2
Fundamental Equations

2.1 INTRODUCTION

The basic equations of fluid mechanics are derived by considering conservation statements (i.e., of mass, momentum, energy, etc.) applied to a finite volume of fluid continuum which is called a *system* or *material volume* and consists of a collection of infinitesimal fluid particles. Quantities involving space and time only are associated with the *kinematics* of the fluid particles. Examples of variables related to the kinematics of the fluid particles are displacement, velocity, acceleration, rate of strain, and rotation. Such variables represent the motion of the fluid particles, in response to applied *forces*. All variables connected with these forces involve space, time, and mass dimensions. These are related to the *dynamics* of the fluid particles.

In the following sections of this chapter we provide information concerning the basic representation of kinematic and dynamic variables and concepts associated with fluid particles and fluid systems.

2.2 FLUID VELOCITY, PATHLINES, STREAMLINES, AND STREAKLINES

A *pathline* represents the trajectory of a fluid particle. At a time of reference t_0, consider a fluid particle to be at position $\vec{r}_0$. In Cartesian coordinates this location is represented by (x_0, y_0, z_0). Due to its motion, the fluid particle is at position $\vec{r}$ at time t, and this new position is represented by coordinates (x, y, z). The functional representation of the pathline is given by

$$\vec{r} = \vec{r}(\vec{r}_0, t) \qquad \text{or} \qquad \vec{x} = \vec{x}(\vec{x}_0, t) \tag{2.2.1}$$

The vector $\vec{r}_0$ (or $\vec{x}_0$) represents the *label* of the particular fluid particle. The concept of pathline is a basic feature of the *Lagrangian* approach, which is explained in greater detail in Sec. 2.4.

As an example of the pathline concept, consider the following description of pathlines in a two-dimensional flow field:

$$x = x_0 e^{-at} \qquad y = y_0 e^{at} \tag{2.2.2}$$

It is possible to eliminate t from these expressions and obtain an equation describing the shape of the pathline in the $x-y$ plane, as

$$xy = x_0 y_0 \tag{2.2.3}$$

This expression shows that pathlines are hyperbolas whose asymptotes are the coordinate axes.

By differentiating the equation of the pathline with regard to time we obtain the Lagrangian expressions for the velocity components. By further differentiating the latter expressions with regard to time, we obtain the Lagrangian expressions for the acceleration components:

$$\vec{V} = \vec{V}(\vec{r}_0, t) = \frac{\partial \vec{r}}{\partial t} \qquad \vec{a} = a(\vec{r}_0, t) = \frac{\partial^2 \vec{r}}{\partial t^2} \tag{2.2.4}$$

For the example pathlines of Eq. (2.2.2), the Lagrangian velocity components are

$$u(x_0, y_0, t) = -ax_0 e^{-at} \qquad v(x_0, y_0, t) = ay_0 e^{at} \tag{2.2.5}$$

By eliminating x_0 and y_0 from Eq. (2.2.5), we obtain the *Eulerian* presentation (which will be discussed hereinafter) of the velocity components,

$$u(x, y, t) = -ax \qquad v(x, y, t) = ay \tag{2.2.6}$$

The Eulerian presentation is the most common way of describing a flow field, where a spatial distribution of velocity values is given (note that velocities do not depend on an initial position in this presentation). It should be further noted that the pathline equation given by Eq. (2.2.2) can be obtained by direct integration of Eq. (2.2.5) or integration of Eq. (2.2.6), while considering that $x = x(x_0, y_0, t)$; $y = y(x_0, y_0, t)$.

By differentiation of Eq. (2.2.5) with regard to time, we obtain the Lagrangian presentation of the acceleration component,

$$a_x(x_0, y_0, t) = a^2 x_0 e^{-at} \qquad a_y(x_0, y_0, t) = a^2 y_0 e^{at} \tag{2.2.7}$$

Again, by eliminating x_0 and y_0 from Eq. (2.2.7), the Eulerian presentation of the acceleration components is

$$a_x(x, y, t) = a^2 x \qquad a_y(x, y, t) = a^2 y \tag{2.2.8}$$

Flow fields are often depicted using *streamlines*. Streamlines are curves that are everywhere tangent to the velocity vector, as shown in Fig. 2.1. A

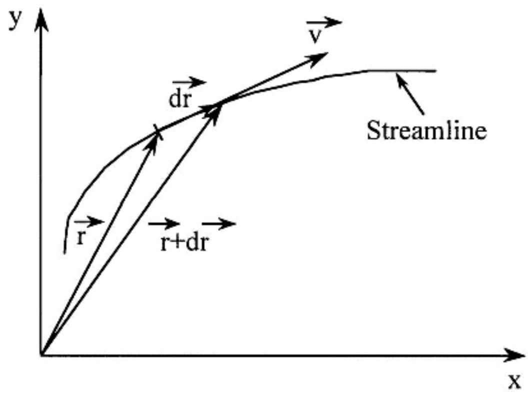

Figure 2.1 Example of streamline.

streamline is associated with a particular time and may be considered as an instantaneous "photograph" of the velocity vector directions for the entire flow field.

As implied in Fig. 2.1 (since the streamlines are tangent to the velocity), a streamline may be described by

$$\vec{V} \times d\vec{r} = 0 \qquad \text{where} \qquad \vec{V} = \vec{V}(\vec{x}, t) \tag{2.2.9}$$

where $\vec{V}$ is the velocity vector, $d\vec{r}$ is an infinitesimal element along the streamline, and $\vec{x}$ is the coordinate vector. In a Cartesian coordinate system, Eq. (2.2.9) yields

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{2.2.10}$$

where u, v, and w are the velocity components in the x, y, and z directions, respectively.

According to Eq. (2.2.10), the shape of the streamlines is constant if the velocity vector can be expressed as a product of a spatial function and a temporal function. Such a case is represented by either one of the following conditions:

$$\vec{V}(\vec{x}, t) = \vec{U}(\vec{x}) f(t) \qquad \frac{\vec{V}}{|\vec{V}|} \neq f(t) \tag{2.2.11}$$

If $\vec{V}$ is solely a spatial function [i.e., $f(t)$ is a constant], then the flow field is subject to *steady state conditions* and the shape of the streamlines is identical to that of the pathlines. As an example, consider the velocity vector represented

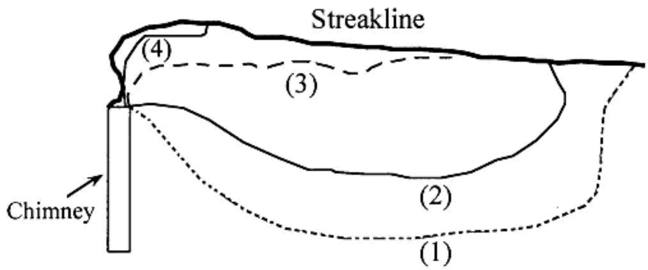

Figure 2.2 Four pathlines and a streakline at a chimney.

by Eq. (2.2.6). The differential equation of the streamlines is

$$-\frac{dx}{x} = \frac{dy}{y} \tag{2.2.12}$$

Direct integration of this equation yields

$$xy = C \tag{2.2.13}$$

where C is a constant of the particular streamline. Since Eq. (2.2.6) refers to steady state conditions, the shape of the streamlines represented by Eq. (2.2.13) is identical to that of the pathlines, which is given by Eq. (2.2.3).

A *streakline* is defined as a line connecting a series of fluid particles with their point source. An example of pathlines and a streakline that might be produced by smoke particles is presented in Fig. 2.2. In this figure the pathlines are enumerated. Pathline (1) refers to the first particle that left the chimney outlet. Pathline (2) refers to the second particle, etc.

2.3 RATE OF STRAIN, VORTICITY, AND CIRCULATION

In this section we discuss variables characterizing the kinematics of the flow field, which are associated with the velocity vector distribution in the domain. All such variables originate from the Eulerian presentation of the velocity vector.

In Fig. 2.3 are described two points in a flow field, A and B. The rates of change of the coordinate intervals between these points are represented by the following expressions given in Cartesian indicial format:

$$\frac{d}{dt}(\Delta x_i) = \Delta u_i = \frac{\partial u_i}{\partial x_j} dx_j \tag{2.3.1}$$

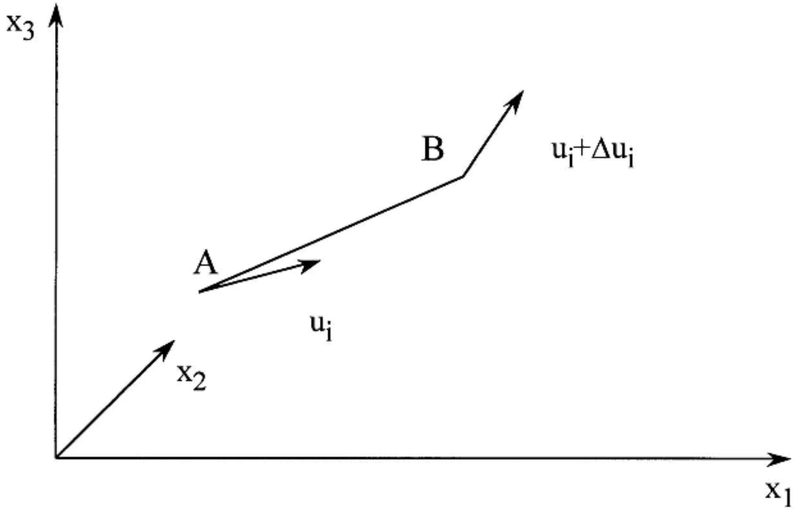

Figure 2.3 Rate of change of distance between two points.

Applying this expression, we obtain a second-order tensor that describes the rate of change of the coordinate intervals per unit length. This second-order tensor can be separated into symmetric and asymmetric tensors,

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.3.2}$$

The first tensor on the right-hand side of Eq. (2.3.2) is the symmetric tensor, called the *rate of strain tensor*. The second tensor is the asymmetric one, called the *vorticity tensor*. Each of these tensors has a distinct physical meaning, as described below.

The rate of strain tensor is represented by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.3.3}$$

In Fig. 2.4 the rate of elongation of an elementary fluid volume in a two-dimensional flow field is illustrated. The rate of elongation per unit length of that elementary volume in the x_i direction is called the *linear* or *normal strain rate*. It is represented by

$$\frac{u_1 + \Delta u_1 - u_1}{\Delta x_1} = \frac{(\partial u_1/\partial x_1)\Delta x_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_1} \tag{2.3.4}$$

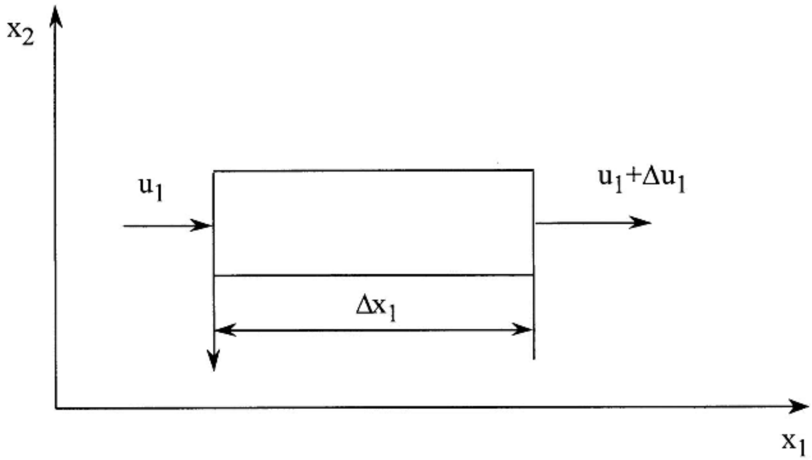

Figure 2.4 Elongation of an elementary fluid volume.

This expression gives the component e_{11} of the strain rate tensor. The components e_{22} and e_{33} represent the linear strain in the x_2 and x_3 directions. They are given, respectively, by

$$e_{22} = \frac{\partial u_2}{\partial x_2} \qquad e_{33} = \frac{\partial u_3}{\partial x_3} \tag{2.3.5}$$

Thus it is seen that diagonal components of the rate of strain tensor describe the linear rate of strain. The *volumetric strain rate* of an elementary volume is given by the *trace* of the strain rate tensor, i.e., the sum of the diagonal components, since

$$\begin{aligned}
&\frac{1}{\Delta x_1 \Delta y_1 \Delta z_1}\frac{d}{dt}(\Delta x_1 \Delta y_1 \Delta z_1) \\
&\quad = \frac{1}{\Delta x_1}\frac{d}{dt}(\Delta x_1) + \frac{1}{\Delta x_2}\frac{d}{dt}(\Delta x_2) + \frac{1}{\Delta x_3}\frac{d}{dt}(\Delta x_3) \\
&\quad = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11} + e_{22} + e_{33}
\end{aligned} \tag{2.3.6}$$

With regard to components of the rate of strain tensor that are not on the diagonal, we consider in Fig. 2.5 the rate of change of the angle of the elementary rectangle, which is called the *shear strain rate*. The expression for the shear strain rate is

$$\begin{aligned}
&\frac{u_1 + \Delta u_1 - u_1}{\Delta x_2} + \frac{u_2 + \Delta u_2 - u_2}{\Delta x_1} \\
&\quad = \frac{(\partial u_1/\partial x_2)\Delta x_2}{\Delta x_2} + \frac{(\partial u_2/\partial x_1)\Delta x_1}{\Delta x_1} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}
\end{aligned} \tag{2.3.7}$$

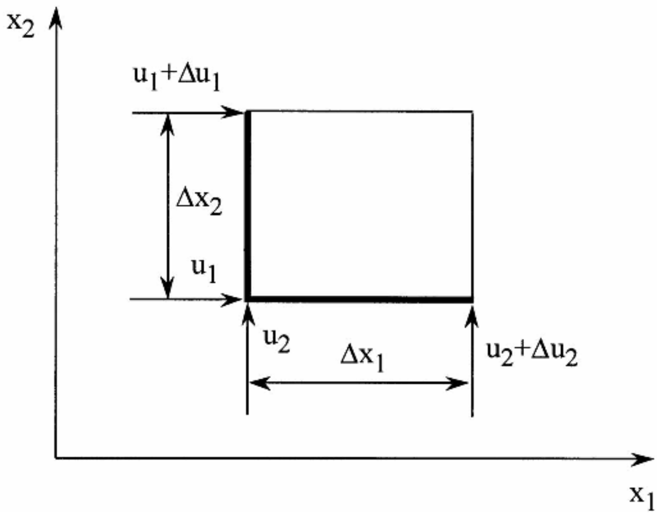

Figure 2.5 Elementary fluid volume subject to shear strain.

This expression is proportional to e_{12}, where

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{2.3.8}$$

Components of the strain rate tensor that are off the main diagonal thus represent deformation of shape. They are equal to half of the corresponding shear rate.

The *vorticity tensor* is an asymmetric tensor given in Cartesian coordinates by

$$\omega_{ij} = \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.3.9}$$

By considering Fig. 2.5, it is possible to visualize the physical meaning of the vorticity tensor. In this figure the velocity components that lead to rotation of an elementary fluid volume in a two-dimensional flow field are shown. The average angular velocity of that volume in the counterclockwise direction is given by

$$\begin{aligned} &\frac{1}{2}\left(\frac{u_2 + \Delta u_2 - u_2}{\Delta x_1} - \frac{u_1 + \Delta u_1 - u_1}{\Delta x_2}\right) \\ &\quad = \frac{1}{2}\left(\frac{(\partial u_2/\partial x_1)\Delta x_1}{\Delta x_1} - \frac{(\partial u_1/\partial x_2)\Delta x_2}{\Delta x_2}\right) \\ &\quad = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \omega_{21} = -\omega_{12} \end{aligned} \tag{2.3.10}$$

This expression indicates that the vorticity tensor is associated with rotation of the fluid particles.

In general, a second-order asymmetric tensor has three pairs of nonzero components. Each pair of components has identical magnitudes but opposite signs. Such a tensor also can be represented by a vector that has three components. Components of the vorticity tensor are proportional to components of the vorticity vector, which is the *curl* of the velocity vector,

$$\vec{\omega} = \nabla \times \vec{V} \qquad \text{or} \qquad \omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{2.3.11}$$

According to this expression, components of the vorticity vector are given by

$$\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \qquad \omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \qquad \omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \tag{2.3.12}$$

Irrotational flow is a flow in which all components of the vorticity vector are equal to zero. In such a flow the velocity vector originates from a potential function, namely

$$\vec{V} = \nabla \Phi \qquad \text{or} \qquad u_i = \frac{\partial \Phi}{\partial x_i} \tag{2.3.13}$$

Potential flows are discussed in greater detail in Chap. 4.

The *circulation* is defined as the line integral of the tangential component of velocity. It is given by

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} \qquad \text{or} \qquad \Gamma = \oint_c u_i \, ds_i \tag{2.3.14}$$

By applying the Stokes theorem, the line integral of Eq. (2.3.14) is converted to an area integral,

$$\oint_c \vec{V} \cdot d\vec{s} = \int_A (\nabla \times \vec{V}) \cdot d\vec{A} \quad \text{or} \quad \oint_c u_i \, ds_i = \int_A \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \, dA_i \tag{2.3.15}$$

This form of the equation is sometimes more useful.

2.4 LAGRANGIAN AND EULERIAN APPROACHES

2.4.1 General Presentation of the Approaches

Some basic concepts of the *Lagrangian* and *Eulerian* approaches have already been represented in the previous section. In the present section we expand on those concepts and describe some derivations of the basic conceptual approaches.

In the Lagrangian approach interest is directed at fluid particles and changes of properties of those particles. The Eulerian approach refers to spatial and temporal distributions of properties in the domain occupied by the fluid. Whereas the Lagrangian approach represents properties of individual fluid particles according to their initial location and time, the Eulerian approach represents the distribution of such properties in the domain with no reference to the history of the fluid particles. The concept of pathlines originates from the Lagrangian approach, while the concept of streamlines is associated with the Eulerian approach.

Every property F of an individual fluid particle can be represented in the Lagrangian approach by

$$F = F(\vec{x}_0, t) \tag{2.4.1}$$

where $\vec{x}_0$ is the location of the fluid particle at time t_0 and t is the time. The property F, according to the Eulerian approach, is distributed in the domain occupied by the fluid. Therefore its functional presentation is given by

$$F = F(\vec{x}, t) \tag{2.4.2}$$

where $\vec{x}$ and t are the spatial coordinates and time, respectively.

According to the Lagrangian approach, the rate of change of the property F of the fluid particle is given by

$$\frac{\partial F(\vec{x}_0, t)}{\partial t} \tag{2.4.3}$$

Therefore the velocity and acceleration of the fluid particle are given by

$$u_i(\vec{x}_0, t) = \frac{\partial x_i(\vec{x}_0, t)}{\partial t} \qquad a_i(\vec{x}_0, t) = \frac{\partial u_i(\vec{x}_0, t)}{\partial t} = \frac{\partial^2 x_i(\vec{x}_0, t)}{\partial t^2} \tag{2.4.4}$$

For example, consider the flow field defined by the pathlines given in Eq. (2.2.2). The Lagrangian velocity components are given by Eq. (2.2.5), and the Lagrangian acceleration components are given by Eq. (2.2.7).

