 are the fluid pressures on the two sides of an interface, then a simple force balance yields

$$\sigma(2\pi R) = (p_1 - p_2)\pi R^2$$

where R is the radius of curvature of the interfacial surface. This result is also written as

$$\sigma = \frac{(p_1 - p_2)R}{2} \tag{1.2.8}$$

For a general surface, the radii of curvature along two orthogonal directions R_1 and R_2 are used to specify the curvature. In this case, the relationship between surface tension and pressure is

$$\sigma = \frac{(p_1 - p_2)R_1 R_2}{R_1 + R_2} \tag{1.2.9}$$

If a fluid and its vapor coexist in equilibrium, the vapor is a *saturated vapor*, and the pressure exerted by this saturated vapor is called the *vapor pressure*, with symbol p_{v}. The vapor pressure depends on the fluid and the temperature.

The *compressibility* of a fluid is defined in terms of the average modulus of elasticity K (Pa), defined as

$$K = -\frac{dp}{dV/V} = \frac{dp}{d\rho/\rho} \tag{1.2.10}$$

where dV is the change in volume accompanying a change in pressure dp, and V and ρ are the original volume and density, respectively. The second expression in Eq. (1.2.10) refers to density changes, but the negative sign is dropped since the density changes in the opposite direction to that of volume.

1.3 MATHEMATICAL PRELIMINARIES

1.3.1 Vectors and Tensors

A point in a three dimensional space is defined by its *coordinates*,

$$x^1, x^2, x^3 \tag{1.3.1}$$

A curve is defined as the totality of points given by the equation

$$x^i = f^i(u) \quad (i = 1, 2, 3) \tag{1.3.2}$$

Here, u is an arbitrary parameter and the f^i are three arbitrary functions.

The point given by Eq. (1.3.1) can be represented by a new set of coordinates (x'^1, x'^2, x'^3), where

$$x'^i = f^i(x^1, x^2, x^3) \tag{1.3.3}$$

The Jacobian of the transformation is

$$J' = \left| \frac{\partial x'^i}{\partial x^j} \right| \quad (i, j = 1, 2, 3) \tag{1.3.4}$$

Eq. (1.3.2) also can be represented by another transformation,

$$x^i = g^i(x'^1, x'^2, x'^3) \tag{1.3.5}$$

Differentiation of Eq. (1.3.3) then yields

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \tag{1.3.6}$$

where index *summation convention* is used. That is, summation is made with regard to the repeating superscript j. Such repeated indices are often referred to as dummy indices. Any such pair may be replaced by any other pair of repeated indices without changing the value of the expression.

For future reference, we introduce the *Kronecker delta*, δ_i^j, defined as

$$\begin{aligned} \delta_i^j &= 1 \quad \text{if} \quad i = j \\ \delta_i^j &= 0 \quad \text{if} \quad i \neq j \end{aligned} \tag{1.3.7}$$

It is evident that

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i \tag{1.3.8}$$

Contravariant Vectors and Tensors, Invariants

Consider a point P with coordinates x^i and a neighboring point Q with coordinates $x^i + dx^i$. These two points define a *vector*, termed the *displacement*, whose components are dx^i. We may still think about the same two points, but apply a different coordinate system x'^i. In this coordinate system the components of the displacement vector are dx'^i. Components of the displacement tensor in the two systems of coordinates are related by Eq. (1.3.6).

If we keep the point P fixed, but vary Q in the neighborhood of P, the coefficient $\partial x'^i/\partial x^j$ remains constant. Under these conditions, Eq. (1.3.6) is a linear homogeneous (or *affine*) transformation.

The vector has an absolute meaning, but the numbers describing this vector depend on the employed coordinate system. The infinitesimal displacements given by Eq. (1.3.6) satisfy the rule of transformation of *contravariant* vectors. Later we also will refer to *covariant* vectors. A contravariant vector is one in which the vector components comprise a set of quantities A^i associated with a point P that transform, on change of coordinates, according to the equation

$$A'^i = A^j \frac{\partial x'^i}{\partial x^j} \qquad (1.3.9)$$

where the partial derivatives are evaluated at point P. The expression for the infinitesimal displacements given by Eq. (1.3.6) represents a particular example of a contravariant vector.

A set of quantities A^{ij} represents components of a contravariant tensor of the *second order* if they transform according to the equation

$$A'^{ij} = A^{km} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^m} \qquad (1.3.10)$$

The product $A^i \times B^j$ of two contravariant vectors is a contravariant tensor of the second order.

Equation (1.3.10) provides a basic format for the definition of contravariant tensors of the third or higher order. We also can conclude that there is a contravariant tensor of the zero order that is a single component quantity, transformed according to the identity relation

$$A' = A \qquad (1.3.11)$$

Such a quantity is called an *invariant*, and its value is independent of the employed coordinate system.

Covariant Vectors and Tensors, Mixed Tensors

If H is an invariant then we may introduce

$$\frac{\partial H}{\partial x'^i} = \frac{\partial H}{\partial x^j} \frac{\partial x^j}{\partial x'^i} \qquad (1.3.12)$$

This transformation is very similar to that of Eq. (1.3.6), but the partial derivative involving the two sets of coordinates is reversed. Equation (1.3.6) indicates that the infinitesimal displacement is the prototype of the contravariant vector. Equation (1.3.12) shows that the partial derivative of an invariant represents a prototype of the general covariant vector. The components of a covariant vector comprise a set of quantities that transform according to

$$A'_i = A_j \frac{\partial x^j}{\partial x'^i} \tag{1.3.13}$$

Suffixes indicating contravariant character are placed as superscripts, and those indicating covariant character are subscripts. This convention means that coordinates should be written x^i rather than x_i, although it is only the differentials of the coordinates, and not the coordinates themselves, that have tensor character.

We may extend Eq. (1.3.13) to define higher order covariant tensors. Following the definitions of contravariant and covariant tensors, *mixed tensors* can be defined. As an example, consider a third-order mixed tensor,

$$A'^{i}_{jk} = A^{m}_{np} \frac{\partial x'^i}{\partial x^m} \frac{\partial x^n}{\partial x'^j} \frac{\partial x^p}{\partial x'^k} \tag{1.3.14}$$

It then follows that the Kronecker delta is a second-order mixed tensor represented by the transformation

$$\delta'^{i}_{j} = \delta^{m}_{n} \frac{\partial x'^i}{\partial x^m} \frac{\partial x^n}{\partial x'^j} \tag{1.3.15}$$

The left-hand side of Eq. (1.3.15) is unity if $i = j$ and zero otherwise. Holding m fixed and summing with respect to n, there is no contribution to the sum unless $n = m$. Therefore the right-hand side of Eq. (1.3.15) reduces to

$$\frac{\partial x'^i}{\partial x^m} \frac{\partial x^m}{\partial x'^j} \tag{1.3.16}$$

and this expression is equal to δ^{i}_{j}.

Addition, Multiplication, and Contraction of Tensors

Two tensors of the same order and type can be added together to give another tensor of the same order and type. For example, we can write

$$C^{i}_{jk} = A^{i}_{jk} + B^{i}_{jk} \tag{1.3.17}$$

A second-order tensor is called a *symmetric tensor* if its components satisfy the relationship

$$A_{ij} = A_{ji} \tag{1.3.18}$$

A second-order tensor is *antisymmetric* or *skew-symmetric* if its components satisfy

$$A_{ij} = -A_{ji} \tag{1.3.19}$$

The definitions given by Eqs. (1.3.18) and (1.3.19) can be extended to more complicated tensors. A tensor is symmetric with respect to a pair of suffixes if the value of the components is unchanged on interchanging these suffixes. A tensor is antisymmetric with respect to a pair of suffixes if interchanging these suffixes leads to a change of sign with no change of absolute value.

Any tensor of the second order can be expressed as the sum of a symmetric and an antisymmetric tensor. As an example, we can write

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}) \tag{1.3.20}$$

The first term on the right-hand side of Eq. (1.3.20) is a symmetric tensor, and the second one is an antisymmetric tensor. This property is useful when discussing stresses in fluid flow (Chap. 2).

Addition or subtraction can be done only with tensors of the same order and type. In multiplication the only restriction is that we never multiply two components with the same literal suffix at the same level in each component. We may take tensors of different types and different literal suffixes. Then the product is a tensor whose order is equal to the sum of orders of the multiplied tensors. As an example,

$$C^m_{ijk} = A_{ij}B^m_k \tag{1.3.21}$$

The product exemplified by Eq. (1.3.21) is called an *outer product*. The *inner product* is associated with *contraction*. It is obtained by multiplication of tensors with lower suffixes identical to lower ones. An example is

$$C^m_i = A_{ij}B^{jm} \tag{1.3.22}$$

The process of contraction cannot be applied to suffixes at the same level. Indices appearing at lower and upper levels represent summation.

The Metric Tensor and the Line Element

Suppose that y^1, y^2, y^3 are rectangular Cartesian coordinates. Then the square of the distance between adjacent points is

$$ds^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 \tag{1.3.23}$$

Any system of curvilinear coordinates is represented by x^1, x^2, x^3 (e.g., cylindrical or spherical polar). The y^i coordinates are functions of the x^i coordinates, and the dy^i components of the infinitesimal displacement are linear homogeneous functions of the dx^i components. We introduce the relationships of Eq. (1.3.6) to obtain a homogeneous quadratic expression in the dx^i components, which may be written as

$$ds^2 = g_{ij} dx^i dx^j \tag{1.3.24}$$

where the coefficients g_{ij} are functions of the x^i coordinates. As the g_{ij} do not occur separately, but only in the combinations $(g_{ij} + g_{ji})$, there is no loss of generality in taking g_{ij} as a symmetric tensor.