The rate of change of the property F of the fluid particles, according to the Eulerian approach, can be expressed through use of the *material* or *absolute derivative*. This derivative expresses the rate of change of the property F by an observer moving with the fluid particle. The expression of the material derivative is given by

$$\frac{DF[\vec{x}(t), t]}{Dt} = \frac{\partial F}{\partial t} + (\nabla F)\frac{d\vec{x}}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_i}\frac{dx_i}{dt} \tag{2.4.5}$$

Therefore the velocity and acceleration distributions in the flow field, according to the Eulerian approach, are given, respectively, by

$$\begin{aligned} &\vec{V} = \frac{d\vec{x}}{dt} \qquad \vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \\ \text{or} \qquad &u_i = \frac{dx_i}{dt} \qquad a_i = \frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \end{aligned} \tag{2.4.6}$$

As an example, consider the Eulerian velocity distribution given by Eq. (2.2.6). By introducing the expressions of Eq. (2.2.6) into Eq. (2.4.6) we obtain the Eulerian acceleration distribution given by Eq. (2.2.8).

2.4.2 System and Control Volume

The previous paragraphs refer to individual fluid particles and their properties. Presently we will refer to aggregates of fluid particles comprising a finite fluid volume. A finite volume of fluid incorporating a constant quantity of fluid particles (or matter) is called a *system* or *material volume*. A system may change shape, position, thermal condition, etc., but it always incorporates the same matter. In contrast, a *control volume* is an arbitrary volume designated in space. A control volume may possess a variable shape, but in most cases it is convenient to consider control volumes of constant shape. Therefore fluid particles may pass into or out of the fixed control volume across its surface.

Figure 2.6 shows an arbitrary flow field. Several streamlines describing the flow direction at time t are depicted. The figure shows a system at time t. A control volume (CV) identical to the system at time t also is shown. At time $t + \Delta t$ the system has a shape different from its shape at time t, but the control volume has its original fixed shape from time t. We may identify three partial volumes, as indicated by Fig. 2.6: volume I represents the portion of the control volume evacuated by particles of the system during the time interval Δt; volume II is the portion of the control volume occupied by particles of the system at time $t + \Delta t$; volume III is the space to which particles of the system have moved during the time interval Δt. Particles of the system also convey properties of the flow. In the following paragraphs we consider the presentation of the rate of change of an arbitrary property η in the system by reference to a control volume.

2.4.3 Reynolds Transport Theorem

The Reynolds transport theorem represents the use of a control volume to calculate the rate of change of a property of a material volume. The rate of

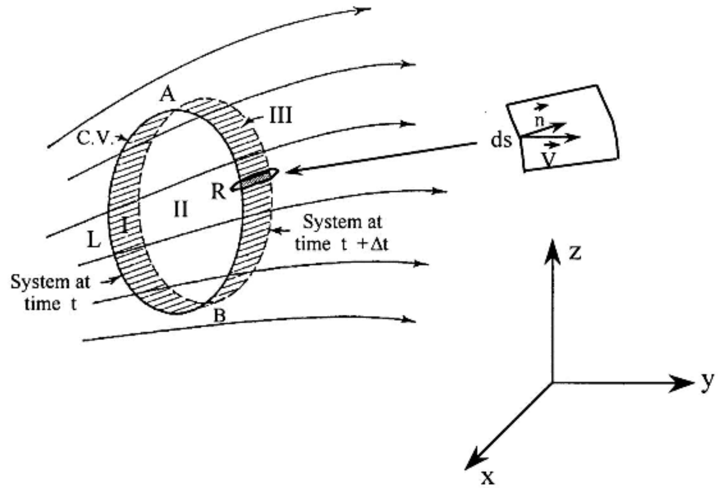

Figure 2.6 System (material volume) and control volume.

change of a property, η, of a material volume is represented by

$$\frac{D}{Dt}\int_{M.V.}\eta\,dU \tag{2.4.7}$$

where $M.V.$ represents material volume and dU is an elementary volume element. In Fig. 2.6, the integral of Eq. (2.4.7) incorporates two parts. One part consists of the control volume, CV, namely volume I and the material volume of Fig. 2.6, and the second part incorporates volumes I and III. An elementary volume ΔU of volumes I and III, as shown in Fig. 2.6, is represented by $\Delta U = (\vec{V}\cdot\vec{n}\,ds)\Delta t$, where $\vec{n}$ is a unit vector normal to the surface of the control volume (by convention, the direction of this vector is outward of the control volume) and ds is an elementary surface element. Summation of all elementary volumes ΔU leads to a surface integral, which is taken over the surface of the control volume, also known as the control surface (S). Therefore the rate of change of the material volume property, η, which is expressed by Eq. (2.4.7), can be given, by reference to the control volume, as

$$\frac{D}{Dt}\int_{M.V.}\eta\,dU = \frac{\partial}{\partial t}\int_{U}\eta\,dU + \int_{S}\eta(\vec{V}\cdot\vec{n})\,ds \tag{2.4.8}$$

where U is the volume of the control volume. If a fixed control volume is considered, then the partial derivative of the first term of the RHS of Eq. (2.4.8) can be moved inside the volume integral of that expression. It should be noted that the property η can be a scalar as well as a vector quantity. This is illustrated in the following sections.

2.5 CONSERVATION OF MASS

2.5.1 The Finite Control Volume Approach

By definition, the total mass of a material volume or system is constant. Therefore,

$$\frac{D}{Dt}\int_{M.V.} \rho\, dU = 0 \tag{2.5.1}$$

Comparison of this expression with Eq. (2.4.7) indicates that the property η of Eq. (2.4.7) was replaced by the density ρ in Eq. (2.4.8). We may, therefore, apply the transport theorem of Reynolds, namely Eq. (2.4.8), to obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\int_U \rho\, dU + \int_S \rho(\vec{V}\cdot\vec{n})\, ds = 0 \qquad \text{or} \\ &\frac{\partial}{\partial t}\int_U \rho\, dU + \int_S \rho(u_i n_i)\, ds = 0 \end{aligned} \tag{2.5.2}$$

Here, the first term represents the rate of change of mass included in the control volume. The second term represents the mass flux flowing through the surface of the control volume. Equation (2.5.2) represents the integral expression for the conservation of mass.

If we refer to a fixed control volume, and the density ρ of the fluid is constant, then the first term of Eq. (2.5.2) vanishes, and

$$\int_S (\vec{V}\cdot\vec{n})ds = 0 \qquad \text{or} \qquad \int_S u_i n_i\, ds = 0 \tag{2.5.3}$$

This equation represents the integral expression for continuity. It indicates that if the fluid density is constant, then the total mass flux entering the control volume is identical to the total mass flux flowing out of the control volume (for a fixed volume). When applied to a control volume of a stream tube, as shown in Fig. 2.7, Eq. (2.5.3) leads to

$$\vec{V}\cdot\vec{n}A = \text{const} \tag{2.5.4}$$

2.5.2 The Differential Approach

Consider again a fixed control volume. We transform the surface integral of the second term on the RHS of Eq. (2.5.2) to a volume integral by the divergence theorem and obtain

$$\int_U \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right] dU = 0 \tag{2.5.5}$$

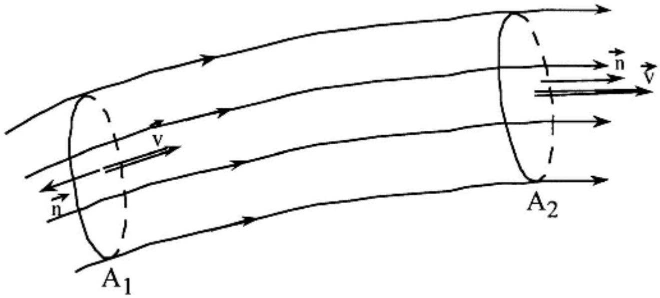

Figure 2.7 The integral continuity expression for a stream tube.

If the control volume is an arbitrarily small elementary volume, then Eq. (2.5.5) yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0 \qquad \text{or}$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0 \qquad \text{or} \tag{2.5.6}$$
$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{V}) = 0$$

This expression represents the differential equation of mass conservation. If the density of the fluid is fixed (i.e., $D\rho/Dt = 0$), then the flow is called *incompressible flow*, and Eq. (2.5.6) gives

$$\nabla \cdot \vec{V} = 0 \qquad \text{or} \qquad \frac{\partial u_i}{\partial x_i} = 0 \tag{2.5.7}$$

This expression represents the differential *continuity equation*.

2.5.3 The Stream Function

If the flow field is two dimensional, and a Cartesian coordinate system is assumed, then Eq. (2.5.7) implies

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.5.8}$$

Then a *stream function* Ψ may be defined that satisfies Eq. (2.5.8),

$$u = \frac{\partial \Psi}{\partial y} \qquad v = \frac{\partial \Psi}{\partial x} \tag{2.5.9}$$

Then, introducing Eq. (2.5.9) into Eq. (2.2.10), it is seen that streamlines are defined by

$$\frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = 0 \qquad (2.5.10)$$

This expression indicates that the differential of the stream function vanishes on the streamlines. Therefore the stream function has a constant value on a streamline, and the value of the stream function can be used for the identification of particular streamlines in the flow field.

Figure 2.8 shows two streamlines, which are identified by Ψ_A and Ψ_B. The discharge per unit width flowing through the stream tube bounded by the streamlines Ψ_A and Ψ_B is given by

$$q = \int_A^B (u\,dy - v\,dx) = \int_A^B \left(\frac{\partial \Psi}{\partial y} dy + \frac{\partial \Psi}{\partial x} dx \right)$$
$$= \int_A^B d\Psi = \Psi_B - \Psi_A \qquad (2.5.11)$$

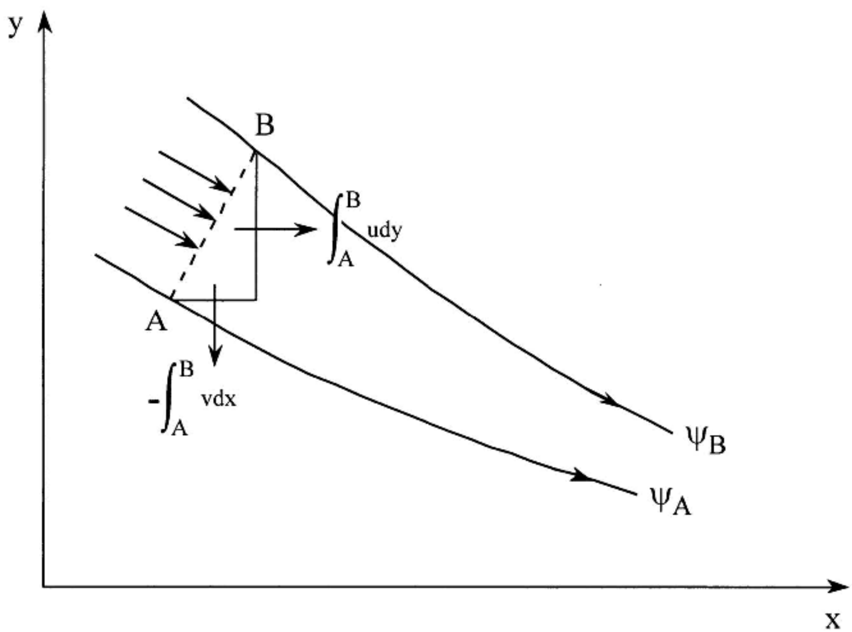

Figure 2.8 Illustration of volumetric flux between two streamlines.

Thus the difference between values of the stream function for two streamlines represents the discharge flowing between those streamlines.

If the flow field is represented by a cylindrical coordinate system, then the employment of the covariant derivative and the relevant scale yield the following expression for the differential continuity equation:

$$\nabla \cdot \vec{V} = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}$$
$$= \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{r}\frac{\partial (r w_z)}{\partial z} = 0 \qquad (2.5.12)$$

where u_r, v_θ, and w_z are physical components of the velocity vector in the r, θ, and z directions, respectively. We may use the concept of stream function in cylindrical coordinates for two types of flow field. One type is a two-dimensional flow field expressed by reference to coordinates r and θ. The other type is an axisymmetric flow field expressed by coordinates r and z.

In the case of two-dimensional flow, there is no flow in the z-direction, and velocity components do not depend on the z coordinate. Therefore the term referring to z and w_z of Eq. (2.5.12) vanishes, and the expressions for u_r and v_θ are given by the stream function as

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \Psi}{\partial r} \qquad (2.5.13)$$

In cases of axisymmetric flow, there is no flow in the θ-direction, and velocity components do not depend on the θ coordinate. Then the presentation of u_r and w_z by the stream function is given as

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial z} \qquad w_z = -\frac{1}{r}\frac{\partial \Psi}{\partial r} \qquad (2.5.14)$$

Note that the stream function of Eq. (2.5.13) has dimensions of discharge per unit width, whereas the stream function of Eq. (2.5.14) has dimensions of volumetric discharge.

2.5.4 Stratified Flow

In cases of *stratified flow*, where the density field is not constant, the differential equation of mass conservation, namely Eq. (2.5.6), is still

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \cdot \vec{V} = 0 \quad \text{or} \quad \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad (2.5.15)$$

(Recall that there were no constraints placed on density in deriving the mass conservation expression.) In particular, consider the second of these expressions, which is rewritten as

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{V} = 0 \qquad \text{or} \qquad \frac{D\rho}{Dt} + \rho\frac{\partial u_i}{\partial x_i} = 0 \tag{2.5.16}$$

This expression indicates that incompressible flow is identified by the vanishing material derivative of the density. In other words, density is constant, *following a fluid particle*. In cases of steady stratified flow, the temporal derivative of the density is zero. If the flow is also incompressible, namely $\nabla \cdot \vec{V} = 0$ [Eq. (2.5.7)], then according to Eq. (2.5.15), the velocity vector is perpendicular to the density gradient.

In cases of steady two-dimensional flow, Eq. (2.5.6) yields

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2.5.17}$$

This equation can be identically satisfied by a stream function defined by

$$\rho u = \frac{\partial \Psi}{\partial y} \qquad \rho v = -\frac{\partial \Psi}{\partial x} \tag{2.5.18}$$

This stream function has dimensions of mass flux per unit width.

2.6 CONSERVATION OF MOMENTUM

The property $\rho\vec{V}$ represents the momentum of a unit volume of the fluid. The rate of change of momentum of a fluid material volume is equal to the sum of forces acting on that material volume. Using the Reynolds transport theorem, Eq. (2.4.8) applied to $\rho\vec{V}$ yields

$$\frac{\partial}{\partial t}\int_U \rho\vec{V}\,dU + \int_S \rho\vec{V}(\vec{V}\cdot\vec{n})\,ds = \int_U \rho\vec{g}\,dU + \int_S \tilde{S}\cdot\vec{n}\,ds + \vec{F}_s \tag{2.6.1a}$$

$$\text{or}\quad \frac{\partial}{\partial t}\int_U \rho u_i\,dU + \int_S \rho u_i(u_k n_k)\,ds = \int_U \rho g_i\,dU + \int_S S_{ik} n_k\,ds + F_{si} \tag{2.6.1b}$$

where $\tilde{S}$ is the stress tensor, which refers to forces acting on the fluid surface of the control volume, and $\vec{F}_s$ represents forces acting on solid surfaces comprising portions of the surface of the control volume.

The first RHS term of Eq. (2.6.1) represents body forces originating from gravity. The gravitational acceleration vector, $\vec{g}$, is equal to the gravity,

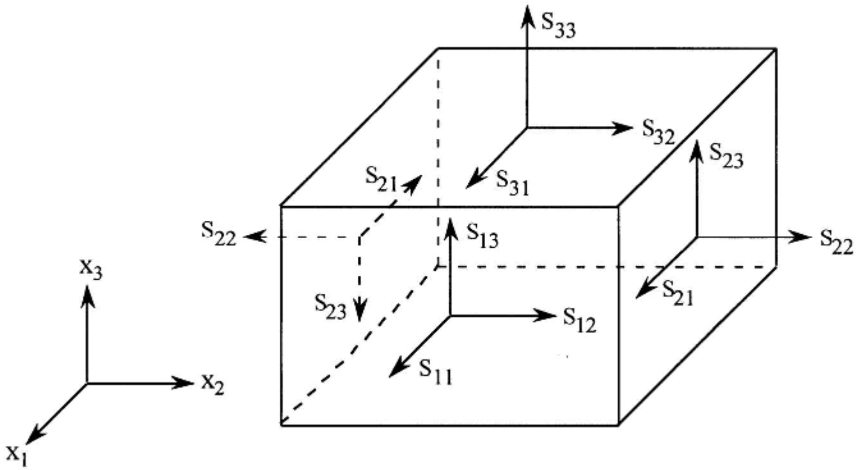

Figure 2.9 Components of the stress tensor acting on a small rectangle.

g, multiplied by a unit vector in the negative direction of the normal to the earth's surface. The second RHS term represents surface forces.

The stress tensor at each point of the surface of the control volume can be completely defined by the nine components of the stress tensor, $\tilde{S}$. Figure 2.9 shows an infinitesimal rectangular parallelepiped with faces having normal unit vectors parallel to the coordinate axes. The force per unit area acting on each face of the parallelepiped is divided into a normal component and two shear components (shear stresses) that are perpendicular to the normal component. Figure 2.9 exemplifies the decomposition of the force per unit area over four different faces. Directions of the stress tensor components shown in Fig. 2.9 are considered positive, by convention. The first subscript of the stress component represents the direction of the normal of the particular face on which the stress acts. The second subscript represents the direction of the component of the stress.

In Fig. 2.10 are shown components of the shear stress creating torque, which may lead to rotation of the elementary rectangle around its center of gravity, G. The total torque is expressed by

$$\begin{aligned}\text{Torque} &= \left(S_{12} + \frac{1}{2}\frac{\partial S_{12}}{\partial x_1}dx_1\right)dx_2\frac{dx_1}{2} + \left(S_{12} - \frac{1}{2}\frac{\partial S_{12}}{\partial x_1}dx_1\right)dx_2\frac{dx_1}{2} \\ &\quad - \left(S_{21} + \frac{1}{2}\frac{\partial S_{21}}{\partial x_2}dx_2\right)dx_1\frac{dx_2}{2} - \left(S_{21} - \frac{1}{2}\frac{\partial S_{21}}{\partial x_2}dx_2\right)dx_1\frac{dx_2}{2}\end{aligned} \tag{2.6.2}$$

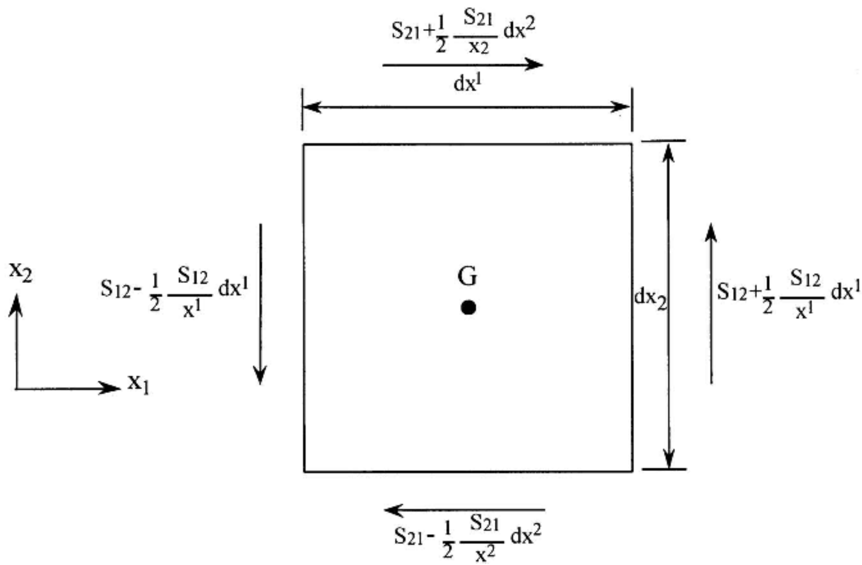

Figure 2.10 Torque applied on an elementary rectangle of fluid.

Also the total torque is equal to the moment of inertia multiplied by the angular acceleration. Therefore, Eq. (2.6.2) yields

$$(S_{12} - S_{21})\, dx_1\, dx_2 = \frac{\rho}{12}\, dx_1\, dx_2 \left[(dx_1)^2 + (dx_2)^2\right] \alpha \tag{2.6.3}$$

where α is the angular acceleration.

Upon dividing Eq. (2.6.3) by the area of the elementary rectangle and allowing dx_1 and dx_2 to approach zero, the RHS of Eq. (2.6.3) vanishes. This result indicates that the stress tensor is a symmetric tensor, namely

$$S_{ij} = S_{ji} \tag{2.6.4}$$

The stress tensor can be decomposed into two tensors, as

$$\tilde{S} = -p\tilde{I} + \tilde{\tau} \qquad \text{or} \qquad S_{ij} = -p\delta_{ij} + \tau_{ij} \tag{2.6.5}$$

where $\tilde{I}$ is a unit matrix, which also can be represented by δ_{ij}, p is the pressure, and $\tilde{\tau}$ is the *deviator stress tensor*, related to shear stresses (see below).

The first term on the RHS of Eq. (2.6.5) is an isotropic tensor, namely a tensor that has components only on its diagonal, and all diagonal components are identical, provided that we apply a Cartesian coordinate system. Components of the isotropic tensor are not modified by rotation of the coordinate system. The pressure, p, is equal to the negative one-third of the *trace* of the stress tensor,

$$p = -\frac{1}{3}(S_{11} + S_{22} + S_{33}) \tag{2.6.6}$$

where the trace of a tensor is defined as the sum of its diagonal components. Note that the trace of the deviator stress tensor is zero. Positive normal stress means tension. However, fluids can only resist and convey negative normal stresses. The definition of Eq. (2.6.6) yields a positive value for the pressure.

Incorporating the definitions and expressions developed in the preceding paragraphs, Eq. (2.6.1) is rewritten to express conservation of momentum in a fluid material volume:

$$\frac{\partial}{\partial t}\int_U \rho u_i \, dU + \int_s \rho u_i (u_k n_k) \, ds = -\int_s p n_i \, ds + \int_s \tau_{ik} n_k \, ds - \int_U \rho g k_i \, dU + F_{Si} \tag{2.6.7}$$

where k_i represents the component of a unit vector perpendicular to the earth, directed toward the atmosphere. For a fixed control volume, the derivative of the first term on the LHS of Eq. (2.6.7) can be moved into the integral of that term.

When Eq. (2.6.7) is applied to an elementary volume of fluid, the last term vanishes since there are no solid surfaces. Then, using the divergence theorem to convert surface integrals to volume integrals, we have

$$\int_{\Delta U}\left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_k)}{\partial x_k} + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ik}}{\partial x_k} + \rho g k_i\right] = 0 \tag{2.6.8}$$

By introducing the conservation of mass, expressed by Eq. (2.5.6), into Eq. (2.6.8), and considering that ΔU is small but different from zero,

$$\rho\left[\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k} - \rho g k_i \tag{2.6.9a}$$

or

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = -\nabla(p + \rho g Z) + \nabla\cdot\tilde{\tau} \tag{2.6.9b}$$

where Z is the elevation with regard to an arbitrary level of reference. Equation (2.6.9) is the *equation of motion*, or the differential equation of conservation of momentum.

The *Bernoulli equation* can be derived by direct integration of Eq. (2.6.9). First, note that the nonlinear term of the LHS of Eq. (2.6.9) can be expressed as

$$(\vec{V}\cdot\nabla)\vec{V} = \nabla\frac{V^2}{2} - \vec{V}\times(\nabla\times\vec{V}) \tag{2.6.10}$$

If the velocity vector is derived from a potential function, then shear stresses also are negligible, and $\nabla \times \vec{V} = 0$. Therefore, in such a case Eqs. (2.6.9) and (2.6.10) yield

$$\rho \left[\frac{\partial}{\partial t}(\nabla \Phi) + \nabla \frac{V^2}{2} \right] = -\nabla(p + \rho g Z) \tag{2.6.11}$$

where Φ is the potential function, defined in Eq. (2.3.13). For steady state cases, direct integration of Eq. (2.6.11) and division by the specific weight of the fluid yield

$$\frac{V^2}{2g} + \frac{p}{\gamma} + Z = const \tag{2.6.12}$$

where $\gamma = \rho g$ is the specific weight of the fluid. This is called the *Bernoulli equation*. The sum of the terms on the LHS of this equation is called the total head, which incorporates the velocity head, the pressure head, and the elevation (or elevation head). The sum of pressure head and elevation is called the *piezometric head*. According to Eq. (2.6.12) the total head is constant in a domain of steady potential flow.

In cases of steady flow with negligible effect of the shear stresses, consider a natural coordinate system that incorporates a coordinate, s, tangential to the streamline, and a coordinate, n, perpendicular to the streamline. The velocity vector has only a component tangential to the streamline. Therefore, Eq. (2.6.9) yields for the tangential direction,

$$\rho \left[V \frac{\partial V}{\partial s} \right] = -\frac{\partial}{\partial s}(p + \rho g Z) \tag{2.6.13}$$

Direct integration of this expression indicates that the total head is constant along the streamline even if the flow is nonpotential flow, provided that the effect of shear stresses is negligible.

A *moving coordinate system* is sometimes applied to calculate momentum conservation. All basic equations applicable to a stationary coordinate system also can be applied to cases in which the coordinate system moves with a constant velocity. It should be noted that the Bernoulli equation, represented by Eq. (2.6.12), is applicable only in cases of steady state. The application of a moving coordinate system may sometimes enable use of Bernoulli's equation in cases of unsteady state conditions.

A *noninertial coordinate system* is one that is subject to acceleration. All momentum quantities in the conservation of momentum equation must be written with respect to an inertial coordinate system. If a noninertial system is used, then the acceleration measured by a fixed observer, $\vec{a}_{\text{F.O.}}$, is given by

$$\begin{aligned} \vec{a}_{\text{F.O.}} = \vec{a}_{\text{M.O.}} + \vec{a}_t + 2\vec{\omega} \times \vec{V}_{\text{M.O}} + \frac{d\vec{\omega}}{dt} \\ \times \vec{r}_{\text{M.O.}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{\text{M.O.}}) \end{aligned} \tag{2.6.14}$$

where subscript F.O. refers to a fixed observer, M.O. refers to an observer moving with the coordinate system, a_t is the translational acceleration of the moving coordinate system, ω is the angular velocity of the moving coordinate system, $V_{\mathrm{M.O.}}$ is the velocity of the fluid particle measured by the moving observer, and $r_{\mathrm{M.O.}}$ is the position of the fluid particle measured by the moving observer. The momentum conservation Eq. (2.6.7) can be applied, with minor modification, to cases in which noninertial coordinate systems are used. In such cases, the integral equation of momentum conservation is given by

$$\begin{aligned}\frac{\partial}{\partial t}\int_U \rho\vec{V}\,dU + \int_s \rho\vec{V}(\vec{V}\cdot\vec{n})\,ds \\ = -\int_s p\vec{n}\,ds + \int_s \vec{\tau}\cdot\vec{n}\,ds - \int_U \rho g\vec{k}\,dU + \vec{F}_s \\ -\int_U \left[\vec{a}_t + 2\vec{\omega}\times\vec{V} + \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right]\rho\,dU \qquad (2.6.15)\end{aligned}$$

The following section provides further discussion of coordinate systems subject to rotational velocity originating from the earth's rotation. This is also described in further detail, using a dimensional scaling approach, in Sec. 2.9.3.

2.7 THE EQUATIONS OF MOTION AND CONSTITUTIVE EQUATIONS

In the preceding section it was shown that the equations of motion represent the conservation of momentum in an elementary fluid volume. The general form of the equations of motion is represented by Eq. (2.6.9), which is again given as

$$\rho\left[\frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\right] = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ik}}{\partial x_k} - \rho g k_i \qquad (2.7.1a)$$

or $$\rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = -\nabla(p+\rho g Z) + \nabla\cdot\tilde{\tau} \qquad (2.7.1b)$$

Different types of fluids are identified by their *constitutive equations*, which provide the relationships between the deviatoric stress tensor, τ_{ij}, and kinematic parameters. For a *Newtonian fluid* the shear stress is assumed to be proportional to the rate of strain, and the constitutive equation for such a fluid is

$$\tau_{ij} = -\left(p + \frac{1}{3}\mu\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + 2\mu e_{ij} \qquad (2.7.2)$$

where e_{ij} is the rate of strain tensor,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.7.3}$$

By introducing Eq. (2.7.2) into Eq. (2.7.1), the general form of the *Navier–Stokes equations* is obtained,

$$\begin{aligned} \rho\frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} - \rho g k_i + 2\mu\left[\frac{\partial e_{ij}}{\partial x_j} - \frac{1}{3}\frac{\partial^2 u_i}{\partial x_i \partial x_j}\right] \\ &= -\frac{\partial p}{\partial x_i} - \rho g k_i + \mu\left[\frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3}\frac{\partial^2 u_i}{\partial x_i \partial x_j}\right] \end{aligned} \tag{2.7.4}$$

For incompressible flow, Eq. (2.7.4) reduces to

$$\rho\frac{D\vec{V}}{Dt} = -\nabla(p + \rho g Z) + \mu\nabla^2\vec{V} \tag{2.7.5a}$$

or

$$\rho\frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}(p + \rho g Z) + \mu\frac{\partial^2 u_i}{\partial x_j^2} \tag{2.7.5b}$$

Non-Newtonian fluids are characterized by constitutive equations different from Eq. (2.7.2). These types of fluids are not considered here.

The equations of motion given in the preceding paragraphs are valid in an inertial or fixed frame of reference. In comparatively small hydraulic systems, it is possible to refer to such equations of motion, while considering that the frame of reference, namely the earth, is stationary. In geophysical applications the rotation of the earth must be considered.

Figure 2.11 shows two coordinate systems: coordinate system (X_1, X_2, X_3), which is stationary, and coordinate system (x_1, x_2, x_3), which rotates at angular velocity Ω with regard to the fixed coordinate system. Any vector associated with the point G has three components in each of the coordinate systems. As an example, the decomposition of the vector $\vec{r}$ into three components of the rotating coordinate system is shown. A general vector $\vec{R}$ is represented in the rotating coordinate system by

$$\vec{R} = R_1\vec{i}_1 + R_2\vec{i}_2 + R_3\vec{i}_3 \tag{2.7.6}$$

A fixed observer, F.O., observes the rate of change of the vector $\vec{R}$ as

$$\begin{aligned} \left(\frac{d\vec{R}}{dt}\right)_{\text{F.O.}} &= \frac{d}{dt}(R_1\vec{i}_1 + R_2\vec{i}_2 + R_3\vec{i}_3) \\ &= \vec{i}_1\frac{dR_1}{dt} + \vec{i}_2\frac{dR_2}{dt} + \vec{i}_3\frac{dR_3}{dt} + R_1\frac{d\vec{i}_1}{dt} + R_2\frac{d\vec{i}_2}{dt} + R_3\frac{d\vec{i}_3}{dt} \end{aligned} \tag{2.7.7}$$

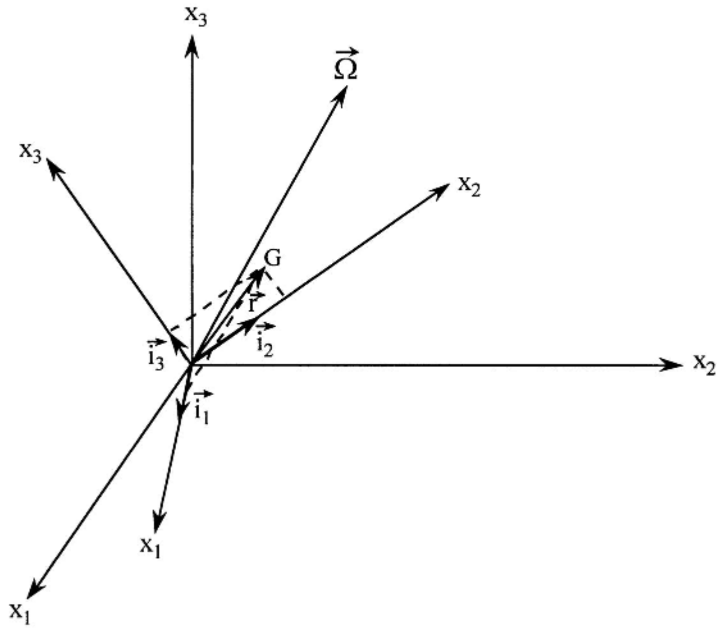

Figure 2.11 Coordinate system x_1, x_2, x_3 rotates with angular velocity Ω with regard to the stationary coordinate system X_1, X_2, X_3.

The first three terms on the RHS represent the rate of change of the vector, as observed by an observer, R.O., rotating with the rotating coordinate system. The second group of three terms represents the rate of change of the vector, originating from rotation of the coordinate system. Therefore Eq. (2.7.7) can be expressed as

$$\left(\frac{d\vec{R}}{dt}\right)_{\text{F.O.}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{R.O.}} + R_1\frac{d\vec{i}_1}{dt} + R_2\frac{d\vec{i}_2}{dt} + R_3\frac{d\vec{i}_3}{dt} \tag{2.7.8}$$

Due to its rotation around the axis, $\vec{\Omega}$, each unit vector $\vec{i}$ traces a cone as shown in Fig. 2.12. The rate of change of this vector is given by

$$\left|\frac{d\vec{i}}{dt}\right| = \sin\beta\left(\frac{d\theta}{dt}\right) = \Omega\sin\beta \tag{2.7.9}$$

The direction of the rate of change of the vector $\vec{i}$ is perpendicular to the plane made by the vectors $\vec{i}$ and $\vec{\Omega}$. Therefore

$$\frac{d\vec{i}}{dt} = \vec{\Omega}\times\vec{i} \tag{2.7.10}$$

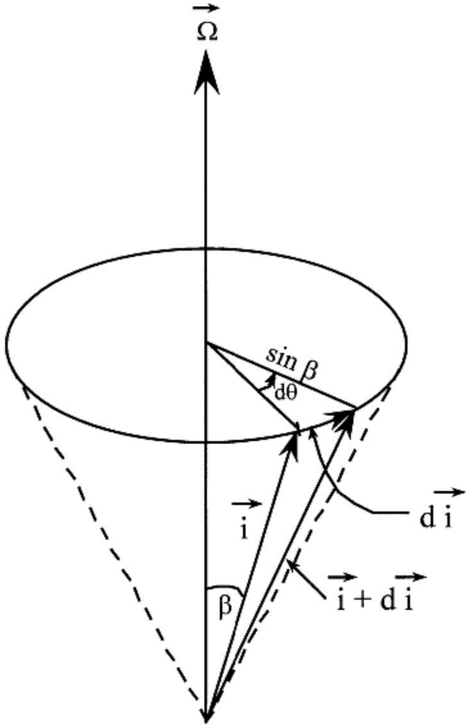

Figure 2.12 Cone of rotation of a unit vector.

The sum of the last three terms of Eq. (2.7.8) is given by

$$R_1\vec{\Omega} \times \vec{i}_1 + R_2\vec{\Omega} \times \vec{i}_2 + R_3\vec{\Omega} \times \vec{i}_3 = \vec{\Omega} \times \vec{R} \tag{2.7.11}$$

Introducing Eq. (2.7.11) into Eq. (2.7.8), we obtain

$$\left(\frac{d\vec{R}}{dt}\right)_{\text{F.O.}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{R.O.}} + \vec{\Omega} \times \vec{R} \tag{2.7.12}$$

This expression gives the relationship between the velocity vector measured by the fixed and rotating observers as

$$\vec{V}_{\text{F.O.}} = \vec{V}_{\text{R.O.}} + \vec{\Omega} \times \vec{r} \tag{2.7.13}$$

Equation (2.7.12) also implies that acceleration can be expressed as

$$\left(\frac{d\vec{V}_{\text{F.O.}}}{dt}\right)_{\text{F.O.}} = \left(\frac{d\vec{V}_{\text{F.O.}}}{dt}\right)_{\text{R.O.}} + \vec{\Omega} \times \vec{V}_{\text{F.O.}} \tag{2.7.14}$$

By introducing Eq. (2.7.13) into Eq. (2.7.14), we obtain

$$\frac{d\vec{V}_{\text{F.O.}}}{dt} = \frac{d}{dt}[\vec{V}_{\text{R.O.}} + \vec{\Omega} \times \vec{r}]_{\text{R.O.}} + \vec{\Omega} \times (\vec{V}_{\text{R.O.}} + \vec{\Omega} \times \vec{r})$$
$$= \left(\frac{d\vec{V}_{\text{R.O.}}}{dt}\right)_{\text{R.O.}} + \vec{\Omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{R.O.}} + \vec{\Omega} \times \vec{V}_{\text{R.O.}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{2.7.15}$$

Thus the relationship between the acceleration in the two coordinate systems is

$$\vec{a}_{\text{F.O.}} = \vec{a}_{\text{R.O.}} + 2\vec{\Omega} \times \vec{V}_{\text{R.O.}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \tag{2.7.16}$$

Upon introducing the vector $\vec{R}$, which is perpendicular to the axis of rotation represented by the vector $\vec{\Omega}$ (also refer to Fig. 2.13), we find

$$\vec{\Omega} \times \vec{r} = \vec{\Omega} \times \vec{R} \tag{2.7.17}$$

Also, using the vector identity,

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = (\vec{\Omega} \cdot \vec{R})\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\vec{R} = -(\vec{\Omega} \cdot \vec{\Omega})\vec{R} = -\Omega^2\vec{R} \tag{2.7.18}$$

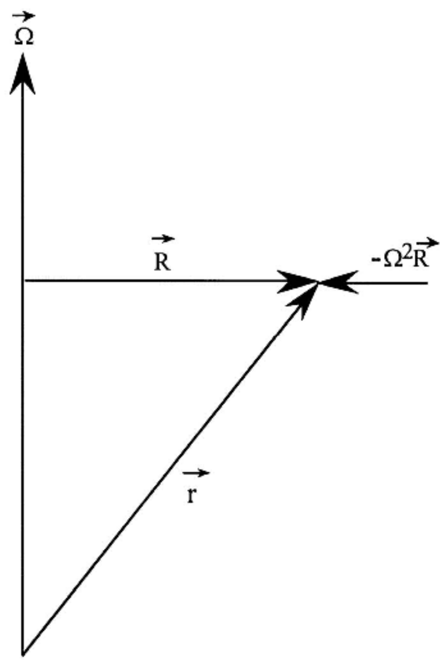

Figure 2.13 Relationships between vectors *r*, *R* and the centripetal acceleration.

with Eq. (2.7.16), we obtain

$$\vec{a}_{\text{F.O.}} = \vec{a} + 2\vec{\Omega} \times \vec{V} - \Omega^2 \vec{R} \tag{2.7.19}$$

where $\vec{V}$ and $\vec{a}$ are the velocity and acceleration vectors, respectively, in the rotating coordinate system. The second term on the RHS of this last result represents the *Coriolis acceleration*. The last term on the RHS of this equation represents *centripetal acceleration*.