As the distance between two given points is not dependent on the applied coordinates, the value of ds or ds^2 is an invariant. According to Eq. (1.3.6), dx^i is a contravariant vector. Therefore, g_{ij} is a second-order covariant tensor. It is called the *metric tensor.*

By applying Eqs. (1.3.23) and (1.3.24), we obtain

$$g_{ij} = \frac{\partial y^1}{\partial x^i}\frac{\partial y^1}{\partial x^j} + \frac{\partial y^2}{\partial x^i}\frac{\partial y^2}{\partial x^j} + \frac{\partial y^3}{\partial x^i}\frac{\partial y^3}{\partial x^j} \tag{1.3.25}$$

As an example, we consider a cylindrical coordinate system in which $x^1 = r$, $x^2 = \theta$, $x^3 = z$. The relationships between the y^i coordinates and x^i coordinates are $y^1 = x^1 \cos x^2$, $y^2 = x^1 \sin x^2$, and $y^3 = x^3$. By introducing these relationships into Eq. (1.3.25), we obtain for the cylindrical coordinate system

$$\begin{aligned} &g_{ij} = 0 \qquad \text{for} \qquad i \neq j \\ &g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = 1 \end{aligned} \tag{1.3.26}$$

The Conjugate Tensor; Lowering and Raising Suffixes

From the covariant metric tensor g_{ij} we can obtain a contravariant tensor g^{ij} given by

$$g^{ij} = \frac{C^{ij}}{g} \tag{1.3.27}$$

where C^{ij} is the *cofactor* of g_{ij} and g is the *determinant* of g_{ij}. The following relationships then hold:

$$g_{ij} C^{ik} = g_{ji} C^{ki} = \delta_j^k \tag{1.3.28}$$

By multiplying both sides of this expression by C^{jm} we obtain

$$g \delta_i^j g^{ik} = \delta_m^k C^{jm} \tag{1.3.29}$$

If $g^{ij} = 0$ for $i \neq j$, then

$$g^{11} = \frac{1}{g_{11}} \qquad g^{22} = \frac{1}{g_{22}} \qquad g^{33} = \frac{1}{g_{33}}$$
$$g^{ij} = 0 \qquad \text{for} \qquad i \neq j \tag{1.3.30}$$

The covariant metric tensor and its contravariant conjugate can be used for *lowering* and *raising* of suffixes. As an example,

$$U_{ijk} = g_{im} V^m_{jk} \tag{1.3.31}$$

Now we may refer to a tensor as a geometrical object that has different representations in different coordinate systems. Until now we could consider that the tensors U^{ij} and U_{ij} were entirely unrelated; one was contravariant and the other covariant, and there was no connection between them. At present we realize that use of the same symbol U for these tensors means that each of them represents the same geometrical object, and internal products with the metric tensors give the relationships between their components.

Geodesics and Christoffel Symbols

A *geodesic* is a curve whose length has a stationary value with respect to arbitrary small variations of the curve while its end points are kept fixed. By using some techniques of variational calculus, it is possible to show that the differential equation of a geodesic is

$$g_{ij} \frac{dp^j}{ds} + [jk, i] p^j p^k = 0 \tag{1.3.32}$$

where s is the arc length along the geodesic and $p^i = dx^i/ds$. The expression given in the square brackets is called the *Christoffel symbol of the first kind*, which is defined by

$$[jk, i] = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right) \tag{1.3.33}$$

The Christoffel of the second kind is defined as

$$\sum_{jk}^{i} = g^{im} [jk, m] \tag{1.3.34}$$

If we multiply Eq. (1.3.32) by g^{in}, we obtain another form for the equation of a geodesic,

$$\frac{dp^i}{ds} + \sum_{jk}^{i} p^j p^k = 0 \tag{1.3.35}$$

This expression also can be represented by

$$\frac{d^2x^i}{ds^2} + \sum_{jk}^{i} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{1.3.36}$$

The differential equation of a geodesic in terms of an arbitrary parameter t is identical to Eq. (1.3.36) in which t replaces s.

Derivatives of Tensors

From Eq. (1.3.12) it is shown that the partial derivative of an invariant with respect to a coordinate is a covariant vector. However, as discussed and shown hereinafter, the partial derivative of a tensor is not a tensor.

We refer to a contravariant vector field U^i, defined along a curve $x^i = x^i(t)$. Then the *absolute derivative of* U^i with regard to t is defined as

$$\frac{\delta U^i}{\delta t} = \frac{dU^i}{dt} + \sum_{jk}^{i} U^j \frac{dx^k}{dt} \tag{1.3.37}$$

This expression is itself a contravariant vector. If the absolute derivative expression of Eq. (1.3.37) vanishes, then the vector U^i is said to be propagated *parallel* along the curve. In the case of a Cartesian coordinate system, the Christoffel symbols vanish and Eq. (1.3.37) yields $dU^i/dt = 0$. In this case the vector passes through a sequence of parallel positions.

The absolute derivative of the vector given by Eq. (1.3.37) means that the vector characteristic is given along a curve. Therefore, Eq. (1.3.37) can be represented by

$$\frac{\delta U^i}{\delta t} = \left(\frac{\partial U^i}{\partial x^k} + \sum_{jk}^{i} U^j \right) \frac{dx^k}{dt} \tag{1.3.38}$$

The left-hand side of Eq. (1.3.38) represents a contravariant vector. The term dx^k/dt also is a contravariant vector. Therefore, the expression between parentheses of Eq. (1.3.38) is a second-order mixed tensor. We call it the *covariant derivative of a contravariant vector*. It is represented as

$$U^i_{,k} = \frac{\partial U^i}{\partial x^k} + \sum_{jk}^{i} U^j \tag{1.3.39}$$

The same method can be applied to obtain the covariant derivative of any tensor from the absolute derivative. In the following equations we provide expressions for the covariant derivative of various types of tensors:

$$U_{i,k} = \frac{\partial U_i}{\partial x^k} - \sum_{ik}^{j} U_j \tag{1.3.40}$$

$$U^{ij}_{,k} = \frac{\partial U^{ij}}{\partial x^k} + \sum_{mk}^{i} U^{mj} + \sum_{mk}^{j} U^{im} \tag{1.3.41}$$

$$U_{ij,k} = \frac{\partial U_{ij}}{\partial x^k} - \sum_{ik}^{m} U_{mj} - \sum_{jk}^{m} U_{im} \tag{1.3.42}$$

$$U^{i}_{j,k} = \frac{\partial U^{i}_{j}}{\partial x^k} + \sum_{mk}^{i} U^{m}_{j} - \sum_{jk}^{m} U^{i}_{m} \tag{1.3.43}$$

Cartesian Tensors

If we refer to two Cartesian coordinate systems z^i and z'^i, then for a contravariant tensor of the second order we may write the following law of transformation:

$$U'^{ij} = U^{mn} \frac{\partial z'^i}{\partial z^m} \frac{\partial z'^j}{\partial z^n} \tag{1.3.44}$$

However, the partial derivatives of Eq. (1.3.44) represent the cosine between the relevant axes of the two Cartesian coordinate systems. Therefore we may write

$$\frac{\partial z'^i}{\partial z^m} = \frac{\partial z^m}{\partial z'^i} = \cos(z'^i z^m) \tag{1.3.45}$$

By introducing the relationships of Eq. (1.3.45) into Eq. (1.3.44), we obtain

$$U'^{ij} = U^{mn} \frac{\partial z^m}{\partial z'^i} \frac{\partial z^n}{\partial z'^j} \tag{1.3.46}$$

This expression is identical to the transformation of a covariant tensor. We may conclude that in every case of Cartesian tensors, the law of transformation remains unchanged when a subscript is raised or a superscript is lowered. Therefore, when dealing with Cartesian tensors, it is common to apply subscripts exclusively. Also, coordinates are represented with subscripts in such cases. The Kronecker delta is identical to the metric tensor and is written as δ_{ij}, which also is identical to the unit matrix.

The *permutation tensor* ε_{ijk} is defined as

$\varepsilon_{ijk} = 0$ if two of the suffixes are equal
$\varepsilon_{ijk} = 1$ if the sequence of numbers ijk is the sequence of 1-2-3, or an even permutation of the sequence
$\varepsilon_{ijk} = -1$ if the sequence of numbers *ijk* is an odd permutation of the sequence 1-2-3

Examples of the application of these rules are

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1 \qquad \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \tag{1.3.47}$$

Using the permutation tensor, the *vector product* C_i of two vectors A_j and B_k is given by

$$C_i = \varepsilon_{ijk} A_j B_k \tag{1.3.48}$$

In addition, the *curl* operator is given by

$$C_i = \varepsilon_{ijk} A_{k,j} \tag{1.3.49}$$

The following useful relation is the *epsilon delta relation*

$$\varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \tag{1.3.50}$$

Physical Components of Tensors

Consider a vector whose components in a Cartesian coordinate system z_i are represented by Z_i. As the coordinate system is a Cartesian one, covariant and contravariant components are identical. The quantities Z_i also are called the *physical components of the vector along the coordinate axes.*

If we introduce curvilinear coordinates x^j, the definition of *contravariant* and *covariant components* X^j and X_j, respectively, of the vector for the coordinate system x^j is given by

$$X^j = Z_i \frac{\partial x^j}{\partial z_i} \qquad X_j = Z_i \frac{\partial z_i}{\partial x^j} \tag{1.3.51}$$

In connection with these components, we do not use the word physical, since in general such components have no direct physical meaning. They may even have physical dimensions different from those of the physical components Z_i.

Let x^j be a curvilinear coordinate system with metric tensor g_{ij}, and let X^j be contravariant components of a vector. We define *the physical components of the vector* X^j *in the direction* λ^j as the invariant

$$g_{ij} X^i \lambda^j = X^i \lambda_i = X_i \lambda^i \tag{1.3.52}$$

If the curvilinear coordinates x^j are orthogonal coordinates, then the line element is given by

$$ds^2 = (h_1 dx^1)^2 + (h_2 dx^2)^2 + (h_3 dx^3)^2 \tag{1.3.53}$$

where h_1, h_2, and h_3 are the *geometrical scales* associated with the respective coordinates. We take a unit vector λ^i in the direction of x^1. Therefore the three components of λ^i are

$$\lambda^1 = \frac{dx^1}{ds} \qquad \lambda^2 = 0 \qquad \lambda^3 = 0 \tag{1.3.54}$$

Since λ^i is a unit vector, we have

$$g_{ij}\lambda^i\lambda^j = h_1^2(\lambda^1)^2 \qquad \lambda^1 = \frac{1}{h_1} \tag{1.3.55}$$

By multiplying by the metric tensor, we lower superscripts and obtain

$$\lambda_1 = h_1 \qquad \lambda_2 = \lambda_3 = 0 \tag{1.3.56}$$

Equations (1.3.52)–(1.3.56) imply that the physical components of the vector X^j along the parametric line of x^1 are X_1/h_1 or h_1X^1. Considering all geometrical scales of the coordinate system we obtain the following expressions for the physical components of the vector:

$$\frac{X_1}{h_1} \qquad \frac{X_2}{h_2} \qquad \frac{X_3}{h_3} \qquad \text{or} \qquad h_1X^1 \qquad h_2X^2 \qquad h_3X^3 \tag{1.3.57}$$

In order to define the physical components of a second order tensor we apply two unit vectors in the directions of two parametric lines of two coordinates. Such an operation leads to the following expressions for the physical components of the second order tensor, in terms of its covariant components:

$$\begin{matrix} \dfrac{X_{11}}{h_1^2} & \dfrac{X_{12}}{h_1h_2} & \dfrac{X_{13}}{h_1h_3} \\[2ex] \dfrac{X_{21}}{h_2h_1} & \dfrac{X_{22}}{h_2^2} & \dfrac{X_{23}}{h_2h_3} \\[2ex] \dfrac{X_{31}}{h_3h_1} & \dfrac{X_{32}}{h_3h_2} & \dfrac{X_{33}}{h_3^2} \end{matrix} \tag{1.3.58}$$

In terms of the contravariant components of the second order tensor, the physical components of Eq. (1.3.58) are given by

$$\begin{matrix} X^{11}h_1^2 & X^{12}h_1h_2 & X^{13}h_1h_3 \\ X^{21}h_2h_1 & X^{22}h_2^2 & X^{23}h_2h_3 \\ X^{31}h_3h_1 & X^{32}h_3h_2 & X^{33}h_3^2 \end{matrix} \tag{1.3.59}$$

As an example, we calculate the relationships between the Cartesian components of the velocity vector and its contravariant, covariant, and physical components in spherical polar coordinates. The spherical polar coordinates are r, θ, and ϕ, which are referred to, respectively, as x^1, x^2, x^3. These coordinates are related to the Cartesian coordinates z_1, z_2, z_3, by

$$z_1 = x^1 \sin x^2 \cos x^3 \qquad z_2 = x^1 \sin x^2 \sin x^3 \qquad z_3 = x^1 \cos x^2 \quad (1.3.60)$$

Components of the velocity vector in the Cartesian and spherical coordinate systems are given, respectively, by

$$V_i = \frac{dz_i}{dt} \qquad v^i = \frac{dx^i}{dt} \quad (1.3.61)$$

The relationships between the contravariant, contravariant spherical coordinate components and Cartesian components are given by

$$v^i = V_j \frac{\partial x^i}{\partial z_j} \qquad v_i = V_j \frac{\partial z_j}{\partial x^i} \quad (1.3.62)$$

By applying Eq. (1.3.61), we calculate the partial derivatives required by Eq. (1.3.62) and define the relationships between the Cartesian and spherical coordinate components of the velocity vector.

The line element in spherical coordinates is given by

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2 \theta d\phi^2 = (dx^1)^2 + (x^1 dx^2)^2 + (x^1 \sin x^2 dx^3)^2 \quad (1.3.63)$$

This expression indicates that the metric tensor components are

$$g_{11} = 1 \quad g_{22} = r^2 \quad g_{33} = r^2 \sin^2 \theta \quad g_{ij} = 0 \quad \text{for} \quad i \neq j \quad (1.3.64)$$

Equation (1.3.61) specifies the various contravariant components of the velocity vector. By multiplying the contravariant velocity vector by the metric tensor we obtain the covariant components of the velocity vector in the spherical coordinate system. The contravariant and covariant components of this vector are given, respectively, by

$$\begin{aligned} v^1 &= \frac{dr}{dt} \qquad v^2 = \frac{d\theta}{dt} \qquad v^3 = \frac{d\phi}{dt} \\ v_1 &= \frac{dr}{dt} \qquad v_2 = r^2 \frac{d\theta}{dt} \qquad v_3 = r^2 \sin^2 \theta \frac{d\phi}{dt} \end{aligned} \quad (1.3.65)$$

According to Eq. (1.3.64) the geometric scales of the spherical coordinate system are

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r \sin \theta \quad (1.3.66)$$

By applying Eqs. (1.3.61) and (1.3.65) we obtain the following expressions for the physical components of the velocity vector in the spherical coordinate system:

$$v_r = \frac{dr}{dt} \qquad v_\theta = r\frac{d\theta}{dt} \qquad v_\phi = r\sin\theta\frac{d\phi}{dt} \tag{1.3.67}$$

We can identify the principal components of a symmetric tensor, and its principal axes. The symmetric tensor has only diagonal components in a coordinate system comprising its principal axes. These components are called principal components. Basically, the principal components are eigenvalues of the matrix representing the symmetric tensor. The principal axes are represented by a set of unit mutually orthogonal vectors called eigenvectors. The principal components λ_i of the symmetric tensor B_{ij} satisfy the equation

$$\det|B_{ij} - \lambda\delta_{ij}| = 0 \tag{1.3.68}$$

This expression represents a third-order equation whose solution provides values of the principal components λ_1, λ_2, and λ_3.

Each of the three eigenvectors is found by solving the following set of equations:

$$(B_{ij} - \lambda\delta_{ij})b_j = 0 \tag{1.3.69}$$

According to Eq. (1.3.69), each of the principal components λ_k is associated with three components of the relevant eigenvector b^k. If the coordinate system is rotated to coincide with the eigenvectors, then the second-order symmetric tensor B_{ij} is transformed to a diagonal matrix with elements λ_1, λ_2, and λ_3. Available computing libraries that include matrix calculation and linear algebra usually include programs aimed at the identification of eigenvalues and eigenvectors of matrices. Such computer codes can be used to identify the principal components and axes of symmetric tensors.

1.3.2 Complex Variables

Complex Numbers

A *complex number* incorporates a *real* and an *imaginary* part. The Cartesian representation of the complex variable z is

$$z = x + iy \tag{1.3.70}$$

Here, x is the real part and y is the imaginary part. The symbol i is given by

$$i = \sqrt{-1} \qquad i^2 = -1 \tag{1.3.71}$$

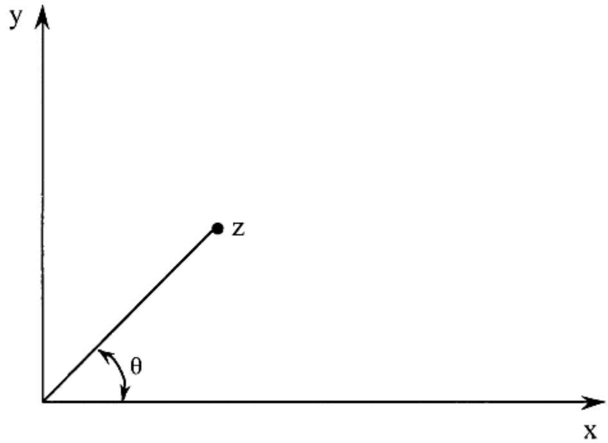

Figure 1.2 Representation of a general complex number, z.