The preceding paragraphs indicate that the equations of motion for *geostrophic* (or, "earth-turned") *scales* should incorporate terms originating from the rotation of earth. Introducing Eq. (2.7.17) into Eq. (2.7.5) yields

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}\nabla(p + \rho g Z) + v\nabla^2\vec{V}^2 + \Omega^2\vec{R} - 2\vec{\Omega} \times \vec{V} \tag{2.7.20}$$

Normally, the centrifugal acceleration term is considered as a minor adjustment to Newtonian gravity, with the sum of these two terms referred to as *effective gravitational acceleration*, $\vec{g}_{\text{eff}}$,

$$\vec{g}_{\text{eff}} = \nabla(-gZ) + \Omega^2\vec{R} \tag{2.7.21}$$

The relationships between the vectors $\vec{\Omega}$, $\vec{R}$, $\vec{g}$, $\Omega^2\vec{R}$, and $\vec{g}_{\text{eff}}$ in the northern hemisphere are shown in Fig. 2.14.

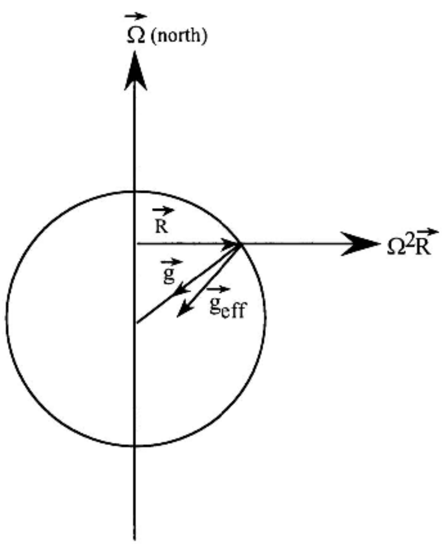

Figure 2.14 Relationships between the vectors Ω, R, g, $\Omega^2 R$, and g_{eff}.

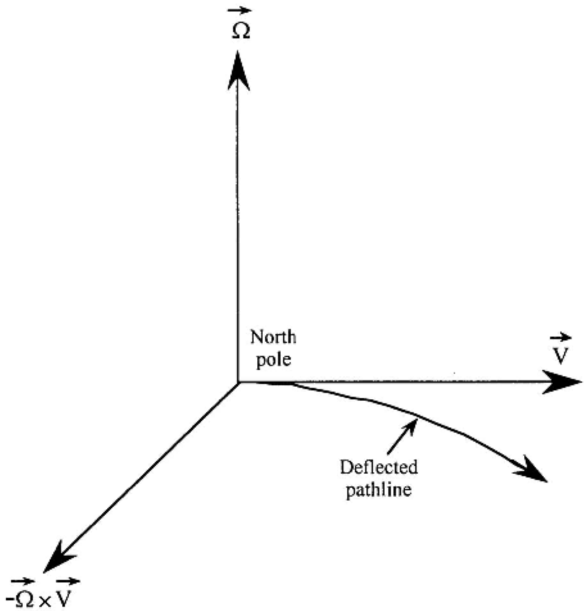

Figure 2.15 Relationships between the vectors Ω, V, and -Ω × V.

In Fig. 2.15 we show the relationships between the vectors $\vec{\Omega}$, $\vec{V}$, and $-\vec{\Omega} \times \vec{V}$. This figure indicates that Coriolis force induces a deflection of pathlines of the fluid particles to the right of their direction in the Northern Hemisphere.

The equation of motion represented by Eq. (2.7.20) is applicable in cases of *geostrophic flows*, in which the effect of the centrifugal acceleration and Coriolis force are significant. For small-scale flows, in small hydraulic systems, such effects are usually negligible. It is usually possible to determine the relative importance of different terms in the equations of motion by scaling analysis, as demonstrated in Sec. 2.9.

2.8 CONSERVATION OF ENERGY

Consider the material volume shown in Fig. 2.16. In general, this material volume may be subject to movement and deformation. The net heat added to the material volume during a short time period dt is dQ. During that time interval, the material volume exerts work dW on its surroundings. According to the first law of thermodynamics,

$$dE = dQ - dW \tag{2.8.1}$$

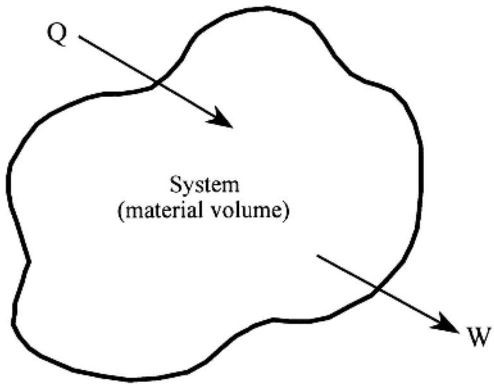

Figure 2.16 Heat Q added to a material volume and work W done by this volume.

where E is the total energy stored within the material volume. This variable incorporates the kinetic, potential, and internal energy [see Eq. (2.8.4) below]. Note that the normal convention is used to express work as a positive quantity when the material volume does work on its surroundings.

The variables Q and W are not point functions, whereas the variable E is a point function distributed within the material volume. Therefore the relationship between the rates of change of the variables given in Eq. (2.8.1) is represented by

$$\frac{DE}{Dt} = \frac{dQ}{dt} - \frac{dW}{dt} \tag{2.8.2}$$

By applying the Reynolds transport theorem, written for energy, we obtain

$$\frac{DE}{Dt} = \frac{\partial}{\partial t} \int_U \rho e \, dU + \int_S \rho e (\vec{V} \cdot \vec{n}) \, dS \tag{2.8.3}$$

where e is the stored energy per unit mass, given by

$$e = \frac{V^2}{2} + gz + u \tag{2.8.4}$$

The first term on the RHS of this equation represents kinetic energy, the second term represents potential energy, and the third term represents internal energy.

The work W done by the control volume on its surroundings incorporates flow work W_{f}, which is associated with stresses acting at the surface of the control volume, and shaft work, which is transferred from the control volume, for instance by turbomachines. The rate of change of the flow work can be represented by

$$\frac{dW_{\mathrm{f}}}{dt} = -\int_S \tilde{S} \cdot \vec{n} \cdot \vec{V}\, dS = \int_S p\vec{V} \cdot \vec{n}\, dS - \int_S \tilde{\tau} \cdot \vec{n} \cdot \vec{V}\, dS \qquad (2.8.5)$$

where $\tilde{S}$ is the stress tensor, p is the pressure, and $\tilde{\tau}$ is the deviator stress tensor. It should be noted that the product $\tilde{\tau} \cdot \vec{n}$ represents stresses normal to the control volume surface. The velocity vector of viscous flow vanishes at solid surfaces, and has no component perpendicular to a solid surface. Therefore, the last term of Eq. (2.8.5) almost vanishes. The only contribution of this term is due to diagonal components of the deviator stress tensor at fluid surfaces subject to flow. In the following development, the last term of Eq. (2.8.5) is neglected.

Introducing Eqs. (2.8.3)–(2.8.5) into Eq. (2.8.2), we obtain

$$\frac{dQ}{dt} - \frac{dW_{\mathrm{s}}}{dt} - \int_S p\vec{V} \cdot \vec{n}\, dS$$
$$= \frac{\partial}{\partial t}\int_U \rho e\, dU + \int_S \left(\frac{V^2}{2} + gz + u\right)(\rho\vec{V} \cdot \vec{n}\, dS) \qquad (2.8.6)$$

Using the divergence theorem to rewrite the last term on the LHS of Eq. (2.8.6), an integral expression for conservation of energy is obtained as

$$\frac{dQ}{dt} - \frac{dW_{\mathrm{s}}}{dt} = \frac{\partial}{\partial t}\int_U \rho e\, dU + \int_S \left(\frac{V^2}{2} + gz + u + \frac{p}{\rho}\right)(\rho\vec{V} \cdot \vec{n}\, dS) \qquad (2.8.7)$$

Application of this equation is illustrated by considering Fig. 2.17, which shows a control volume with two openings. The fluid enters the control volume through one of the openings, of cross-sectional area A_1, with velocity V_1, pressure p_1, and temperature T_1. The fluid flows out of the control volume through the second opening, of cross-sectional area A_2, with velocity V_2, pressure p_2, and temperature T_2.

Referring to this control volume, under steady state conditions Eq. (2.8.7) yields

$$\frac{dQ}{dt} - \frac{dW_{\mathrm{s}}}{dt} = -\left[\frac{V_1^2}{2} + g(z_{\mathrm{c}})_1 + h_1\right]\rho_1 V_1 A_1$$
$$+ \left[\frac{V_2^2}{2} + g(z_{\mathrm{c}})_2 + h_2\right]\rho_2 V_2 A_2 \qquad (2.8.8)$$

where z_{c} is the elevation of the center of gravity of the cross-sectional area, and h is the specific enthalpy, which is defined by

$$h = u + \frac{p}{\rho} = C_{\mathrm{p}}T = C_{\mathrm{v}}T + \frac{p}{\rho} \qquad (2.8.9)$$

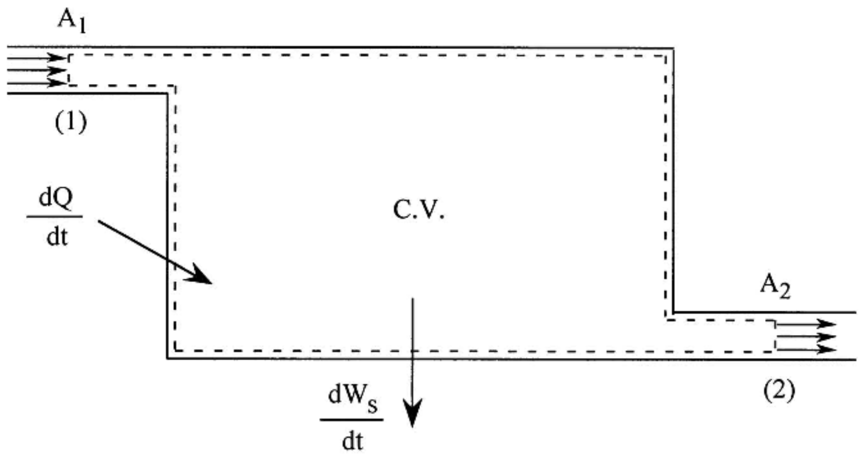

Figure 2.17 Energy conservation in a control volume (C.V.) with a single entrance and a single exit.

where C_p and C_v are the specific heats for constant pressure and constant volume, respectively.

Due to conservation of mass, $\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = dm/dt$, where dm/dt is the mass flow rate which enters and leaves the control volume of Fig. 2.17. Dividing Eq. (2.8.8) by the mass flow rate and rearranging terms,

$$\left[\frac{V_1^2}{2} + g(z_c)_1 + h_1\right] + \frac{dQ/dt}{dm/dt} = \left[\frac{V_2^2}{2} + g(z_c)_2 + h_2\right] + \frac{dW_s/dt}{dm/dt} \quad (2.8.10)$$

The second term on the LHS of this equation represents the ratio between the heat flux into the control volume and the mass flow rate through the control volume. It also can be represented by dQ/dm, namely the net heat added to the control volume per unit mass of flow. The last term of Eq. (2.8.10) can be represented by dW_s/dm, namely the net work done by the control volume per unit mass of flow through the control volume. In the case of incompressible fluid, if the control volume is insulated and does not perform work on its surrounding, then Eq. (2.8.10) indicates

$$\left[\frac{V_1^2}{2} + g(z_c)_1 + \frac{p_1}{\rho}\right] - \left[\frac{V_2^2}{2} + g(z_c)_2 + \frac{p_2}{\rho}\right] = C(T_2 - T_1) \quad (2.8.11)$$

where C is the specific heat of the incompressible fluid. For both Eq. (2.8.10) and Eq. (2.8.11), terms within the square brackets represent the total head in the entrance and exit cross sections, respectively.

Equation (2.8.11) indicates that the difference in total head between cross section 1 and cross section 2, in an insulated control volume, is represented by a raise in temperature multiplied by the specific heat of the fluid. On the other hand, if the control volume is kept at constant temperature, namely *isothermal* conditions, then Eq. (2.8.10) yields

$$\left[\frac{V_1^2}{2} + g(z_c)_1 + \frac{p_1}{\rho}\right] - \left[\frac{V_2^2}{2} + g(z_c)_2 + \frac{p_2}{\rho}\right] = -\frac{dQ}{dm} \tag{2.8.12}$$

This expression shows that for an isothermal control volume of incompressible fluid, the head difference between the entrance and exit represents the net heat per unit mass of flow that is transferred from the control volume into its surrounding. The heat transferred from the control volume into the surrounding is created in the control volume due to friction (viscous) forces.

Equations (2.8.11) and (2.8.12) indicate that Bernoulli's equation is approximately satisfied if the control volume does not perform any work on its surrounding and if heat transfer between the control volume and the surroundings is negligible. These equations also show that the conservation of energy with some approximation leads to Bernoulli's equation. Section 2.9.3 extends this discussion with the basic issues of thermal energy sources and transport in the environment.

2.9 SCALING ANALYSES FOR GOVERNING EQUATIONS

As described in Sec. 1.4, it is possible to apply dimensional reasoning to the general governing equations in order to simplify them for most ordinary applications. This process requires that characteristic values for various quantities must be defined *(characteristic scales)* and that the analysis be based on developing order-of-magnitude estimates for different terms in the equation. For now, we define the following characteristic scales for a fluid flow problem:

L = length (for some problems both vertical and horizontal length scales are needed)
U = velocity
Δp_0 = pressure difference
T = time
ρ_0 = density
$\Delta \rho_0$ = density difference
$\Delta \theta_0$ = temperature difference
ΔC_0 = dissolved solids concentration difference
Ω_0 = rotation rate

These scales will be used in the following discussion to estimate the typical order of magnitude for various terms in each of the basic equations discussed in the preceding sections of this chapter. To some extent, the material is parallel to the previous discussions, though the emphasis here is on relative orders of magnitude of different terms in the equations. First, we consider the mass conservation, or continuity equation.

2.9.1 Mass Conservation

The general statement of continuity, or mass conservation, is given by Eq. (2.5.6),

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \rho + \rho \vec{\nabla} \cdot \vec{V} = 0$$

or, dividing by ρ,

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{\rho}\vec{V} \cdot \vec{\nabla} \rho + \vec{\nabla} \cdot \vec{V} = 0 \tag{2.9.1}$$

The scaling quantities defined above are then substituted to estimate the relative magnitudes for each of the terms and, to provide a simpler means of comparison, we divide all the terms in Eq. (2.9.1) by the divergence term, so that the first and second terms will be compared with 1. The respective relative magnitudes for each of the terms are then

$$\begin{aligned} &\left[\frac{1}{T}\frac{\Delta\rho_0}{\rho_0}\right] + \left[\frac{U}{L}\frac{\Delta\rho_0}{\rho_0}\right] + \left[\frac{U}{L}\right] \approx 0 \\ &\quad \Rightarrow \left[\frac{L}{UT}\frac{\Delta\rho_0}{\rho_0}\right] + \left[\frac{\Delta\rho_0}{\rho_0}\right] + [1] \approx 0 \end{aligned} \tag{2.9.2}$$

The procedure is then to compare the probable magnitudes of the first two terms in brackets with [1]. Except in certain cases, where compressible effects become important, the controlling factor is the possible relative change in density that may exist in a flow. Thus it is necessary to estimate the expected changes in density resulting from changes in environmental conditions.

In general, the density of natural water depends on its temperature, salinity and, to a much lesser extent, pressure. Other dissolved solids may affect water density, but the largest variations are due to salt. The rate of change of density with temperature is given by the thermal expansion coefficient,

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial \theta} \tag{2.9.3}$$

where the negative sign indicates that density decreases with increasing temperature. (It should be noted that this is true only when temperature is above the temperature of maximum density, which for pure water is 4°C, so there is the potential that α changes sign for certain problems.) In terms of the scaling quantities defined above, the magnitude of the relative change in density is

$$\left[\frac{\Delta\rho_0}{\rho_0}\right] \approx [\alpha\Delta\theta_0] \tag{2.9.4}$$

In water, α is generally a function of temperature (water density is a parabolic function of temperature, at least over a range of normal environmental temperatures), with magnitude approximately $10^{-4}\,°\mathrm{C}^{-1}$. A typical large temperature variation might be of order 10°C so, using Eq. (2.9.4), the expected magnitude of relative density variations is of order 0.001 (0.1%), which is insignificant compared with 1. Even temperature changes as high as 30–50°C would produce only a relatively negligible change in density for water.