The *Argand diagram* shown in Fig. 1.2 provides a geometric presentation of complex numbers. The length between the coordinate origin and the point represented by z is the *modulus* of the complex variable. It can be represented by r or $|z|$. As shown in Fig. 1.2,

$$|z| = \sqrt{x^2 + y^2} \tag{1.3.72}$$

Also, it is seen that $x = r\cos\theta$ and $y = r\sin\theta$. Therefore the complex variable z can be given by its trigonometric representation as

$$z = r(\cos\theta + i\sin\theta) \tag{1.3.73}$$

Complex variables z_1 and z_2 are added like vectors, i.e., the real part of z_1 is added to the real part of z_2, and the imaginary part of z_1 is added to the imaginary part of z_2. Thus

$$z = z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2) \tag{1.3.74}$$

The factor i is an operator that upon multiplication rotates a complex number through 90°. Powers of i are as follows:

$$i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \tag{1.3.75}$$

Also, the product of two complex variables z_1 and z_2 is

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1) \tag{1.3.76}$$

A complex number also can be expressed in an exponential form. It is based on an infinite series expansion of the exponential and trigonometric functions.

For example, the Maclaurin series expansions for e^x, $\sin x$, and $\cos x$ are given, respectively, by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{1.3.77}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{1.3.78}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{1.3.79}$$

All these series are convergent for all values of x. Replacing x by $i\theta$ in Eq. (1.3.77) and using Eq. (1.3.75), we obtain

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \tag{1.3.80}$$

Or, upon rearranging,

$$e^{i\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \tag{1.3.81}$$

By applying Eqs. (1.3.78) and (1.3.79), Eq. (1.3.81) becomes

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.3.82}$$

All three forms of a complex number are then

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{1.3.83}$$

Following these definitions, the n^{th} power of a complex number is given by

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta) \tag{1.3.84}$$

The product of two complex numbers is

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \tag{1.3.85}$$

and the division of two complex numbers yields

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \tag{1.3.86}$$

Alternatively, the division of two complex variables can be represented by

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \tag{1.3.87}$$

In order to avoid the presence of imaginary terms in the denominator of Eq. (1.3.87), its numerator and denominator have been multiplied by the *complex conjugate* of z_2. The complex conjugate of a complex variable is defined by replacing i by $-i$. Finally, the logarithm of a complex number can be written

$$\ln z = \ln(re^{i\theta}) = \ln r + i\theta = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}\frac{y}{x} \qquad (1.3.88)$$

A function of a complex variable is defined as

$$w = f(z) = f(x + iy) \qquad (1.3.89)$$

The complex function w can be separated into real and imaginary parts, called ϕ and ψ, respectively,

$$w = \phi(x, y) + i\psi(x, y) \qquad (1.3.90)$$

where ϕ and ψ are both real functions of x and y. The function w is called *holomorphic, regular*, or *analytic* in a region, provided that within this region (1) there is one and only one value of w for each value of z and that value is finite, and (2) w has a single-valued derivative at each point of the region.

The derivative of $f(z)$ is also a complex function, given by

$$\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} \qquad (1.3.91)$$

where the infinitesimal value δz is given by

$$\delta z = \delta x + i\delta y \qquad (1.3.92)$$

There is no limitation on the relationship between δx and δy. We may choose paths of $\delta z \to 0$ in which $\delta x = 0$ or $\delta y = 0$, for example. These options imply

$$\lim_{\delta x \to 0;\ \delta y = 0} \frac{f(z + \delta z) - f(z)}{\delta x + i\delta y} = \lim_{\delta x \to 0} \frac{f(z + \delta x) - f(z)}{\delta x} = \frac{\delta f}{\delta x} \qquad (1.3.93)$$

$$\lim_{\delta x = 0;\ \delta y \to 0} \frac{f(z + \delta z) - f(z)}{\delta x + i\delta y} = \lim_{\delta x \to 0} \frac{f(z + i\delta y) - f(z)}{i\delta y} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y} \qquad (1.3.94)$$

As the derivative of the analytic function does not depend on the path of $\delta z \to 0$, Eqs. (1.3.93) and (1.3.94) imply

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \qquad (1.3.95)$$

The derivative of f comprises real and imaginary parts given by

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} \tag{1.3.96}$$

Introducing Eqs. (1.3.90) and (1.3.92) into Eq. (1.3.96), we obtain

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{1.3.97}$$

These relations are called the *Cauchy–Riemann* relations.

Differentiating the first of Eq. (1.3.97) with respect to x and the second with respect to y and adding, and differentiating the first of Eq. (1.3.97) with respect to y and the second with respect to x and subtracting one from the other, we obtain, respectively,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \tag{1.3.98}$$

These expressions indicate that both functions ϕ and ψ satisfy the *Laplace* equation in two-dimensional Cartesian coordinates.

1.3.3 Partial Differential Equations

All basic processes typical of environmental fluid mechanics can be formulated as partial differential equations (PDEs). Partial differential equations arise because the functions for which solutions are sought (e.g., concentrations, velocities, temperature, etc.) tend to depend on one or more spatial coordinates as well as time. As will be seen in subsequent chapters, most equations of interest contain diffusion processes, which involve second-order spatial derivatives. The solution of the relevant differential equation(s) subject to appropriate initial and boundary conditions provides the basis for mathematical simulation of the physical problem. In the following paragraphs, we review the basic types of partial differential equations encountered with environmental fluid mechanics issues.

Identification of the partial differential equation connected with the particular problem of interest is of major importance. Different criteria of convergence and stability are typical of each type of partial differential equation, as described below. The equation provides the basic guideline for the development of a mathematical model that can be applied to the solution of that problem. In cases of numerical simulations, particular rules for the development of the numerical scheme are used for the particular differential equation that is associated with a given problem. Problems of environmental fluid mechanics can be classified into two general categories: problems of equilibrium and problems of propagation.

The general format of a second-order linear PDE in a two dimensional domain is given by

$$a\frac{\partial^2\varphi}{\partial x^2}+b\frac{\partial^2\varphi}{\partial x\partial y}+c\frac{\partial^2\varphi}{\partial y^2}=f \tag{1.3.99}$$

where a, b, and c are constant coefficients and f represents a linear combination of coefficients multiplied by lower order derivatives of the dependent variable φ.

The method and form of the solution of a PDE subject to initial and boundary conditions depends on the type of the PDE. It is common to classify PDEs according to the relationships between the coefficients of Eq. (1.3.99) as follows:

If $b^2-4ac>0$ then the PDE is *hyperbolic* (1.3.100a)

If $b^2-4ac=0$ then the PDE is *parabolic* (1.3.100b)

If $b^2-4ac<0$ then the PDE is *elliptic* (1.3.100c)

The terms hyperbolic, parabolic, and elliptic chosen to classify partial differential equations stems from the analogy between the form of the discriminant (b^2-4ac) for partial differential equations and the form of the discriminant that classifies conic sections. There is no other significance to this terminology. If the PDE refers to a domain with n dimensions, then the characteristics, if real characteristics exist, are surfaces of $(n-1)$ dimensions, along which signals, or information, propagate. If no real characteristics exist, then there are no preferred paths of information propagation. Therefore the existence or absence of characteristics has a significant impact on the solution of the partial differential equation.

First-order partial differential equations refer to advection or convection of a property φ, such as solute concentration or heat. The general form of such an equation in the (x, t) domain, where x refers to a spatial coordinate and t refers to time, is given by

$$\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}=0 \tag{1.3.101}$$

where u is the advection velocity. If φ refers to dissolved mass of a solute, then the second term in Eq. (1.3.101) incorporates the process of solute mass being carried (advected) by a fluid particle as it moves through the domain. The location of any fluid particle is related to its velocity u by a simple relationship representing the differential equation of the particle pathline:

$$\frac{dx}{dt}=u \tag{1.3.102}$$

Thus the pathline of a fluid particle is given by

$$x = x_0 + \int_{t_0}^{t} u\,dt \tag{1.3.103}$$

Along the pathline of the fluid particle the advection equation can be written as

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial t} + \frac{dx}{dt}\frac{\partial \varphi}{\partial x} = \frac{d\varphi}{dt} = 0 \tag{1.3.104}$$

The last part of this equation shows that φ is constant along the pathline of the fluid particle. This pathline is the characteristic path associated with the advection equation. The first-order differential equation of the form given by Eq. (1.3.101) is termed a first-order hyperbolic partial differential equation, and it has a single family of characteristic curves, along which the information propagates in the domain. A single first-order partial differential equation is always hyperbolic. In second-order hyperbolic partial differential equations there are two families of characteristic curves, along which the information propagates.

Parabolic and hyperbolic differential equations are typical of *propagation problems*. The propagation is in time and space. This means that parabolic and hyperbolic differential equations usually refer to problems of a property propagating in the domain. The features of the propagation of the property in cases of parabolic differential equations are different from those of hyperbolic differential equations. Elliptic partial differential equations generally concern equilibrium problems, i.e., ones that do not involve time derivatives.

A typical parabolic equation associated with environmental fluid mechanics is the equation of diffusion. In the (x, t) domain, the form of this equation is given by

$$\frac{\partial \varphi}{\partial t} = \alpha \frac{\partial^2 \varphi}{\partial x^2} \tag{1.3.105}$$

where α is the *diffusion coefficient*, or *diffusivity*. In many applications an advective term is added, forming an advection–diffusion equation (see Chap. 10).

A typical hyperbolic equation associated with environmental fluid mechanics is the *wave equation*. In the (x, t) domain, the form of this equation is given by

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \frac{\partial^2 \varphi}{\partial x^2} \tag{1.3.106}$$

where c is the propagation speed of the wave.

A typical elliptic equation, associated with environmental fluid mechanics is the Laplace equation. In the (x, y) domain, the form of this equation is given by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{1.3.107}$$

The solution of a parabolic or hyperbolic partial differential equation, of the types given by Eqs. (1.3.105) and (1.3.106), can be obtained, provided that adequate initial and boundary conditions are given. Initial conditions refer to values of the unknown variables and possibly their space derivatives at a time of reference. Boundary conditions refer to values of the unknown variables and their space derivatives at the boundaries or other specific locations of the domain. The solution of an elliptic partial differential equation of the type given by Eq. (1.3.107) can be obtained, provided that adequate boundary conditions of the domain are given. For elliptic partial differential equations there are no initial conditions, since time derivatives are not involved.

There are three types of linear boundary conditions that can be applied to the solution of partial differential equations:

1. All values of the dependent variable, φ, are specified on the boundaries of the domain:

$$\varphi = f(x, y) \qquad \text{where} \qquad (x, y) \in G \tag{1.3.108}$$

where G is the surface of the domain. Boundary conditions of this type are referred to as *Dirichlet* boundary conditions.

2. All values of the gradient of the dependent variable, φ, are specified on the boundaries of the domain:

$$\frac{\partial \varphi}{\partial n} = f(x, y) \qquad \text{where} \qquad f(x, y) \in G \tag{1.3.109}$$

where n represents a coordinate normal to the boundary G. Boundary conditions of this type are referred to as *Neumann* boundary conditions.

3. A general linear combination of Dirichlet and Neumann boundary conditions:

$$a\varphi + b\frac{\partial \varphi}{\partial n} = c \tag{1.3.110}$$

where a, b, and c are functions of (x, y). This type of boundary condition can be used to specify total flux, as will be described in later chapters.

It should be noted that besides linear boundary conditions, the domain may be subject to nonlinear boundary conditions. An example is application of boundary conditions at a water-free surface, which may be part of the solution of the problem. Application of such conditions is generally very complicated.

1.4 DIMENSIONAL REASONING

1.4.1 Uses of Dimensional Analysis

Dimensional analysis provides a powerful tool to evaluate relationships between various parameters of a problem when the governing equation is not known from some other source, such as a theoretical result. The basic premise underlying any dimensional reasoning is that all physically realistic expressions must be dimensionally consistent. In fact, the *Buckingham π theorem*, introduced in the following section, can be seen as a formal statement of a relationship between variables based simply on their dimensional units. Following this idea, any physical equation that is dimensionally balanced (that is, the dimensional units are the same for each of the terms in the equation) can be written in nondimensional form. The easiest way to see this is to divide all the terms of the equation by one of the terms. Done properly, this usually results in equations expressed in terms of common *dimensionless parameters*. Since all the terms have the same physical dimensions, the result of this process is a relationship between these dimensionless variables, which can be used to evaluate the relative importance of different terms in any given equation. For example, it would be possible to gain some understanding of the relative importance of different forces in a particular flow field by looking at the values of the parameters in dimensionless forms of the momentum equations. This process sometimes allows simplification of a general governing equation, by eliminating terms that are seen as being of lesser importance, compared with others.

A common example of a dimensionless number is the *Reynolds number*, defined as

$$\mathrm{Re} = \frac{UL}{\nu} \tag{1.4.1}$$

where U is a characteristic velocity and L is a characteristic length of the problem being studied, and ν is kinematic viscosity of the fluid. Re represents the relative importance of inertia to viscous forces. For example, a high value of Re indicates that viscous forces are not very important. (As will be seen later, a high Re is associated with *turbulent* flow.)

The result of dimensional analysis is a definition of a relationship between the appropriate dimensionless variables resulting from grouping the parameters of the problem. The specific form of the relationship is not revealed using dimensional analysis — physical experiments must be performed to provide additional information. For example, dimensional analysis can be used to show that a dimensionless group incorporating the drag on a sphere moving at constant velocity through a fluid should depend on Re. However, the actual form of the relationship is determined from experimental results.

One other important application of dimensional analysis is in providing a means of scaling the results of a model study to prototype conditions. This is necessary, for instance, in extrapolating results from laboratory physical modeling studies to field conditions. In order to do this, conditions of *similarity* must be satisfied. There are three kinds of similarity. Intuitively, a model or experiment should be *geometrically* similar to the field situation, which means that the ratio of all length scales is the same between the model and the prototype. *Kinematic* similarity incorporates similarity of length and time quantities. *Dynamic* similarity also must be satisfied in order to properly scale results concerning forces and stresses. Kinematic and dynamic similarity are obtained when appropriate dimensionless parameters are the same in the model and in the prototype. Dynamic similarity is equivalent to saying the ratios of relevant forces are the same.

For example, consider an open channel flow, as sketched in Fig. 1.3. For simplicity, we assume a rectangular cross section of width b and flow depth h. Geometric similarity implies

$$L_{\mathrm{r}} = \frac{L_1}{L_2} \tag{1.4.2}$$

where L_{r} is the length scale ratio and L represents any length for the problem, in this case either b or h. Subscripts 1 and 2 refer to the two systems (prototype and model — expressing the ratio in this way avoids very small values for L_{r}). Thus $h_1/h_2 = b_1/b_2$ and $h_1/b_1 = h_2/b_2$ (i.e., the flow aspect ratio is the same in the two systems). In some cases *distorted scale* models are necessary,

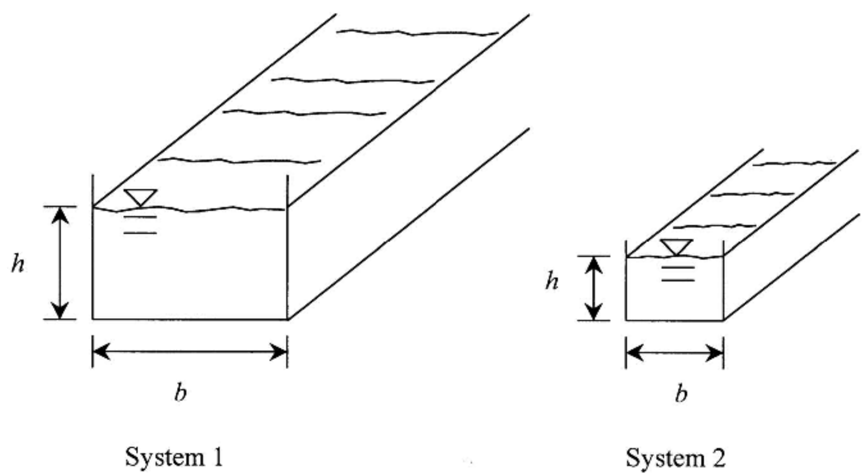

Figure 1.3 Open channel flow in two geometrically similar rectangular channels.

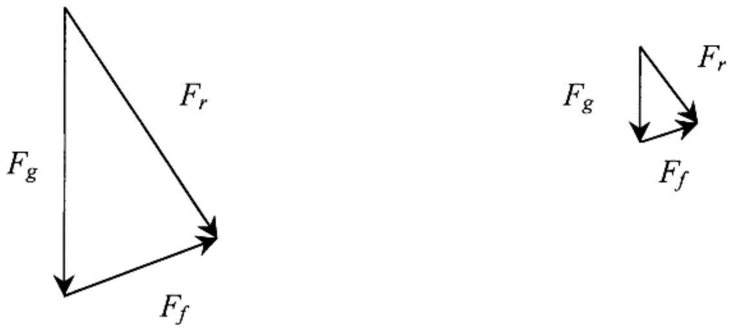

Figure 1.4 Force diagrams for the two systems shown in Fig. 1.3; F_g is the gravity force, F_f is friction, and F_r is the resultant force.

such as when a physical model of a large lake is used. In this case, because the horizontal dimensions are generally much greater than the vertical scale, the horizontal scale ratio is chosen to be much larger than the vertical ratio. This is to avoid models with extremely shallow water layers. Scaling is otherwise similar to that in nondistorted models; one should be careful to maintain common values of the relevant dimensionless parameters between the model and prototype.

If we consider a small fluid element in either system and assume that the important forces for this problem are gravity and friction, the resultant forces on the fluid element can be calculated, and we obtain force diagrams like those in Fig. 1.4. The shapes of these force diagrams must be similar for the two systems if there is dynamic similarity. As shown in the following section, this condition is satisfied when the corresponding values of properly defined dimensionless variables are the same.

1.4.2 Dimensionless Parameters

Buckingham π Theorem

The Buckingham π theorem states that a group of physical variables defined for a given problem may be combined in such a way as to form a non-dimensional representation of the same problem. Moreover, since the original variables are functionally related, i.e.,

$$f(x_1, x_2, \ldots, x_n) = 0 \quad (1.4.3)$$

where the $x_1, x_2, \ldots, x_n$ represent the n physical variables of a problem, then the nondimensional variables also are functionally related. If there are k physical dimensional units involved with the n variables, then (n-k) dimensionless parameters should be formed, and

$$f(\pi_1, \pi_2, \ldots, \pi_{(n-k)}) = 0 \tag{1.4.4}$$

where the $\pi_1, \pi_2, \ldots, \pi_{(n-k)}$ are the dimensionless groupings.