As with temperature, a salinity expansion coefficient can be defined by

$$\beta = \frac{1}{\rho}\frac{\partial\rho}{\partial C} \tag{2.9.5}$$

and

$$\left[\frac{\Delta\rho_0}{\rho_0}\right] \approx [\beta\Delta C_0] \tag{2.9.6}$$

where C indicates the concentration of dissolved solids, primarily salts. Relatively sophisticated expressions have been developed to calculate density in the ocean as a function of temperature and salinity, and a typical value for β is about $8 \times 10^{-4}\ \mathrm{ppt}^{-1}$. Density is approximately linearly related to salinity except when concentrations start to approach saturation, but that is not a condition of major interest for most environmental applications. Typical ocean salinity is approximately 30 ppt (parts per thousand) ($C = 0.03$), so the relative density variation is estimated according to Eq. (2.9.6) as 0.024, or 2.4%. Hypersaline lakes exist in some parts of the world, where C may be as high as 200 or 250 ppt. This would result in $(\Delta\rho_0/\rho_0)$ being of order 20%, but for most natural conditions this result is much less than 1 and may be ignored.

The possible effect of pressure is somewhat more complicated. First, we note that the definition of *sonic velocity*,

$$c_0 = \sqrt{\frac{\partial p}{\partial\rho}} \tag{2.9.7}$$

can be rearranged to obtain

$$\left[\frac{\Delta p_0}{\Delta\rho_0 c_0^2}\right] \approx 1 \Rightarrow \left[\frac{\Delta\rho_0}{\rho_0}\right] \approx \left[\frac{\Delta p_0}{\rho_0 c_0^2}\right] \tag{2.9.8}$$

The value for c_0 is approximately 1,500 m/s in water and with $\rho_0 = 1{,}000\ \mathrm{kg/m^3}$, a pressure difference of order 2.25×10^6 kPa is needed before $(\delta\rho_0/\rho_0)$ becomes of order 1. This is equivalent to the pressure at a depth of 225 km under water, which is clearly unreasonable. This result is, however, consistent with the assumption of incompressible flow that is normally applied for water. Further estimates for δp_0 or $(\delta\rho_0/\rho_0)$ can be obtained under special conditions by looking at possible balances between terms in scaling analyses of the momentum equation. Results from such an exercise show that pressure effects can be neglected for normal environmental conditions in water. In fact, the only circumstances under which this term becomes important are with high-speed flows, when U approaches c_0, with very high frequency oscillatory flow, or with large-scale atmospheric motions or temperature changes.

Thus it may be concluded that $(\delta\rho_0/\rho_0)$ is small for normal environmental conditions. Also, the factor (LU/T) appears in Eq. (2.9.2), but this ratio is usually of order 1, and when it is multiplied by $(\delta\rho_0/\rho_0)$, it becomes very small and may be neglected. Since both the first two terms in Eq. (2.9.2) are negligibly small, and the right-hand side is zero, the only way to balance the equation is to have the third term also equal 0, i.e.,

$$\vec{\nabla} \cdot \vec{V} = 0 \tag{2.9.9}$$

which is the continuity equation for an incompressible fluid, as defined previously in Eq. (2.5.7). Equivalently, referring back to Eq. (2.9.1), we may conclude that $D\rho/Dt = 0$, i.e., the density "following a fluid particle" remains constant. This is consistent with the conclusion found in Sec. 2.5.4.

2.9.2 Momentum Conservation

In vector notation, the general momentum equation is (refer to Sec. 2.7)

$$\begin{aligned}\frac{D\vec{V}}{Dt} + 2\vec{\Omega} \times \frac{D\vec{r}}{Dt} + \frac{D\vec{\Omega}}{Dt} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r})\\ = \vec{g} - \frac{1}{\rho}\vec{\nabla}p + \frac{\mu}{\rho}\left[\nabla^2\vec{V} + \frac{1}{3}\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{V})\right]\end{aligned} \tag{2.9.10}$$

In general, this equation would have a term added to the LHS, $D^2\vec{R}/Dt^2$, to account for translational acceleration of the coordinate system, but for problems of practical interest this term can be neglected. The time derivative term for position also can be replaced by $D\vec{r}/Dt = \vec{V}$, and incompressible fluid will be assumed, as shown above. With these assumptions, Eq. (2.9.10) reduces to

$$\frac{D\vec{V}}{Dt} + 2\vec{\Omega} \times \vec{V} + \frac{D\vec{\Omega}}{Dt} \times \vec{r} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = \vec{g} - \frac{1}{\rho}\vec{\nabla} p + \upsilon \nabla^2 \vec{V} \tag{2.9.11}$$

For problems in environmental fluid mechanics, the frame of reference is the earth's surface, so that $\vec{\Omega}$ represents the rotation of the earth. The earth rotates at a nearly constant rate, so the time derivative term for $\vec{\Omega}$ vanishes. The resulting equation is then similar to Eq. (2.7.20). We now consider the remaining terms.

Figure 2.18 shows a cross section of the earth along a north–south axis, along with the centripetal acceleration vector. The total magnitude of this term is ($\Omega^2 R \cos\theta$), where θ is the latitude. The components, normal (pointing towards the earth's center) and tangential to the earth's surface, are ($\Omega^2 R \cos^2\theta$) and ($\Omega^2 R \cos\theta \sin\theta$), respectively. Similarly, Fig. 2.19 illustrates the components of the Coriolis term, $\vec{\Omega} \times \vec{V}$. The normal and

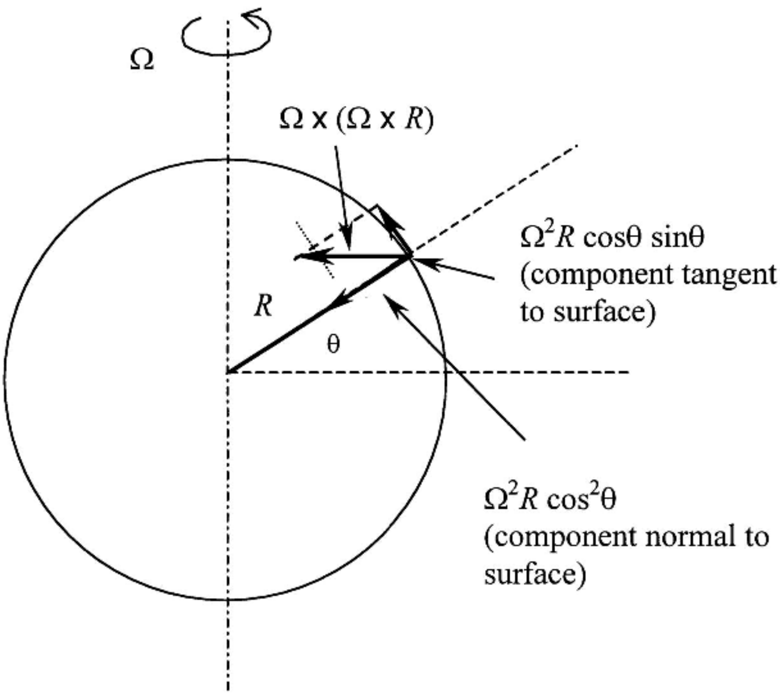

Figure 2.18 Cross section of earth, showing centripetal acceleration term.

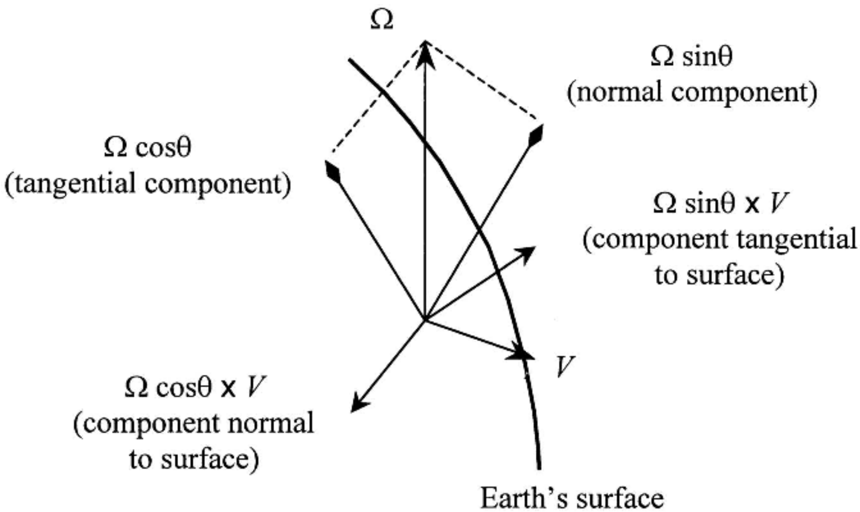

Figure 2.19 Components of Coriolis acceleration, for velocity tangent to surface (note: Coriolis term is $\vec{\Omega} \times \vec{V}$).

tangential components of this term are $((\vec{\Omega}\cos\theta) \times \vec{V})$ and $((\vec{\Omega}\sin\theta) \times \vec{V})$, respectively.

We first compare the normal components with gravity, using values for Ω and R appropriate for rotation of the earth: $\Omega = 2\pi(\text{rad/day}) \cong 7 \times 10^{-5}\ (\text{s}^{-1})$ and $R \cong 6 \times 10^6$ m. The magnitude of the centripetal term is then $(\Omega^2 R) \cong 0.03\ \text{m/s}^2$, which is much less than g ($\cong 10\ \text{m/s}^2$). Also, in order for the normal Coriolis term to be comparable to g, the velocity magnitude would have to be of order $O(10^5\ \text{m/s})$, which is obviously too large for practical consideration.

For the tangential components, first note that the centripetal term is a constant, while the Coriolis term depends on the magnitude of $\vec{V}$. The centripetal term is usually considered as a minor adjustment to gravity, as previously noted (see Eq. 2.7.21) and as shown in Fig. 2.20 (see also Fig. 2.14). For now, we retain the Coriolis term and show in the following discussion under what circumstances it needs to be included. A simplified version of Eq. (2.9.11) is thus

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} + 2\vec{\Omega} \times \vec{V} = \vec{g} - \frac{1}{\rho}\vec{\nabla} p + \upsilon \nabla^2 \vec{V} \qquad (2.9.12)$$

Note that this is essentially the same result as Eq. (2.7.20), with Eq. (2.7.21) substituted for $\vec{g}_{\text{eff}}$ (note also that $\vec{g}_{\text{eff}} \cong \vec{g}$).

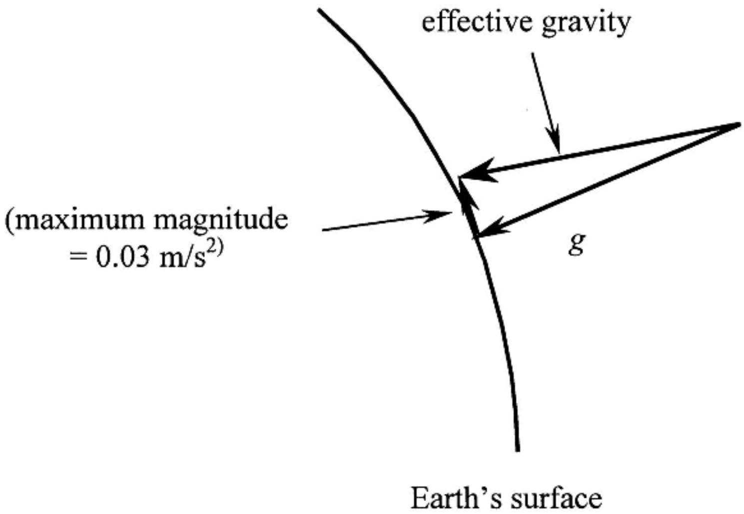

Figure 2.20 Relative importance of the effect of centripetal acceleration as an adjustment to gravity.

This analysis can be extended by considering the pressure term as consisting of hydrostatic and dynamic contributions. Referring to Fig. 2.21, hydrostatic pressure is defined by

$$p_z = p_r - \int_{z_r}^{z} \rho g \, dz \tag{2.9.13}$$

where p_r is a reference value.
The total pressure is the sum of p_z and p_d = dynamic pressure, so the pressure term in Eq. (2.9.12) can be written as

$$\begin{aligned} \frac{1}{\rho}\vec{\nabla} p &= \frac{1}{\rho}\vec{\nabla} p_r - \frac{g}{\rho}\vec{\nabla}\int_{z_r}^{z} \rho dz + \frac{1}{\rho}\vec{\nabla} p_d \\ &= \frac{1}{\rho}\vec{\nabla} p_r - \frac{g}{\rho}\int_{z_r}^{z} \vec{\nabla}\rho dz - g\vec{\nabla} z + g\frac{\rho_r}{\rho}\vec{\nabla}_{z_r} + \frac{1}{\rho}\vec{\nabla} p_d \end{aligned} \tag{2.9.14}$$

where this last result is obtained using the fact that $\rho = \rho_r$ at $z = z_r$. Then, substituting Eq. (2.9.14) into Eq. (2.9.12), we obtain

$$\begin{aligned} &\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} + 2\vec{\Omega}\times\vec{V} \\ &\quad = -\frac{1}{\rho}\vec{\nabla} p_r + \frac{g}{\rho}\int_{z_r}^{z} \vec{\nabla}\rho dh - \frac{\rho_r}{\rho} g\vec{\nabla} z_r - \frac{1}{\rho}\vec{\nabla} p_d + \upsilon\nabla^2\vec{V} \end{aligned} \tag{2.9.15}$$

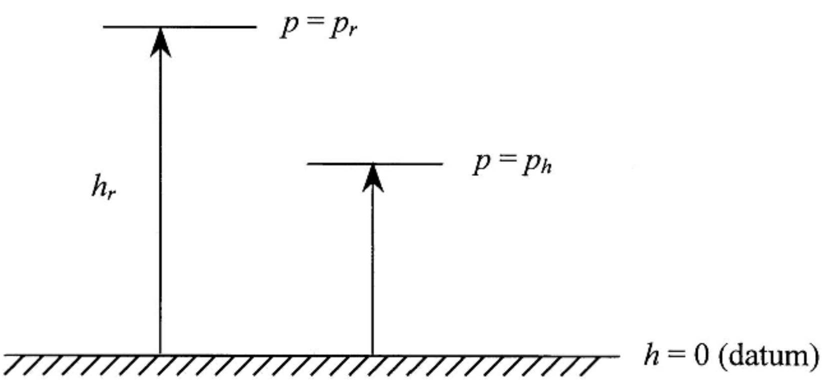

Figure 2.21 Illustration of hydrostatic pressure variations.

If we now let $\rho = \rho_0 + \delta\rho$, where ρ_0 is the constant base, or characteristic value previously defined for density, then

$$\frac{1}{\rho} = \frac{1}{\rho_0 + \Delta\rho} = \frac{1}{\rho_0}\left(\frac{1}{1 + \dfrac{\Delta\rho}{\rho_0}}\right) \qquad \text{and} \qquad \frac{\Delta\rho}{\rho_0} \ll 1$$

(from previous scaling of mass conservation equation), so

$$\frac{1}{\rho} \cong \frac{1}{\rho_0} \qquad \text{and} \qquad \rho_r \cong \rho_0 \tag{2.9.16}$$

This last result is a statement of the *Boussinesq approximation*, which says density variations are negligible except in the buoyancy terms, as will be shown below. Eq. (2.9.15) is thus written as

$$\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} + 2\vec{\Omega}\times\vec{V}$$
$$= -\frac{1}{\rho_0}\vec{\nabla}(p_r + p_d) + \frac{g}{\rho_0}\int_{Z_r}^{Z}\vec{\nabla}\rho dz - g\vec{\nabla}z_r + v\nabla^2\vec{V} \tag{2.9.17}$$

The first term on the RHS of Eq. (2.9.17) is the net force due to pressure gradients, the second term is the effect of density variations (important for stratified fluids), and the third term is the effect of reference surface gradients (such as waves).

Using the same characteristic scaling variables as in Sec. 2.9.1, the magnitudes of the terms in Eq. (2.9.17) may be compared under different scenarios. Dividing by the convective term ($\vec{V}\cdot\vec{\nabla}\vec{V}$), which has characteristic magnitude (U^2/L), results in relative magnitudes as

$$\left[\frac{L}{UT}\right] + [1] + \left[\frac{\Omega_0 L}{U}\right] \approx \left[\frac{\Delta p_0}{\rho_0 U^2}\right] + \left[\frac{g(\Delta\rho/\rho_0)L}{U^2}\right] - \left[\frac{gL}{U^2}\right] + \left[\frac{v}{UL}\right] \tag{2.9.18}$$

where

$$\left[\frac{U}{\Omega_0 L}\right] = \text{Rossby number, Ro}$$

$$\left[\frac{\Delta p_0}{\rho_0 U^2}\right] = \text{Euler number, Eu}$$

$$\left[\frac{U}{\sqrt{g(\Delta\rho_0/\rho_0)L}}\right] = \text{densimetric Froude number, Fr}_\text{d}$$

$$\left[\frac{U}{\sqrt{gL}}\right] = \text{Froude number, Fr}$$

$$\left[\frac{UL}{v}\right] = \text{Reynolds number, Re}$$

Thus, for example, if Ro is large, Coriolis effects should be negligible in the momentum equation. Similarly, pressure effects are small if Eu is small, density effects are negligible if $\mathrm{Fr_d}$ is very large, changes in surface elevation may be neglected if Fr is very large, and viscous effects are small when Re is large.

The time-dependent term $[L/UT]$ is the *Strouhal number*, and it should be clear that a problem may be treated as being steady for large times, $T \to \infty$, when this ratio is small. The values of Ro (order of magnitude) for several representative situations are listed in Table 2.1 using $\Omega_0 \approx 10^{-4}\mathrm{s}^{-1}$, which is valid for mid-latitudes. It is clear from these examples that the Coriolis effect is expected to be important only in systems with larger L (estuaries, large lakes, and ocean currents), depending also on U and Ω_0.

Table 2.1 Estimates of Ro for Different Environmental Systems

	L(m)	U(m/s)	Ro
Stream	1–10	0.1–1	10^2–10^4
Pond	10–100	0.1	10–10^2
River	100	0.1–1	10–10^2
Estuary	10^3–10^4	1	1–10
Large lake	10^3–10^5	0.1	10^{-2}–1
Ocean current	10^5–10^6	0.01–0.1	10^{-4}–10^{-2}

The relative importance of density gradients can be estimated by assuming $\mathrm{Fr_d} \approx 1$. This is easily shown to be equivalent to assuming that the convective term is balanced by the buoyancy term in Eq. (2.9.18), i.e.,

$$\vec{V} \cdot \vec{\nabla}\vec{V} \approx \frac{g}{\rho_0} \int_{z_r}^{z} \vec{\nabla}\rho \, dz \Rightarrow U \approx \left[g \frac{\Delta\rho}{\rho_0} L \right]^{1/2} \tag{2.9.19}$$

Then, for a typical value of $\Delta\rho/\rho_0 \cong 10^{-3}$ (corresponding to a temperature difference of about 10°C), and $g \cong 10 \text{ m/s}^2$, $L = (1\text{–}100 \text{ m})$, gives $U \cong (0.1\text{–}1 \text{ m/s})$. Thus, at least in the buoyancy term, even a small density difference can generate an appreciable velocity. The Boussinesq approach of neglecting density variations does not apply to the buoyancy term, unless very small characteristic lengths (L) are involved.