Consider the force diagrams indicated in Fig. 1.4. The shapes will be the same when the ratios of any two of the forces are the same in each system. The resultant or inertial force is represented by $(\rho \forall U/t)$, where ρ is the fluid density, $\forall$ is the volume of a fluid element, U is its velocity, and t is some appropriate time scale. Here, (U/t) has been used to approximate acceleration, and t will be estimated as $t = L/U$, where L is a characteristic length scale. The viscous force, acting on area A, is approximately $(\mu U/L)(A)$, where (U/L) has been used to estimate the velocity gradient. Substituting L^3 for $\forall$ and L^2 for A, the ratio of inertial to viscous force is then

$$\frac{\rho(L^3)U^2/L}{\mu(U/L)L^2} = \frac{\rho LU}{\mu} = \mathrm{Re} \tag{1.4.5}$$

In other words, this is the Reynolds number as defined in Eq. (1.4.1). By going through a similar procedure for the ratio of inertial to gravity force, where the force of gravity F_g is approximated by $\rho \forall g$, we obtain

$$\frac{\rho L^2 U^2}{\rho L^3 g} = \mathrm{Fr} \tag{1.4.6}$$

which defines Fr as a second dimensionless parameter, the *Froude number*.

Thus by insuring that the Reynolds numbers and the Froude numbers are the same for both systems, the shape of the resulting force diagrams will be the same, and dynamic similarity will be achieved. This type of reasoning may be applied to problems with a greater number of relevant forces, with the result that additional dimensionless parameters would need to be defined. Many different dimensionless parameters have been defined for various problems in fluid mechanics. Rather than attempting to list them all here, we shall introduce them as needed within the context of a given problem or derivation.

In order to illustrate the application of the Buckingham π theorem, let us consider the problem of finding the drag on a smooth sphere fully immersed and moving at constant velocity through a fluid. It is assumed that the drag is a function of the velocity and diameter of the sphere, and the density and viscosity of the fluid. Note that one limitation of the Buckingham π theorem is that it does not provide specific guidance on which parameters should be chosen for a given problem. These must be chosen on the basis of experience and physical intuition, with perhaps some trial and error to be expected in some cases. Usually, it will be clear when the wrong set of parameters is chosen, since it will be difficult to perform experimental tests to obtain a clear relationship between the dimensionless parameters defined. For the present problem, a functional relationship is defined by

$$f(D, U, d, \rho, \mu) = 0 \tag{1.4.7}$$

where D is drag, U is velocity, d is the sphere diameter, and ρ and μ are fluid density and viscosity, respectively. There are five variables and three physical dimensional units, mass (M), length (L), and time (T), so two π's will be defined. First, a subset of variables is defined, called the *basis set*, with the following characteristics:

- The number of variables in the basis set is equal to the number of physical dimensions.
- All the dimensions of the problem are represented by the variables, as simply as possible.
- Variables are chosen so that recognizable dimensionless groupings are found.
- The main parameter of interest (the dependent variable) is not chosen for the basis.

The third of these conditions is not absolutely necessary, but it usually helps to interpret results, particularly in view of the above interpretation of many of these dimensionless groups as force ratios. In many cases there is not a unique basis set, and different basis sets will result in definitions of different sets of dimensionless numbers. This is acceptable, from a purely dimensional analysis point of view, but is it preferable to form common dimensionless groupings whenever possible.

For the current example, U, d, and ρ are chosen as the basis variables. These are combined with D and μ, in turn, to form two π groups. The first of these is found from

$$\pi_1 = D(U)^a (d)^b (\rho)^c = \left(\frac{ML}{T^2}\right)\left(\frac{L}{T}\right)^a (L)^b \left(\frac{M}{L^3}\right)^c$$

Separate equations are then formed for each of the dimensional units, to find a, b, and c so that π_1 is dimensionless. For mass M, $(1 + c = 0)$ gives $c = -1$. The equation for time T is $(-2 - a = 0)$, or $a = -2$. The last equation for length L gives $(1 + a + b - 3c = 0)$, or $(a + b = -4)$. Then $b = -2$ and

$$\pi_1 = \frac{D}{\rho U^2 d^2}$$

This is a dimensionless drag and is commonly referred to as a *drag coefficient*, C_D. Following a similar procedure using μ, it is easily shown that a Reynolds number results for π_2. The dimensionless result analogous to

Eq. (1.4.7) is then

$$f(C_D, \mathrm{Re}) = 0 \tag{1.4.8}$$

and experimental results are needed to describe the specific form of the functional relationship. It should be noted that an equally valid dimensional analysis result would be obtained by using one of the π's raised to some power, or using the inverse. However, use of common dimensionless parameters is preferred, as noted above.

1.4.3 Scales of Motion

It is evident from the above discussion that it is necessary to define certain parameters of a problem in order to carry out dimensional analysis. A similar requirement is to define certain characteristic scales to represent a problem, in order to use dimensional reasoning to carry out scaling analyses. The usual scales of interest for kinematic problems are those for length, velocity, or time, though other types of parameters are sometimes needed. Often the choices for these scales are obvious. A simple example is with open channel flow, where the flow mean velocity is usually chosen as the velocity scale, and depth (or hydraulic radius) is chosen as the characteristic length scale. The choice for these scales determines values for the nondimensional variables discussed in the previous section, so some care must be taken. As shown in Sec. 2.7, one of the principal applications of scaling analysis is in developing an understanding of the relative importance of the various terms of a relationship, with a view to simplifying the equation whenever possible. In addition to possibly simplifying the equation, the main advantage of developing nondimensional forms of equations is that the actual scale becomes secondary — it is only the dimensionless groups that are important. Nondimensional equations and parameters apply equally to systems with very different scales (e.g., values of L and U), as long as the values of the dimensionless groupings are similar. This idea forms the basis for physical modeling tests and provides the means for scaling model results to estimate prototype conditions.

PROBLEMS

Solved Problems

Problem 1.1 The *material* or *substantial derivative* of the velocity vector represents the acceleration of the fluid particles. In Cartesian coordinates the acceleration is expressed by

$$a_i = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

where u_i are components of the velocity vector. What is the expression for the acceleration in a general coordinate system? What are the expressions for the contravariant, covariant, and physical components of the acceleration in a cylindrical coordinate system?

Solution

The expression for the contravariant acceleration vector is

$$a^i = \frac{\partial u^i}{\partial t} + u^j u^i_{,j}$$

Multiplying this expression by the metric tensor and replacing indices, we obtain the following expression for the covariant acceleration vector:

$$a_i = \frac{\partial u_i}{\partial t} + u^j u_{i,j}$$

In a cylindrical coordinate system the physical components of the velocity and acceleration vectors are given by the following symbols, respectively:

u, v, w (physical components in directions r, θ, z, respectively)

a_r, a_θ, a_z

The line element in a cylindrical coordinate system is given by

$$ds^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2$$

Therefore components of the metric tensor and geometrical scales in the r, θ, and z directions are given by

$$g_{11} = 1 \qquad g_{22} = r^2 \qquad g_{33} = 1$$

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = 1$$

By applying Eq. (1.3.57), the following relationships between physical, covariant, and covariant components of the velocity and acceleration vectors are obtained:

$$u = u_1 = u^1 \qquad v = \frac{u_2}{r} = ru^2 \qquad w = u_3 = u^3$$

$$a_r = a_1 = a^1 \qquad a_\theta = \frac{a_1}{r} = ra^1 \qquad a_z = a_3 = a^3$$

By applying the general expressions for Christoffel symbols given by Eqs. (1.3.33) and (1.3.34), we obtain values of the second symbols of Christoffel. The only nonzero symbols in a cylindrical coordinate system are

$$\sum_{22}^{1} = -r \qquad \sum_{12}^{2} = \sum_{21}^{2} = \frac{1}{r}$$

We apply the general expression for the contravariant acceleration vector with these expressions for the second symbols of Christoffel to obtain

$$a^1 = \frac{\partial u^1}{\partial t} + u^1 \frac{\partial u^1}{\partial r} + u^2 \left(\frac{\partial u^1}{\partial \theta} - r u^2 \right) + u^3 \frac{\partial u^1}{\partial z}$$
$$a^2 = \frac{\partial u^2}{\partial t} + u^1 \left(\frac{\partial u^2}{\partial r} + \frac{u^2}{r} \right) + u^2 \left(\frac{\partial u^2}{\partial \theta} + \frac{u^1}{r} \right) + u^3 \frac{\partial u^2}{\partial z}$$
$$a^3 = \frac{\partial u^3}{\partial t} + u^1 \frac{\partial u^3}{\partial r} + u^2 \frac{\partial u^3}{\partial \theta} + u^3 \frac{\partial u^3}{\partial z}$$

By introducing the physical components of the velocity and acceleration vectors into these expressions we obtain

$$a_r = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} + w \frac{\partial u}{\partial z}$$
$$a_\theta = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} + w \frac{\partial v}{\partial z}$$
$$a_z = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} + w \frac{\partial w}{\partial z}$$

Problem 1.2 Develop the expression for div $\vec{V}$ in cylindrical coordinates by applying the contravariant as well as covariant components of the velocity vector $\vec{V}$.