As a special case of the general result shown in Eq. (2.9.17), consider a situation of steady, constant density flow, with $\Omega = \nabla z_r = 0$. Then

$$\vec{V} \cdot \vec{\nabla}\vec{V} = -\vec{\nabla}\left(\frac{p}{\rho_0} + gz \right) + \upsilon \nabla^2 \vec{V} \tag{2.9.20}$$

where (p/ρ_0) represents dynamic pressure and (gz) is the hydrostatic pressure. This equation is then multiplied by (i.e., take dot product with) $\vec{V}$, to obtain a mechanical energy equation,

$$\vec{V} \cdot \vec{\nabla}\left(\frac{1}{2} V^2 \right) + \frac{1}{\rho_0}(\vec{V} \cdot \vec{\nabla} p) + \vec{V} \cdot \vec{\nabla}(gz) = \upsilon(\vec{\nabla}V)^2 = -\varepsilon \tag{2.9.20}$$

where ε is the viscous dissipation rate for mechanical energy. If we now define total head as

$$H = \frac{V^2}{2g} + \frac{p}{\rho_0 g} + z \tag{2.9.21}$$

(refer to Eq. 2.8.11), then Eq. (2.9.20) becomes

$$\vec{V} \cdot \vec{\nabla}(gH) = -\varepsilon \tag{2.9.22}$$

If $\varepsilon \cong 0$, then this is the *Bernoulli equation*, also derived in Sec. 2.8.

Note that for steady flow, the left-hand-side of Eq. (2.9.22) is the same as the material derivative, (gDH/Dt), and if inviscid conditions are assumed $(\varepsilon = 0)$, then

$$\frac{DH}{Dt} = 0 \Rightarrow \frac{V^2}{2g} + \frac{p}{\rho_0 g} + z = K \text{ (a constant)} \tag{2.9.23}$$

which is the usual form of the Bernoulli equation used in many introductory textbooks. In general, this result holds along a streamline (i.e., following a fluid particle). If, however, the flow is also irrotational ($\vec{\omega} = \vec{0}$), the vector identity

$$\nabla^2 \vec{V} = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{V}) \tag{2.9.24}$$

can be used to show that the Bernoulli result (2.9.23) is valid everywhere in the flow field. This is because the RHS of Eq. (2.9.24) is 0, due to continuity for the first term and irrotationality for the second.

It is interesting to note that $\nabla^2 \vec{V} = 0$ for an irrotational flow field, independent of the value of Re. However, the value of Re controls the rate at which vorticity grows outward from solid boundaries, which may be important for boundary layer analysis (see Chap. 6).

Another special case of interest is when the velocity vanishes, so Eq. (2.9.17) becomes

$$\vec{0} = -\frac{1}{\rho_0} \vec{\nabla} p + g \vec{\nabla} z \Rightarrow 0 = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - g \tag{2.9.25}$$

which gives the hydrostatic pressure field (assuming boundary conditions are known).

One additional case of interest is that of *geostrophic flow*. For this case, there is a balance in the momentum equation between the Coriolis and pressure terms, so

$$2\vec{\Omega} \times \vec{V} \approx -\frac{1}{\rho_0} \vec{\nabla}_p \tag{2.9.26}$$

This balance has many applications in meteorology and in the oceans. When this balance occurs, large-scale pressure differences (gradients), for example, can be related to corresponding characteristic velocities by

$$\frac{\Delta p_0}{L} \approx \rho_0 \Omega_0 U \tag{2.9.27}$$

These flows are discussed further in Chap. 9.

2.9.3 Thermal Energy Equation

The thermal energy equation is derived from the general conservation of energy equation and may be written as (see Sec. 12.3.1 for further discussion)

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \vec{V} \cdot \vec{\nabla} \theta = \frac{\kappa}{\rho c} \nabla^2 \theta - \frac{1}{\rho c} \vec{\nabla} \cdot \vec{\varphi}_{\mathrm{r}} + \frac{1}{\rho c} (\rho_0 \varepsilon) - \frac{\alpha c_0^2 \theta}{c} (\vec{\nabla} \cdot \vec{V}) \tag{2.9.28}$$

where θ is temperature, c is *specific heat*, κ is *thermal conductivity*, φ_{r} is radiation heat flux, ε is the kinematic viscous dissipation rate of mechanical energy, α is the *thermal expansion coefficient*, and c_0 is the *sonic velocity*. The terms on the RHS of this equation relate to conduction (diffusion), radiative heat transfer, viscous heating, and compression or expansion heating, respectively. Following a similar procedure as in the preceding sections, we introduce characteristic values for this equation to derive

$$\left[\frac{\delta\theta_0}{T}\right] + \left[\frac{U\Delta\theta_0}{L}\right] \approx \left[\frac{\kappa\Delta\theta_0}{\rho c L^2}\right] - \left[\frac{\varphi_0}{\rho c L}\right] + \left[\frac{\varepsilon}{c}\right] - \left[\frac{\alpha c_0^2\theta_0}{c}\frac{U}{L}\frac{\Delta\rho_0}{\rho_0}\right] \tag{2.9.29}$$

where the compression/expansion term is scaled as in Sec. 2.9.1, to substitute $(\delta\rho_0/\rho_0)$ for $(\vec{\nabla}\cdot\vec{V})$. To nondimensionalize the equation, each term is divided by the advection term, so

$$\left[\frac{L}{UT}\right] + [1] \approx \left[\frac{\kappa/\rho c}{UL}\right] - \left[\frac{\varphi_0}{\rho_0 c U\Delta\theta_0}\right] + \left[\frac{U^2}{c\Delta\theta_0}\right] + \left[\frac{\alpha^2 c_0^2\theta_0}{c}\frac{1}{\alpha\Delta\theta_0}\frac{\Delta\rho_0}{\rho_0}\right] \tag{2.9.30}$$

where $\varepsilon \approx U^3/L$ has been substituted for the dissipation rate (see Chap. 5).

Typical magnitudes for the terms on the right-hand side are estimated as follows:

Heat conduction: First, note that a *thermal diffusivity* may be defined as

$$k_t = \frac{\kappa}{\rho c} \tag{2.9.31}$$

and the conduction term may be rewritten as

$$\left[\frac{k_t}{UL}\right] = \left[\frac{k_t}{v}\right]\left[\frac{v}{UL}\right] = \left(\frac{1}{\mathrm{Pr}}\right)\left(\frac{1}{\mathrm{Re}}\right) \tag{2.9.32}$$

where Pr is the Prandtl number and signifies the ratio of heat transport to momentum transport. Re is the Reynolds number as defined previously. In water, Pr has a value of about 7 (a fixed value), so conductive heat transfer depends on Re.

Radiative heating: The prime heating source by radiation is the sun, and a typical value for φ_0 in temperate latitudes is about 200–250 $\mathrm{W/m^2}$. The amount of heating that takes place for any given radiative input depends on the length of time over which the heating takes place and, of course, the depth (or volume) of the water body under consideration.

Viscous heating: In water the specific heat is $c \cong 1\mathrm{J/g^\circ C}$. If $(U^2/c\Delta\theta_0)$ is to be of order 1 (i.e., the magnitude of the viscous heating term would be sufficient to require it to be included in the temperature equation), then estimates for $\Delta\theta_0$ may be obtained based on U. At the upper range of environmental flow conditions, water velocities may be of order 1–10 m/s. The characteristic temperature change associated with this range of values is 2.5×10^{-4} to 2.5×10^{-2}°C, which may be ignored under most circumstances.

Compression heating: Because of its dependence on fluid compressibility, this term is generally important only for atmospheric studies, or possibly in the deep oceans. Otherwise it can be neglected.

Thus the final usual form of the temperature equation is

$$\frac{\partial\theta}{\partial t} + \vec{V} \cdot \nabla\theta = k_t \nabla^2 \theta - \frac{1}{\rho c} \nabla \cdot \vec{\varphi}_r \qquad (2.9.33)$$

and this is examined further in Chap. 12.

PROBLEMS

Solved Problems

Problem 2.1 A two-dimensional flow field is given by the following velocity components:

$$u = V\cos(\omega t) \qquad v = V\sin(\omega t)$$

where u and v represent the velocity in the x and y directions, respectively; V and ω are constant coefficients. Provide expressions for the streamlines and pathlines.

Solution

As velocity components are time dependent, the flow is unsteady. The differential equation for the streamlines is

$$\frac{dx}{V\cos(\omega t)} = \frac{dy}{V\sin(\omega t)}$$

By rearranging this expression to solve for dy, we obtain

$$dy = \tan(\omega t)dx$$

Direct integration of this expression then gives the equation of the streamlines as

$$y = \tan(\omega t)x + C$$

where C is an integration constant. This expression indicates that streamlines are straight lines whose slope is time dependent. The differential equations of the streamlines are

$$\frac{dx}{dt} = V\cos(\omega t) \qquad \frac{dy}{dt} = V\sin(\omega t)$$

Direct integration of these expressions, and considering that at time $t = 0$ the fluid particle is located at $x = x_0$ and $y = y_0$, yields

$$x = x_0 + \frac{V}{\omega}\sin(\omega t) \qquad y = y_0 + \frac{V}{\omega} - \frac{V}{\omega}\cos(\omega t)$$

Eliminating time from these expressions, we obtain

$$(x - x_0)^2 + \left(y - y_0 - \frac{V}{\omega}\right)^2 = \left(\frac{V}{\omega}\right)^2$$

This expression indicates that the pathlines are circles with radius V/ω and that the center of each pathline is located at $x = x_0$ and $y = y_0 + V/\omega$.

Problem 2.2 A two-dimensional flow field is given by the following velocity components:

$$u = \alpha y \qquad v = \alpha x$$

where u and v represent the velocity in the x and y directions, respectively, and α is a constant. Provide expressions for the streamlines and pathlines.

Solution

As velocity components are not time dependent, the flow is steady. Therefore the shape of the streamlines does not change with time, and that shape is identical to that of the pathlines. The differential equation for the streamlines is

$$\frac{dx}{\alpha y} = \frac{dy}{\alpha x}$$

and upon rearranging,

$$y\,dy = x\,dx$$

Direct integration of this expression yields the following equation of the streamlines:

$$x^2 - y^2 = C$$

where C is an integration constant. This equation indicates that streamlines are hyperbolas whose asymptotes have a slope of 45°. As expected, the shape of the streamlines does not change with time (since the flow is steady).

The differential equations of the pathlines are

$$\frac{dx}{dt} = \alpha y \qquad \frac{dy}{dt} = \alpha x$$

Differentiating the first expression with regard to time, we obtain

$$\frac{d^2x}{dt^2} = \alpha \frac{dy}{dt}$$

Introducing the first two expressions into the last one then gives

$$\frac{d^2x}{dt^2} - \alpha^2 x = 0$$

The solution of this differential equation is

$$x = C_1 \exp(\alpha t) + C_2 \exp(-\alpha t)$$

where $C_1 + C_2 = x_0$.
Introducing this expression into the basic equation of $dy/dt = \alpha x$ and integrating, we obtain

$$y = C_1 \exp(\alpha t) - C_2 \exp(-\alpha t)$$

where $C_1 - C_2 = y_0$.

We may eliminate time from the expressions of x and y and obtain

$$x^2 - y^2 = 4C_1C_2$$

This expression indicates that pathlines and streamlines have identical shapes, as found previously.

Problem 2.3 A two-dimensional flow field is given by the following velocity components:

$$u = \alpha yt \qquad v = \alpha xt$$

where u and v represent the velocity in the x and y directions, respectively; t is the time and α is a constant. Provide expressions for the streamlines.

Solution

As velocity components are time dependent, the flow is unsteady. However, the velocity vector can be expressed as a product of a space vector with a time function. Therefore the shape of the streamlines does not change with time, and that shape is identical to that of the pathlines, as shown below. The differential equation for the streamlines is

$$\frac{dx}{\alpha yt} = \frac{dy}{\alpha xt}$$

Upon rearranging, this gives

$$y\,dy = x\,dx$$

Direct integration of this expression yields the equation of the streamlines as

$$x^2 - y^2 = C$$

where C is an integration constant. This equation indicates that streamlines are hyperbolas whose asymptotes have a slope of 45°. As previously noted, the shape of the streamlines does not change with time.

Problem 2.4 For each of the following flow fields, calculate components of the rate of strain, vorticity tensor and vector, and the circulation on the sides of a small square with sides of length $2b$ centered on the origin.

(a) $u = ax \qquad v = -ay$
(b) $u = ay \qquad v = ax$
(c) $u = ay \qquad v = -ax$

Solution

Components of the rate of strain tensor:

$$\text{(a)}\quad e_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = a \qquad e_{12} = e_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$e_{22} = \frac{1}{2}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) = -a$$

$$\text{(b)}\quad e_{11} = 0 \qquad e_{12} = e_{21} = a \qquad e_{22} = 0$$

$$\text{(c)}\quad e_{11} = 0 \qquad e_{12} = e_{21} = 0 \qquad e_{22} = 0$$

Components of the vorticity tensor:

$$\text{(a)}\quad \zeta_{11} = \zeta_{22} = 0 \qquad \zeta_{12} = -\zeta_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

$$\text{(b)}\quad \zeta_{11} = \zeta_{22} = 0 \qquad \zeta_{12} = -\zeta_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0$$

$$\text{(c)}\quad \zeta_{11} = \zeta_{22} = 0 \qquad \zeta_{12} = -\zeta_{21} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = a$$

Components of the vorticity vector: Only case (c) is relevant, as the flow is two-dimensional [no vorticity for cases (a) or (b)]. Thus

$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = -2a\vec{k}$$

where $\vec{k}$ is a unit vector in the z-direction.

Note that the component ζ_{21} is equal to half of the corresponding vorticity component.

Circulation values: First, note that the circulation is defined by

$$\Gamma = \oint_c \vec{V} \cdot d\vec{l}$$

where C is a closed curve and dl is a line element. As required, the closed line integral should be performed in the counterclockwise direction along the four sides of the small square as shown in Fig. 2.22.
For flow fields (a) and (b) the circulation vanishes. In case (c), we obtain

$$\Gamma = \left[\int_{-b}^{b} v\,dy\right]_{x=b} + \left[\int_{b}^{-b} u\,dx\right]_{y=b} + \left[\int_{b}^{-b} v\,dy\right]_{x=-b} + \left[\int_{-b}^{b} u\,dx\right]_{y=-b}$$
$$= \int_{-b}^{b} -ab\,dy + \int_{b}^{-b} ab\,dx + \int_{b}^{-b} ab\,dy + \int_{-b}^{b} -ab\,dy = 8ab^2$$

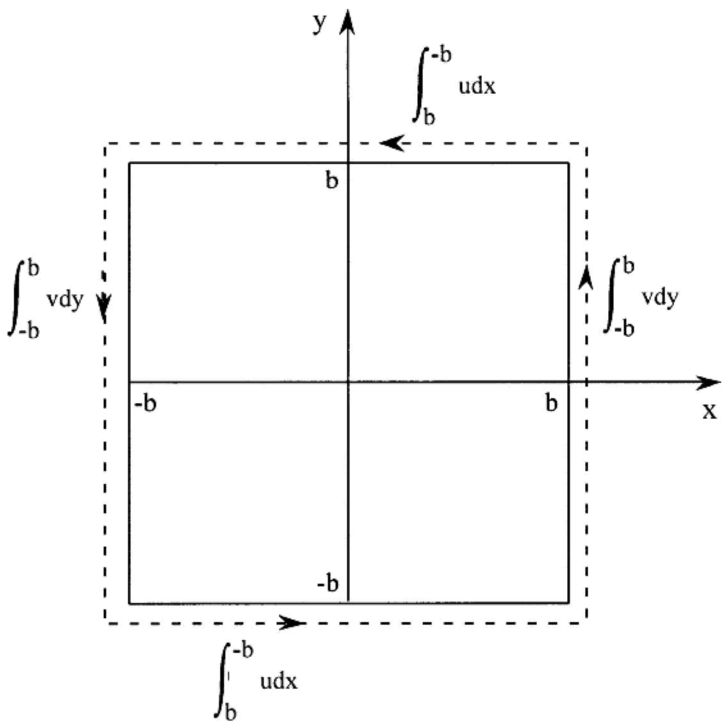

Figure 2.22 Line integration around square element, Problem 2.4.

Problem 2.5 A fluid flow is given by the following pathlines:

$$x = x_0(1 + \alpha t) \qquad y = y_0/(1 + \alpha t)$$

where α is a constant. Calculate the components of the velocity and acceleration vectors by applying the Lagrangian and Eulerian approaches.

Solution

Lagrangian components of velocity:

$$u = \frac{dx}{dt} = \alpha x_0 \qquad v = \frac{dy}{dt} = \frac{-\alpha y_0}{(1 + \alpha t)^2}$$

Eulerian components of velocity: These are obtained by the elimination of x_0 and y_0 from the Lagrangian expressions. According to the pathline equations,

$$x_0 = \frac{x}{1 + \alpha t} \qquad y_0 = y(1 + \alpha t)$$

We introduce these equations into the Lagrangian expressions of the velocity components to obtain

$$u = \frac{\alpha x}{1 + \alpha t} \qquad v = \frac{-\alpha y}{1 + \alpha t}$$

Lagrangian components of the acceleration:

$$a_x = \frac{d^2 x}{dt^2} = 0 \qquad a_y = \frac{d^2 y}{dt^2} = \frac{2\alpha^2 y_0}{(1 + \alpha t)^3}$$

Eulerian components of the acceleration: It is possible to introduce x_0 and y_0 into the Lagrangian expressions by x, y, t to obtain the Eulerian expressions of the acceleration components. Alternatively, the accelerations are obtained by direct application of the substantial derivative, as

$$a_x(x, y, t) = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-\alpha^2 x}{(1 + \alpha t)^2} + \frac{\alpha^2 x}{(1 + \alpha t)^2} = 0$$

$$a_y(x, y, t) = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{\alpha^2 y}{(1 + \alpha t)^2} + \frac{\alpha^2 y}{(1 + \alpha t)^2} = \frac{2\alpha^2 y}{(1 + \alpha t)^2}$$

Problem 2.6 Derive the differential form of the continuity equation directly by considering a small fluid element as shown in Fig. 2.23. Density ρ and fluid velocity (u, v, w) are defined at the center of the element. Use a Taylor series expansion to express the densities and velocities on each face in terms of ρ, u, v, and w.

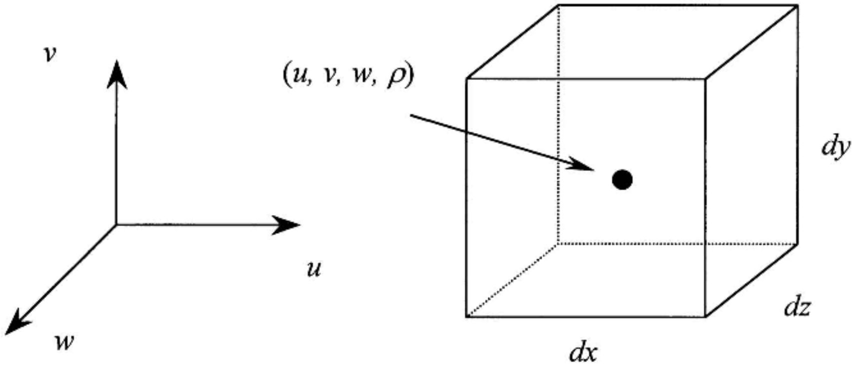

Figure 2.23 Definition sketch, Problem 2.6.

Solution

A general statement of conservation of mass for any control volume is the rate of change of mass in the volume is equal to the rate at which mass is transported into the volume across the control surface, minus the rate at which mass is transported out of the volume, plus or minus the rates at which mass is either created or destroyed in the volume. When applied to the fluid element shown in Fig. 2.23 and noting that water is neither created nor destroyed, this statement is written in mathematical terms as

$$\begin{aligned}\frac{\partial(\rho\forall)}{\partial t} &= \left\{\left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right] - \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\right]\right\} dy\,dz \\ &\quad + \left\{\left[\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right] - \left[\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\right]\right\} dx\,dz \\ &\quad + \left\{\left[\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right] - \left[\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\right]\right\} dx\,dy\end{aligned}$$

where $\forall = dx\,dy\,dz$ is the element volume and higher order terms in the Taylor series expansions have been neglected, with the assumption that dx, dy, and dz are all small. Each of the terms on the right-hand side of this equation represents the net transport of fluid mass across the control surface in each of the three coordinate directions. Nothing that the volume is independent of time, then by combining terms and simplifying, we have

$$\frac{\partial\rho}{\partial t}(dx\,dy\,dz) = -\frac{\partial(\rho u)}{\partial x}(dx\,dy\,dz) - \frac{\partial(\rho v)}{\partial y}(dx\,dy\,dz) - \frac{\partial(\rho w)}{\partial z}(dx\,dy\,dz)$$

Dividing by the volume, $dx\,dy\,dz$, and bringing all terms to the left-hand side then leads to Eq. (2.5.6), which is the desired continuity equation.

Problem 2.7 Figure 2.24 shows a reservoir of volume U, which includes for time $t \leq 0$ pure water with density ρ_0. At time $t = 0$, effluent with volumetric discharge $2Q$ and density ρ_1 starts flowing into the reservoir. The reservoir volume is kept constant due to infiltration of the reservoir water into the ground with volumetric discharge Q, and evaporation of pure water (density ρ_0), also with volumetric discharge rate Q. What is the value of the reservoir fluid density as a function of time? What is the value of that density as $t \to \infty$? Assume that the fluid is kept completely mixed in the reservoir.

Solution

The fluid is incompressible, and density is subject to variation due to dissolved solids, which are assumed to not affect the volume of the water. Therefore, we may refer to Eq. (2.5.2) with regard to volumetric quantities, namely, the reservoir volume is kept constant, and volumetric discharge into the reservoir is identical to the total flow out of the reservoir. Using the integral equation of mass conservation (2.5.2), we obtain

$$\frac{d\rho}{dt}U + Q(\rho + \rho_0) - 2Q\rho_1 = 0$$

Using separation of variables, this expression yields

$$\frac{d\rho}{2\rho_1 - \rho_0 - \rho} = \frac{Q}{U}\,dt$$

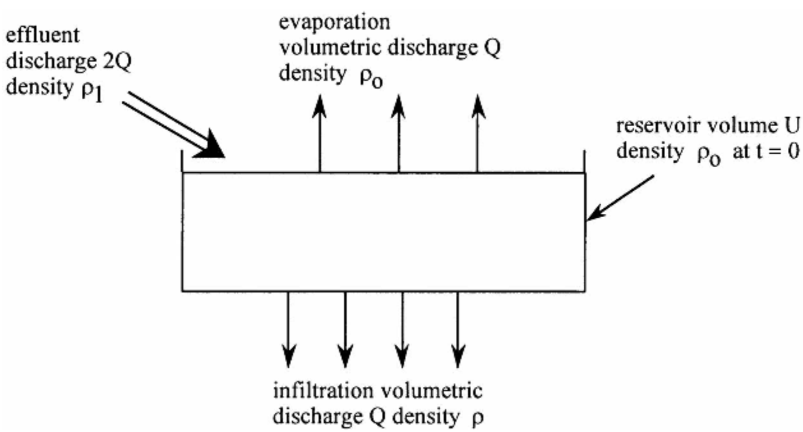

Figure 2.24 Definition sketch, Problem 2.7.

Direct integration of this expression, while considering that $\rho = \rho_0$ at $t = t_0$, yields

$$\frac{2\rho_1 - \rho_0 - \rho}{2\rho_1 - 2\rho_0} = \exp\left(-\frac{Q}{U}t\right)$$

For $t \to \infty$ the RHS of this expression vanishes. Therefore the asymptotic limit for the fluid density is

$$\rho = 2\rho_1 - \rho_0$$

Problem 2.8 Figure 2.25 shows a system of two stagnant plates and a plate that moves downward with velocity V. Due to the movement of the third plate, the incompressible fluid, which is located between the plates, is subject to flow. The velocity in the x-direction is distributed uniformly between the two horizontal plates. Calculate the velocity distribution in the fluid domain when the gap between the two horizontal plates is h. Find the expression for the stream function. Is the fluid domain subject to steady flow?

Solution

The velocity u in the x-direction is independent of the y-coordinate. The integral equation of continuity (2.5.3), applied to the control volume (C.V.)

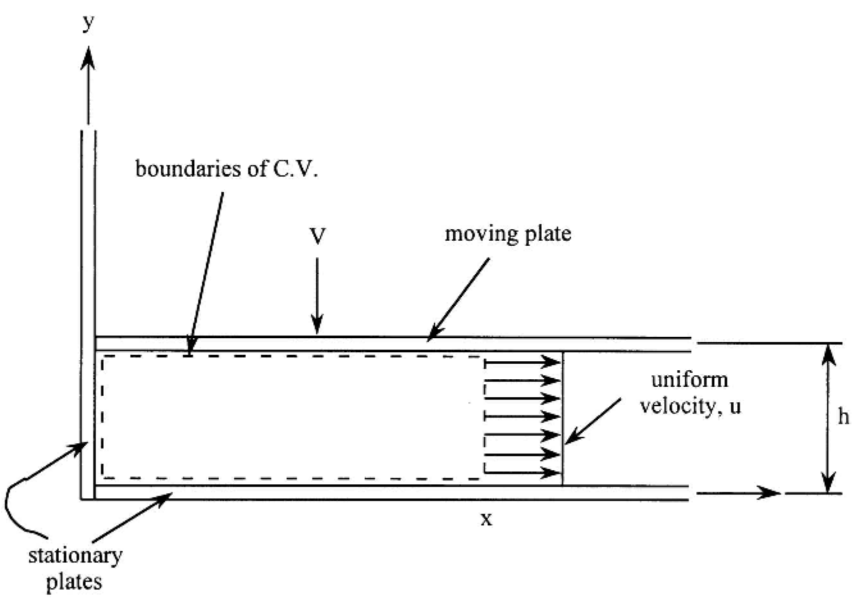

Figure 2.25 Definition sketch, Problem 2.8.

shown in the figure yields

$$-Vx + uh = 0 \Rightarrow u = \frac{V}{h}x$$

Introducing this expression into the differential equation of continuity (2.5.8), we obtain

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -\frac{V}{h}$$

Direct integration of this expression yields

$$v = -\frac{V}{h}y + f(x,\ t)$$

Considering that at $y = 0$, the velocity component v vanishes, we obtain

$$v = -\frac{V}{h}y$$

According to Eq. (2.5.9), the following relationship for the stream function is found:

$$\Psi = \int u\,dy = \frac{V}{h}xy + f(x)$$

The derivative of this expression with regard to the y coordinate is

$$\frac{\partial \Psi}{\partial x} = \frac{V}{h}y + f'(x) = -v = \frac{V}{h}y \Rightarrow f'(x) = 0 \Rightarrow f(x) = const$$

By choosing $f(x) = 0$, we obtain

$$\Psi = \frac{V}{h}xy$$

The flow is subject to unsteady state, as the value of h is time dependent.

Problem 2.9 An incompressible fluid flows past a corner making an angle of $(3\pi/4)$ as shown in Fig. 2.26. It is proposed to describe this flow by a stream function,

$$\Psi = 2r^{4/3}\sin\left(\frac{4}{3}\theta\right)$$

(a) What is the magnitude of the velocity at any point in the flow field (as a function of r)?
(b) Show that there is no flow across the solid boundaries shown.

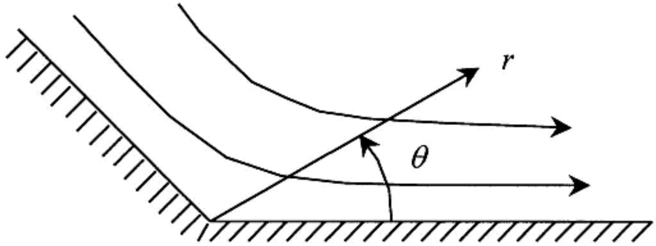

Figure 2.26 Flow of incompressible fluid past a corner, Problem 2.9.

Solution

(a) From Eq. (2.5.13), the velocity components are

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{8}{3}r^{1/3}\cos\left(\frac{4}{3}\theta\right)$$
$$v_\theta = -\frac{\partial \Psi}{\partial r} = -\frac{8}{3}r^{1/3}\sin\left(\frac{4}{3}\theta\right)$$

The velocity magnitude is the square root of the sum of the squares of each of the velocity components,

$$V = \frac{8}{3}r^{1/3}$$

(b) Since both boundaries represent radial arms with respect to the origin at the corner, it is sufficient for this problem to show simply that $v_\theta = 0$ when $\theta = 0$ or $\theta = 3\pi/4$. That this is the case is immediately seen when we use the expression for v_θ from part (a). It should also be noted that this result shows that the proposed stream function satisfactorily describes the flow past this corner.

Problem 2.10 Figure 2.27 shows a cylinder, with weight W_c, with a piston standing on a table. Due to a downward movement of the piston, fluid flows out of the cylinder through a nozzle located at the bottom of the cylinder. The cross-sectional area of the cylinder is A_1, the cross-sectional area of the nozzle outlet is A_2, and the fluid density is ρ. Calculate the forces F_H and F_V, which are needed to hold the cylinder, when the depth of the fluid volume is h.

Solution

A Cartesian coordinate system (x, y) is added for reference. We start with the choice of the control volume $(C.V.)$ as shown in Fig. 2.27. It should be noted that other types of control volumes could be used as well.

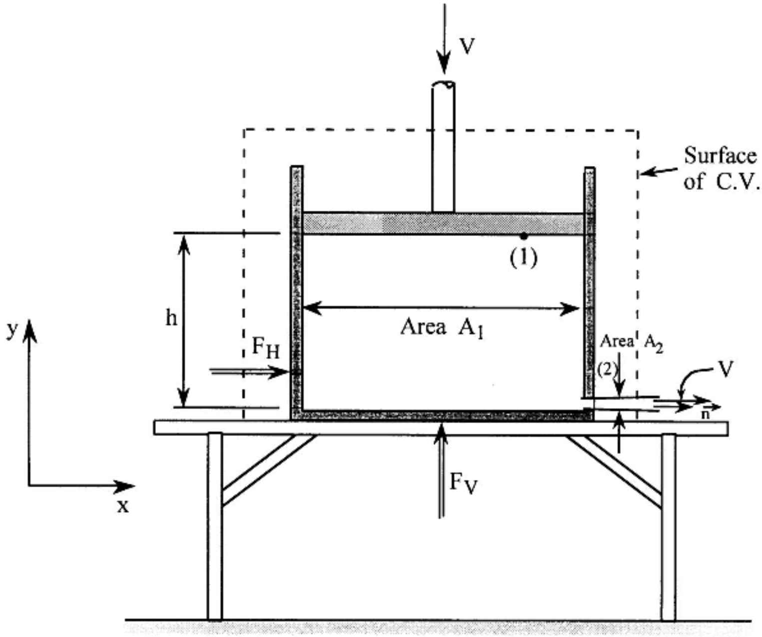

Figure 2.27 Definition sketch, Problem 2.10.

In the x-direction there is no momentum of the control volume. The force F_H is applied by the "solid hand" which is cut by the surface of the control volume. At the exit of the nozzle, the velocity vector and the normal vector have identical directions. Due to continuity, the speed of the jet flowing out of the nozzle is $V(A_1/A_2)$. At the nozzle exit the pressure is equal to atmospheric pressure. Equation (2.6.7) yields for the x-direction:

$$\rho\left[V\frac{A_1}{A_2}\right]^2 = F_{\mathrm{H}}$$

In the y-direction there is a negative momentum. Its values at times t and $t+\Delta t$ are given, respectively, by:

$$(\text{Momentum})_t = -\rho h A_1 V \qquad (\text{Momentum})_{t+\Delta t} = -\rho(h - V\Delta t)A_1 V$$

The difference in momentum between times t and $t+\Delta t$, divided by Δt, provides the first RHS term of Eq. (2.6.7), namely,

$$\frac{\partial}{\partial t}\int_U \rho V_y \, dU = \lim_{\Delta t\to 0}\frac{-\rho(h - V\Delta t)A_1 V + \rho h A_1 V}{\Delta t} = \rho V^2 A_1$$

Two solid surfaces comprise a portion of the surface of the control volume. Through these surfaces two forces are applied. One of them is F_V and the other one is applied through the shaft of the piston. The force applied through the shaft of the piston can be calculated using the Bernoulli equation. We consider that the piston movement is slow, and approximately steady state conditions prevail in the fluid. Then Bernoulli's equation applied between point (1) and point (2) yields

$$\frac{V^2}{2g} + \frac{p_1}{\gamma} + h = \frac{[V(A_1/A_2)]^2}{2g} \Rightarrow p_1 = \rho\frac{V^2}{2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] - \rho g h$$

Considering equilibrium of the piston, we obtain

$$\begin{aligned} p_1 A_1 &= W_p + F_p; \Rightarrow F_p = p_1 A_1 - W_p \\ &= \rho\frac{V^2 A_1}{2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] - \rho g h A_1 - W_p \end{aligned}$$

where W_p is the weight of the piston and F_p is the force applied through the shaft of the piston.

Introducing all the expressions developed in the preceding paragraphs with regard to the y-direction into Eq. (2.6.7), we obtain

$$\rho V^2 A_1 = -\rho g h A_1 - W_\mathrm{c} - W_\mathrm{p} - F_\mathrm{p} + F_\mathrm{V}$$

where W_c is the weight of the cylinder. Therefore the force F_V is given by

$$\begin{aligned} F_\mathrm{V} &= \rho V^2 A_1 + \rho g h A_1 + W_\mathrm{c} + W_\mathrm{p} + \rho\frac{V^2 A_1}{2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] \\ &\quad - \rho g h A_1 - W_p \\ &= W_\mathrm{c} + \rho\frac{V^2 A_1}{2}\left[\left(\frac{A_1}{A_2}\right)^2 + 1\right] \end{aligned}$$

Problem 2.11 Figure 2.28 shows a small cart moving with velocity V_v due to the impact of a two-dimensional water jet on a plate oriented at an angle α with respect to the jet direction. The velocity and width of the jet are V_1 and b_1, respectively. The water jet is divided into two smaller jets, whose widths are b_1 and b_2. The force applied by the water jet on the cart is perpendicular to the impacted plate. Assuming that the effect of gravitation is negligible when applying Bernoulli's equation, calculate (a) The widths b_1 and b_2 of the two jets-(b) the vertical and horizontal forces acting on the cart, and (c) the power transferred to the moving cart.

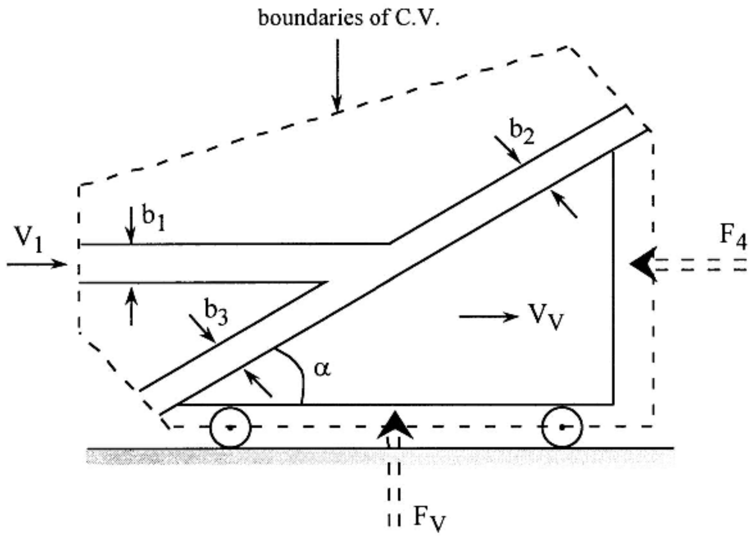

Figure 2.28 Water jet driving cart motion, Problem 2.11.

Solution

We apply a coordinate system that moves with the cart. In such a coordinate system the domain is subject to steady state, and Bernoulli's equation is applicable. The velocity of the jet that hits the cart, in the new coordinate system, is $V_1 - V_v$. As the effect of gravitation is negligible at the jet division, the velocities of the two jets created by the jet division are also $V_1 - V_v$. We apply the control volume with boundaries as shown in the figure. The forces F_{H} and F_{V} are needed to keep the control volume in its appropriate position. By applying the equation of momentum conservation (2.6.1) in the horizontal direction, we obtain

$$-\rho(V_1 - V_v)^2 b_1 + \rho(V_1 - V_v)^2 (b_2 - b_3)\cos\alpha = -F_H$$

Applying the conservation of momentum in the y-direction gives

$$\rho(V_1 - V_v)^2 (b_2 - b_3)\sin\alpha = F_{\mathrm{V}}$$

As the resultant force is perpendicular to the oblique plate, we obtain

$$\frac{F_{\mathrm{V}}}{F_{\mathrm{H}}} = \tan\alpha$$

From continuity, we have

$$b_1 = b_2 + b_3$$

The last four equations allow the determination of the four unknown quantities b_2, b_3, F_H, and F_V. The results of the calculation are

$$b_2 = \frac{b_1}{2}\left(1 + \frac{1}{2\cos\alpha}\right) \quad b_3 = \frac{b_1}{2}\left(1 - \frac{1}{2\cos\alpha}\right)$$

$$F_H = \rho\,(V_1 - V_v)^2\,\frac{b_1}{2} \qquad F_v = \rho\,(V_1 - V_v)^2\,\frac{b_1}{2}\tan\alpha$$

The force that leads to the cart movement is equal to F_H and acts in the positive x direction. The power transferred from the water jet into the cart is equal to the product of this force with the velocity V_v of the cart in the horizontal direction. Therefore the power N is given by

$$N = \rho\,(V_1 - V_v)^2\,\frac{b_1}{2} V_v$$

Problem 2.12 Figure 2.29 shows a rocket fired from rest in outer space along a horizontal straight line where air friction is negligible. The mass of the body of the rocket is M and it carries an original fuel mass M_f which burns at a mass flow rate α. The exhaust cross-sectional area and velocity relative to the rocket are A_e and V_e, respectively, and the density of the fluid at the exhaust is ρ_e. The velocity of the rocket relative to a fixed observer is V. Our objective is to determine the value of V as a function of time.