Solution

The general required expressions for div $\vec{V}$ are

$$\nabla \cdot \vec{V} = u^i_{,i} = g^{ij} u_{i,j}$$

The expression with contravariant components of the velocity vector is

$$u^i_{,i} = \frac{\partial u^i}{\partial x^i} + \sum_{ji} u^j = \frac{\partial u^1}{\partial r} + \frac{\partial u^2}{\partial \theta} + \frac{u^1}{r} + \frac{\partial u^3}{\partial z} = \frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} + \frac{\partial w}{\partial z}$$

The expression with covariant components of the velocity vector is

$$g^{ij} u_{i,j} = g^{ij} \frac{\partial u_i}{\partial x^j} - g^{ij} \sum_{ij}^{k} u_k = \frac{\partial u_1}{\partial r} + \frac{1}{r^2}\frac{\partial u_2}{\partial \theta} + \frac{u_1 r}{r^2} + \frac{\partial u_3}{\partial z}$$
$$= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$$

Problem 1.3 Prove the following relationships:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sinh ix = i \sin x$$

Solution

We apply the Euler relationships,

$$e^{ix} = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x$$

Introducing these expressions into the expressions for $\sin x$ and $\cos x$, we obtain the relationships written above. Introducing the explicit expression for $\sinh ix$, we obtain the last identity.

Problem 1.4 Find the complex numbers given by

$$\text{(a)}\ \ (2+i)(3-2i) \quad \text{(b)}\ \ \frac{1+3i}{1-i} \quad \text{(c)}\ \ \ln(3+4i)$$

Solution

(a) $(2+i)(3-2i) = 6 + 3i - 4i + 2 = 8 - i$

(b) $\dfrac{1+3i}{1-i} = \dfrac{(1+3i)(1+i)}{(1-i)(1+i)} = \dfrac{1+3i+i-3}{1-i+i+1} = \dfrac{-2+4i}{2} = -1+2i$

(c) $\ln(3+4i) = \dfrac{1}{2}\ln(3^2+4^2) + i\tan^{-1}\dfrac{4}{3} = 1.61 + i0.93$

Problem 1.5 Separate the following functions of z into their real and imaginary parts ϕ and ψ:

$$\text{(a)}\ \ \frac{1}{z} \quad \text{(b)}\ \ \ln z^2 \quad \text{(c)}\ \ e^{iz}$$

Solution

(a) $\dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{(x+iy)(x-iy)} = \dfrac{x-iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

Therefore

$$\phi = \frac{x}{x^2+y^2}; \ \psi = \frac{y}{x^2+y^2}$$

(b)
$$\ln z^2 = \ln[(x+iy)(x+iy)] = \ln(x^2+y^2+i2xy)$$
$$= \frac{1}{2}\ln[(x^2+y^2)^2 + 4x^2y^2] + i\tan^{-1}\frac{2xy}{x^2+y^2}$$

Therefore

$$\phi = \frac{1}{2}\ln[(x^2 + y^2)^2 + 4x^2y^2]; \psi = \tan^{-1}\frac{2xy}{x^2 + y^2}$$

(c) $e^{iz} = \exp[i(x + iy)] = \exp(ix - y) = e^{-y}e^{ix} = e^{-y}(\cos x + i\sin x)$

Problem 1.6 Consider the problem of dumping sewage from a barge into a linearly stratified ocean, as illustrated in Fig. 1.5.

A volume V_s of sludge of density ρ_s is released suddenly from the barge into water of density ρ_0 and density gradient $(-d\rho_a/dz)$. Find the maximum depth of penetration, d_{max}, the minimum dilution at that depth, and the time of descent. (Note that the sludge cloud seeks its density equilibrium position, which also depends on entrainment.)

Solution

First, we define

$S =$ (total sample volume)/(volume of effluent in sample)

$P = 1/S =$ volume fraction of effluent(= relative concentration)

A definition of dilution is

$$D = \frac{1 - P}{P}$$

where $D =$ (volume ambient water in sample)/(volume effluent in sample)= $S - 1$.

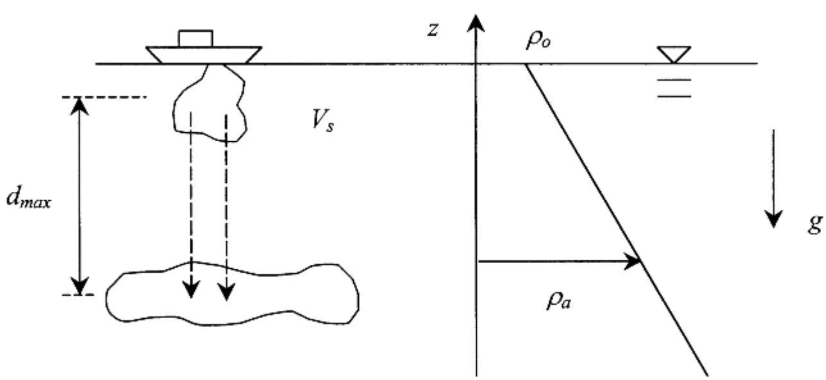

Figure 1.5 Definition sketch, Problem 1.6.

The variables of the problem are

$$d_{\max} \qquad \forall s \qquad g \qquad \rho_0 \qquad \rho_s \qquad -d\rho_a/dz$$

with corresponding units

$$(L) \qquad (L^3) \qquad (L/T^2) \qquad (M/L^3) \qquad (M/L^3) \qquad (M/L^4)$$

According to the Buckingham π theorem, with six variables and three dimensional quantities, there should be three dimensionless groupings. However, it is usually more convenient to work with one or two dimensionless groups, for ease of analysis. Therefore, define

$$\Delta\rho = \rho_s - \rho_a$$

and

$$V_s g \Delta\rho = \text{submerged weight of sludge}$$

It also will be convenient to combine g and $(-d\rho_a/dz)$. Then we have for variables

$$d_{\max} \qquad \forall_s g \Delta\rho \qquad -g d\rho_a/dz \qquad \rho_0$$

the corresponding units

$$(L) \qquad (ML/T^2) \qquad (M/L^3T^2) \qquad (M/L^3)$$

There are now four variables and three dimensions, so only one dimensionless grouping (π) is needed. If we now set

$$\begin{aligned}\pi &= (d_{\max})^a (\forall_s g \Delta\rho)^b \left(-g\frac{d\rho_a}{dz}\right)^c (\rho_0)^d \\ &= (L)^a \left(\frac{ML}{T^2}\right)^b \left(\frac{M}{L^3T^2}\right)^c \left(\frac{M}{L^3}\right)^d\end{aligned}$$

and solve individually for each of the power coefficients,

$$\begin{aligned}&(\text{M}):\ b = -c - d \\ &(\text{L}):\ a + b - 3c - 3d = 0 \\ &(\text{T}):\ b = -c\end{aligned}$$

then we can solve for the power coefficients to define π. However, we have four power coefficients and only three equations. Therefore it is necessary to set the value for one of the power coefficients arbitrarily. Anticipating the desired result, we set $c = 1$ and solve for the remaining values based on this assumption. Note that an equally valid result could be obtained starting with other values for c. Also, from examination of the equations for M and T, it is seen that the only solution that can satisfy both is when $d = 0$; however, again anticipating the result, we include ρ_o, twice, so that it cancels. This is because of the desire to develop a solution that has recognizable parameters. The final result is

$$\pi = (d_{\max})^4 \frac{\left(-\dfrac{g}{\rho_0}\dfrac{d\rho_a}{dz}\right)}{\left(V_s g \dfrac{\Delta\rho}{\rho_0}\right)}$$

Note that ρ_0 cancels in the numerator and denominator, so that effectively $d = 0$. It is also evident that g cancels, but it, too, is kept because of conventional definitions. For example, the square root of the term in parentheses in the numerator is called the *buoyancy frequency*, N, and the term in the denominator is the buoyancy force per unit mass acting on the submerged sludge. If we further define $g' = g\Delta\rho/\rho_0 =$ *reduced gravity*, and note that, since there is only one π for this problem, then it must equal a constant (say A^4), then the final result is

$$d_{\max} = A\left(\frac{\forall_s g'}{N^2}\right)^{1/4}$$

Now, if experiments are done, measuring $d_{\max}$ while varying the other parameters in this expression, then a plot of $d_{\max}$ versus $(V_s g'/gN^2)^{1/4}$ should result in a straight line with slope corresponding to the value for A, such as is illustrated in Fig. 1.6.

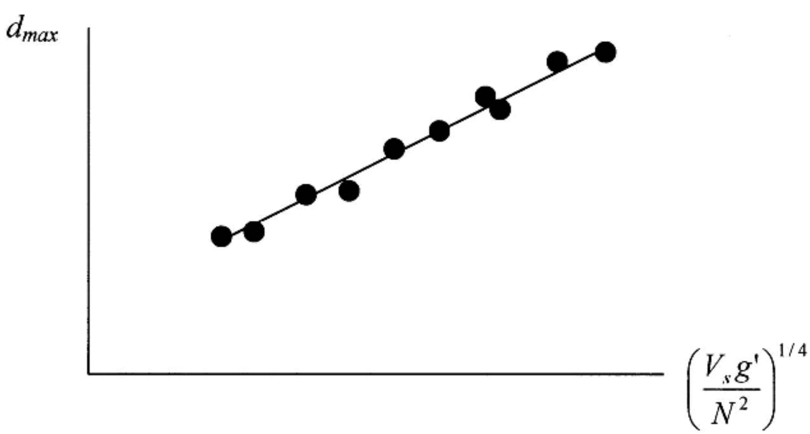

Figure 1.6 Variation of $d_{\max}$, Problem 1.6.

From experimental studies, the value of A is found to be approximately 2.66. The remainder of this problem solution is left as an exercise for the student.

Unsolved Problems

Problem 1.7 The expression for the vorticity (ω_{ij}) tensor is

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$$

where the u_i are components of the velocity vector. Find the components of the vorticity tensor in cylindrical and spherical coordinates.

Problem 1.8 The divergence of a second-order tensor is a vector expressed as

$$(\nabla \cdot \stackrel{\approx}{B})_i = B^j_{i,j}$$

Find the expressions for the components of div$B$ in Cartesian, cylindrical, and spherical coordinates.

Problem 1.9 The stress tensor for a Newtonian incompressible fluid is given by

$$\tau_{ij} = -pg_{ij} + \mu(u_{i,j} + u_{j,i})$$

where p is the pressure, μ is the fluid viscosity, and the u_i are components of the velocity vector. Find expressions for components of τ_{ij} and div τ in Cartesian and cylindrical coordinates.

Problem 1.10 Prove the following expressions by using indicial notation:

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})$$

$$\vec{V} \cdot \nabla \vec{V} = \nabla \frac{|V|^2}{2} - \vec{V} \times \nabla \times \vec{V}$$

Problem 1.11 Prove the vector identity

$$\nabla^2 \vec{V} = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{V})$$

Problem 1.12 How many separate quantities are represented by each of the following expressions?

(a) $\varepsilon_{ijk}\dfrac{\partial u_k}{\partial x_j}$ (b) $\dfrac{\partial u_j}{\partial x_j}$

(c) $\dfrac{\partial^2 u_i}{\partial x_j^2}+\dfrac{1}{3}\dfrac{\partial^2 u_j}{\partial x_i \partial x_j}$ (d) $\dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j}+\dfrac{\partial u_j}{\partial x_i}\right)$

Problem 1.13 Use the properties of the alternating tensor ε_{ijk} to prove the vector identity

$$\overrightarrow{\nabla}\cdot(\overrightarrow{\nabla}\times\overrightarrow{V})=0 \qquad \text{for } \overrightarrow{V} = \text{any vector}$$

Problem 1.14 Find the complex numbers given by

(a) $5e^{i}+3e^{1.5i}$ (b) $(1+i)(2-i)(1+3i)$

(c) $\ln(i)$ (d) $\ln(-1)$

Problem 1.15 Separate the following functions into their real and imaginary parts:

(a) $\dfrac{z}{\tilde{z}}$ where $\tilde{z}=x-iy$ (b) $\dfrac{z-\tilde{z}}{z+\tilde{z}}$

(c) $\tilde{z}+\dfrac{1}{z}$ (d) $\ln\left(\dfrac{1}{z}\right)$ (e) $\tilde{z}^2 z$

Problem 1.16 Show that Cauchy–Riemann relations in two-dimensional cylindrical coordinates are

$$\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}$$

Problem 1.17 Which of the following functions are analytic functions?:

(a) $r\cos\dfrac{\theta}{2}+ir\sin\dfrac{\theta}{2}$ (b) $\sqrt{r}\cos\dfrac{\theta}{2}+i\sqrt{r}\sin\dfrac{\theta}{2}$

(c) $\dfrac{1}{x^2}+i\dfrac{1}{y^2}$ (d) $\dfrac{x^2+y^2}{x-iy}$

(e) $\dfrac{x}{x^2+y^2}-i\dfrac{y}{x^2+y^2}$ (f) $x^2-y^2-x+i(2xy-y)$

Problem 1.18 Determine the derivatives of the following analytic functions and separate the derivatives into their real and imaginary parts:

(a) $w = (1+i)\ln z$ (b) $z = \ln w$

(c) $w = \dfrac{i}{z} + z$ (d) $w = z^2 + iz$

(e) $w = \sqrt{z}$ (f) $w = \ln z + z$

Problem 1.19 Prove the following identities:

(a) $\cosh ix = \cos x$

(b) $\sinh z = \sinh x \cos y + i \sinh x \sin y$

(c) $\sin z = \sin x \cosh y + i \cos x \sinh y$

(d) $\cos z = \cos x \cosh y - i \sin x \sinh y$

Problem 1.20 Assuming that the drag (D) experienced by an object moving through a fluid is a function of its projected area (A) in the direction of motion, its velocity (V), and the density (ρ) and viscosity (μ) of the fluid, develop a dimensionless relationship to show how the drag should be related to the other variables of the problem.

Problem 1.21 Use dimensional analysis to develop an expression for the vertical velocity (w) produced in a container of a fluid of depth h, when heated from below with input power P (= energy input per unit time, $\mathrm{ML^2/T^3}$). Assume that w is a function of h and P, as well as fluid density (ρ, $\mathrm{M/L^3}$), thermal expansion coefficient (α, $1/T$), and specific heat (c, energy per unit mass, per unit temperature, $\mathrm{L^2/T^2}\theta$). To simplify, combine α, ρ, and c as $(\alpha/\rho c)$.

Problem 1.22 It is desired to formulate an expression to predict the mixing generated by wind blowing over a stratified water body, as shown in Fig. 1.7. Specifically, the wind transfers energy into the water by a surface shear stress, which may be characterized by the friction velocity, $u* = (\tau/\rho_0)^{1/2}$. Part of this energy is used to mix fluid across the density interface, resulting in a deepening of the upper layer. Formulate a nondimensional expression that could be used to relate the entrainment velocity, $u_\mathrm{e} = dh/dt$, to other variables of the problem (remember to include g). The result should be written in terms of the *bulk Richardson number*,

$$\mathrm{Ri} = \frac{g'h}{u_*^2}$$

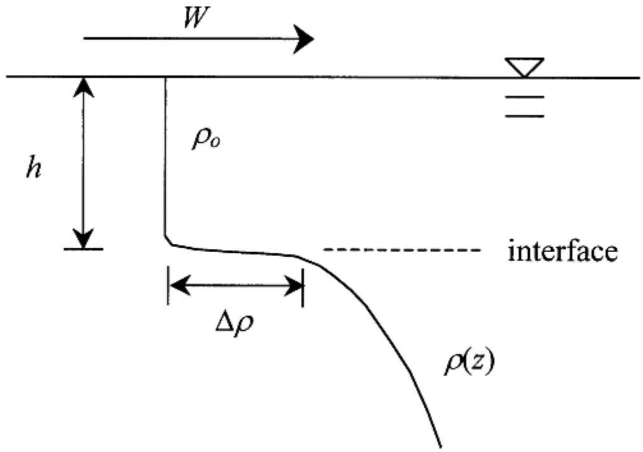

Figure 1.7 Mixed layer structure, Problem 1.22.

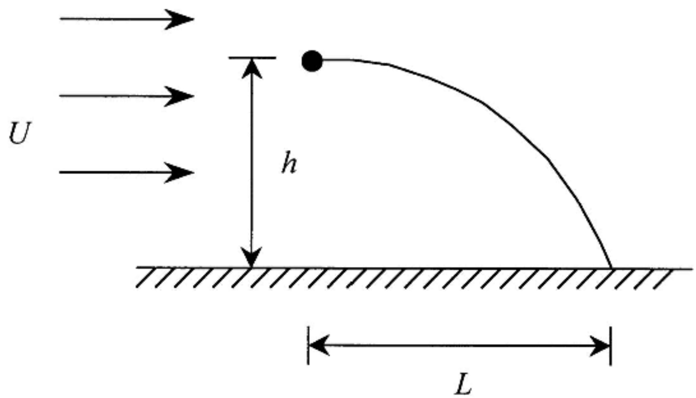

Figure 1.8 Definition sketch, Problem 1.23.

Problem 1.23 A 1 : 10 scale model is to be used to test the distance (L) a sphere of diameter (d) will travel when released at a height (H) in a fluid stream moving at velocity (U) (see Fig. 1.8). It is assumed that L is a function of these other variables, as well as the fluid viscosity and specific weight, i.e., $L = f(H, d, U, \gamma, \mu)$. The model and prototype viscosities are the same, but the model specific weight is nine times the specific weight of the prototype.

(a) Determine an appropriate set of dimensionless parameters to characterize this problem.

(b) If the prototype velocity is 50 mph, what should be the model velocity?

(c) If L is measured for a particular test with the model to be 0.1 m, what would the corresponding L be for the prototype?

SUPPLEMENTAL READING

Aris, R., 1962. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Prentice-Hall, Englewood Cliffs, New Jersey. (This book provides an easy treatment of tensors and their application to fluid mechanics.)

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Rouse, H., and Ince, S., 1957. *History of Hydraulics*. Iowa Institute of Hydraulic Research, State University of Iowa.

Sommerfeld, A., 1964. *Mechanics of Deformable Bodies*. Academic Press, New York. (Contains coverage of Cartesian tensors.)

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