Solution

We apply Eq. (2.6.15) to solve this problem. The momentum due to the flow inside the control volume is assumed to be negligible. Therefore the first LHS term of this equation vanishes. Also, all terms of the RHS of Eq. (2.6.16) vanish, except for the volume integral associated with the translational acceleration. Therefore we obtain

$$-\rho_e V_e^2 A_e = -(M + M_f - \alpha t)\frac{dV}{dt}$$

Conservation of mass yields

$$\rho_e V_e A_e = \alpha$$

We introduce this relationship into the equation of momentum conservation. Separation of variables of the resulting expression yields.

$$\frac{dV}{\alpha V_e} = \frac{dt}{M + M_f - \alpha t}$$

Direct integration of this expression and assuming that $V = 0$ at $t = 0$ yields

$$V = V_e \ln\left(\frac{M + M_f}{M + M_f - \alpha t}\right)$$

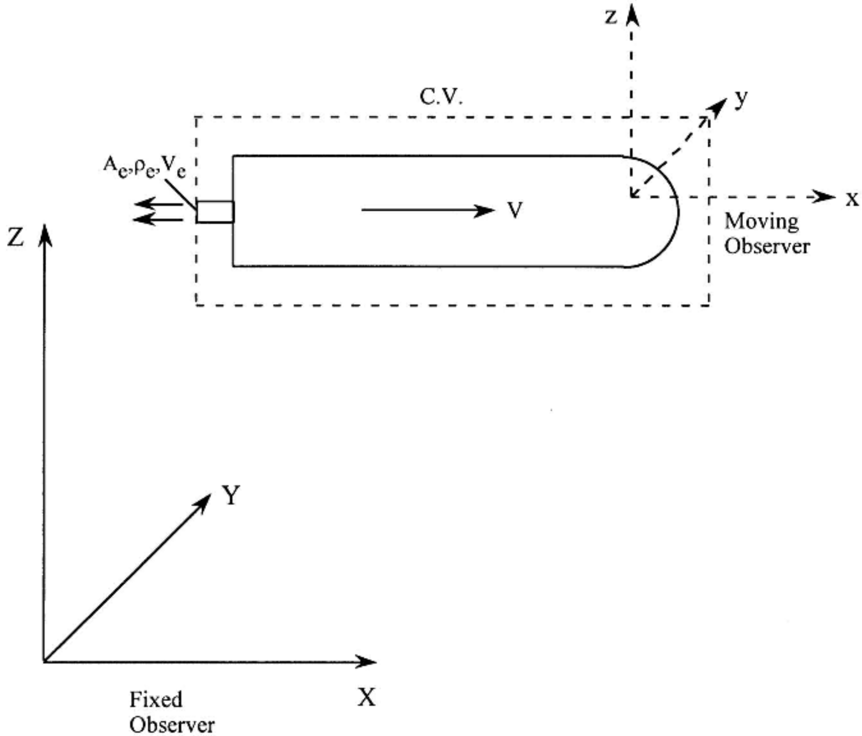

Figure 2.29 Rocket motion, Problem 2.12.

According to this expression, the maximum value of the rocket velocity is obtained when all the fuel is burnt, namely when $t = M_f/\alpha$. At that time the rocket velocity is given by

$$V = V_e \ln\left(\frac{M + M_f}{M}\right)$$

Problem 2.13 Figure 2.30 shows a pump that delivers a water discharge Q from a tank through a pipe of total length L, which is ended with a nozzle. The pipe diameter is D_1, the nozzle diameter D_2. The Darcy–Weissbach friction coefficient for the pipe flow is f. Water level in the tank is h_1 and its value is given. The exit of the nozzle is located at an elevation h_2 above the pump, which is also given. Calculate the power delivered by the pump into the flowing water. The system is at constant temperature.

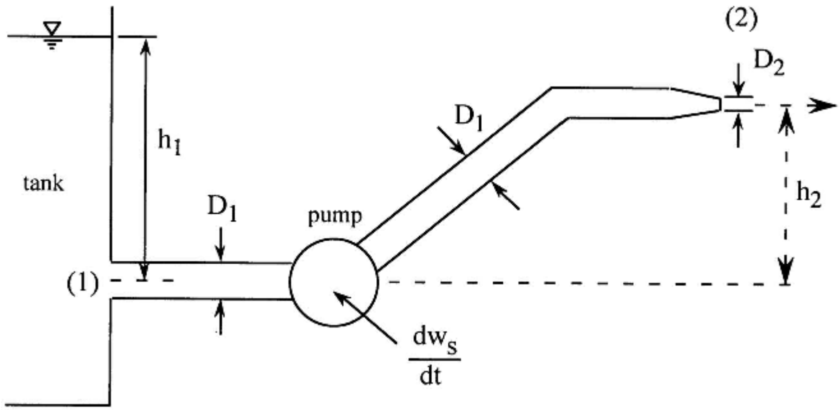

Figure 2.30 Pumped water jet, Problem 2.13.

Solution

The total head at the exit of the nozzle, at cross section (2), is

$$H_2 = \frac{V_2^2}{2g} + h_2 \qquad \text{where} \qquad V_2 = \frac{4Q}{\pi D_2^2}$$

The total head at the entrance cross section (1) is equal to h_1. The power of the pump is needed to increase the water head from its initial value h_1 to its final value at the exit cross section. Part of this power is converted to heat, which is transferred into the surroundings (so that the system remains at constant temperature). The head loss between the entrance and exit multiplied by the weight discharge is equal to the rate of heat transferred into the environment. Therefore the power delivered from the pump into the flowing water is given by

$$N = \rho g Q \left(H_2 - h_1 + f \frac{L}{D_1} \frac{V_1^2}{2g} \right) \qquad \text{where} \qquad V_1 = \frac{4Q}{\pi D_1^2}$$

Problem 2.14 Considering the flow given in Problem 2.9, find the pressure at any point in the flow field, relative to $p = p_0$ at the corner. Neglect gravity.

Solution

It is already known that this flow is steady and incompressible. It can also be shown to be irrotational. In this case, pressures are found using Bernoulli's equation. Since gravity effects are neglected, we have

$$\frac{p_0}{\gamma} + \frac{V_0^2}{2g} = \frac{p}{\gamma} + \frac{V^2}{2g}$$

From the velocity components found in Problem 2.9, it is easily seen that $V_0 = 0$. Then, substituting the general expression for the velocity, we find

$$p = p_0 - \frac{1}{2}\rho V^2 = p_0 - \frac{32}{9}\rho r^{2/3}$$

Unsolved Problems

Problem 2.15 A two-dimensional flow field is given by the following velocity distribution:

$$u = a(y - b) \qquad v = a(x - b)$$

where a and b are constant coefficients.

(a) Develop the expression for the pathlines in the domain.
(b) Develop the expression for the streamlines. Show that streamlines and pathlines have the same shape. Provide a schematic of the streamlines.

Problem 2.16 Using the velocity distribution of Problem 2.15,

(a) Calculate values of components of the rate of strain tensor.
(b) Show that the fluid is subject to irrotational flow and develop the expression for the potential function.

Problem 2.17 A two-dimensional flow field is given by

$$u = -a(y - b) \qquad v = a(x - b)$$

where a and b are constant coefficients.

(a) Determine values of components of the rate of strain tensor and the vorticity tensor.
(b) Calculate the value of the circulation along a circle whose center is at point (b, b), with radius b.

Problem 2.18 The velocity field for a two-dimensional flow is given by

$$u_1 = U \exp\left(-\frac{x_1}{L}\right) \sec h^2\left(\frac{x_2}{L}\right) \qquad \text{and}$$

$$u_2 = Cx_2 + U \exp\left(-\frac{x_1}{L}\right) \tanh\left(\frac{x_2}{L}\right)$$

Where U, C, and L are constants. Find

(a) Acceleration of a fluid particle
(b) Variation of density of a fluid particle
(c) Components of fluid vorticity
(d) Components of fluid rate of strain

Problem 2.19 For each of the velocity distributions of Problems 2.15 and 2.17,

(a) Calculate the Lagrangian components of the velocity and acceleration.
(b) Calculate the Eulerian components of the velocity and acceleration.

Problem 2.20 Velocity components of the flow and density of a fluid are given by

$$u = x(1 + \alpha xy) \qquad v = -y(1 + \alpha xy) \qquad \rho = \frac{\rho_0}{1 + \alpha xy}$$

where α and ρ_0 are constants.

(a) Calculate the components of the acceleration.
(b) Calculate the rate of change of density of the fluid particles, assuming that α is small and has negligible effect on u and v.

Problem 2.21 Starting with the fluid element shown in Fig. 2.31, demonstrate graphically that the divergence $(\vec{\nabla} \cdot \vec{V})$ must be zero if the fluid is incompressible. Is it necessary that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial z} = 0$ in order to make the same conclusion?

Problem 2.22 For each of the velocity distributions of Problems 2.15 and 2.17,

(a) Show that continuity is satisfied for incompressible flow.
(b) Determine the expression for the stream function.
(c) Calculate the flow rate between two streamlines of your choice.

Problem 2.23 For each of the velocity and density distributions of Problem 2.20,

(a) Show that the equation of mass conservation is satisfied.
(b) Develop the expression for the stream function of the mass flux.

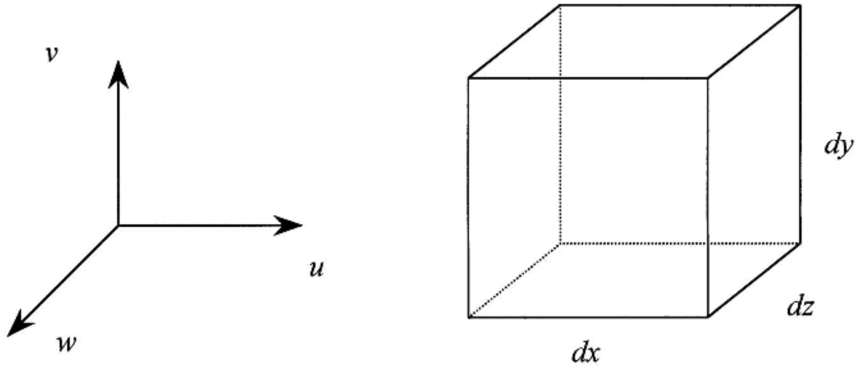

Figure 2.31 Fluid element, Problem 2.21.

Problem 2.24 Derive an integral statement of the equation expressing conservation of dissolved mass (concentration C) following the Reynolds transport theorem approach. Where might such an equation be useful?

Problem 2.25 Figure 2.32 shows a section of a two-dimensional channel with walls described by

$$y = \pm \frac{h}{2(x+1)}$$

where h is the width of the channel at its entrance where $x = 0$. A fluid of constant density flows through the channel. The velocity component in the x-direction is solely a function of x. At the channel entrance the velocity in the x-direction is given by $u = u_0$.

(a) Determine the velocity component in the x-direction.
(b) Determine the velocity component in the y-direction.
(c) Develop the expression for the stream function in the channel. What are reasonable values of the stream function along the walls of the channel?

Problem 2.26 Fluid is subject to steady-state flow in an infinite domain. In every vertical cross section of the domain, the velocity component in the horizontal x-direction is not a function of y. In every horizontal cross section of the domain, the velocity component in the vertical y-direction is not a function of x. At the point ($x = 8$ m, $y = 0$) it was found that there is only velocity in the x-direction, whose value is $u = 0.1$ m/s. At the point ($x = -12$ m, $y = 0$) it was found that there is also only velocity in the x-direction, with a value of $u = -0.1$ m/s.

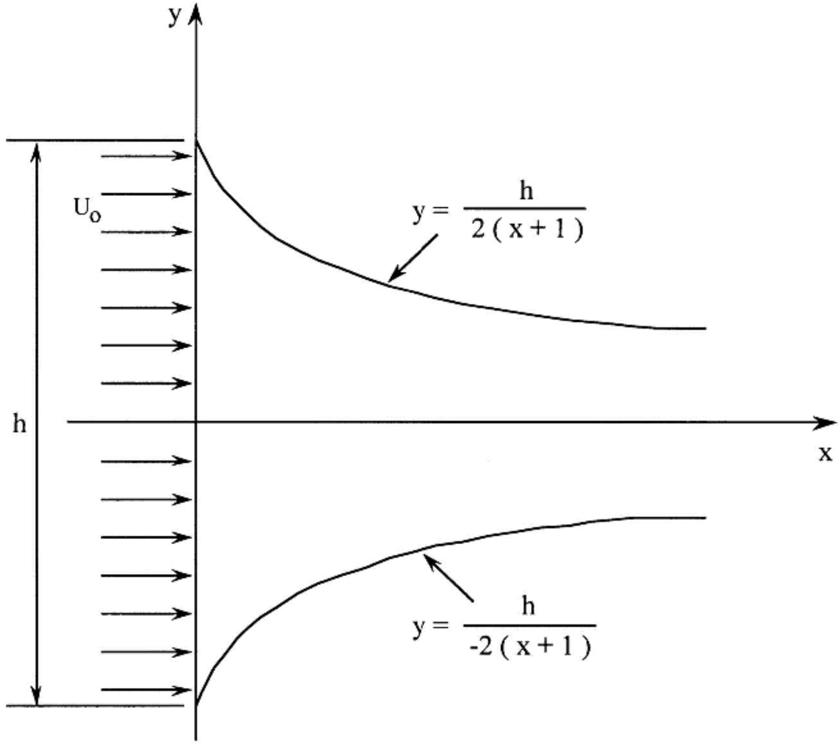

Figure 2.32 Two-dimensional converging flow, Problem 2.25.

(a) Apply the integral continuity equation to determine the distribution of the velocity component in the y-direction.
(b) Apply the differential continuity equation to determine the distribution of the velocity component in the x-direction.
(c) Develop the expression for the stream function. Find stagnation points, and provide a schematic of the streamlines.
(d) Check whether the flow is a potential flow. If it is a potential flow, determine the expression for the potential function.
(e) Determine components of the rate of strain tensor.
(f) Determine components of the rate of strain tensor in the entire domain in the coordinate system $(\overline{x}, \overline{y})$ whose $\overline{x}$ axis bisects the angle between the axes x and y.

Problem 2.27 A water reservoir has a volume $U = 50{,}000\ \text{m}^3$. At time $t = 0$ the density of the water is $\rho_0 = 1000\ \text{kg/m}^3$. At that time two effluent sources start to divert water into the reservoir. Both sources provide an identical volumetric discharge of $Q = 36\ \text{m}^3/\text{s}$ (for each source). The density of the fluid of the first source is $\rho_1 = 1{,}020\ \text{kg/m}^3$. The density of the fluid of the second source is $\rho_2 = 1{,}010\ \text{kg/m}^3$. These sources may be assumed to be rapidly mixed throughout the reservoir. Fluid percolates into the ground through the bottom of the reservoir with flow rate Q and with density equal to that of the reservoir water ρ. At the reservoir surface water evaporates, with discharge Q and density ρ_0.

(a) Prove that the reservoir volume is kept constant.
(b) Develop a general equation for the variation of the density of the reservoir water. What is the value of this density for time $t \to \infty$?
(c) Substitute numerical values of the variables, and find the time at which the water density becomes 99% of its value at $t \to \infty$.

Problem 2.28 A model is needed to predict the transient response of a constant volume mixing tank due to a step change in influent concentration of a conservative substance. The model is to be used to quantify the degree of mixing and short-circuiting in the tank. Assume that a fraction m of the total tank volume V is actually well mixed and that only a fraction n of the inflow Q enters the zone of perfect mixing, while the remaining portion of the inflow short-circuits directly to the outlet (i.e., it is not mixed at all inside the tank). The concentration at any time t in the mixed zone is C'. The material exiting from this zone is mixed with the portion of inflow that is short-circuited and the mixture leaves the tank at flow rate Q and concentration C. The initial concentration in the tank is C_0 (everywhere). At time $t = 0$ the influent concentration C_i is changed suddenly from $C_i = C_0$ to $C_i = 0$.

(a) Sketch the problem, showing how n and m are incorporated.
(b) Show that, in general, the outflow concentration may be calculated as $C = nC' + (1 - n)C_i$
(c) Write the general mass balance equation for C' (in the mixed zone) — include C_i in the formulation.
(d) Substitute the result from part (b) into your result from part (c) and develop a differential equation that describes the rate of change of C with time.
(e) Solve the equation to calculate (C/C_0) as a function of n, m, and (t/t^*), where $t^* = V/Q$ is the overall tank residence time.
(f) Using the experimental data plotted in Fig. 2.33, estimate the values for n and m (note that values for C' are obtained from the middle of the tank, which is expected to be in the fully mixed region).

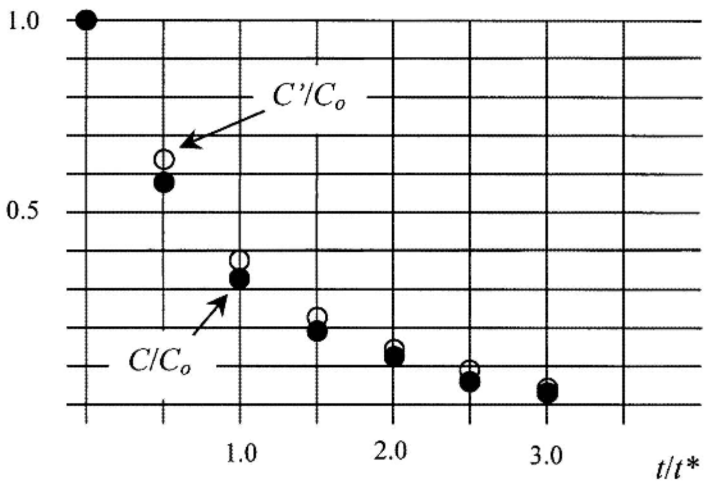

Figure 2.33 Nondimensional concentration data, Problem 2.28.

Problem 2.29 A shallow lake has mean (depth-averaged) horizontal velocity components U and V, in the x and y coordinate directions, respectively, and U and V are in general functions of (x, y, t), where $t =$ time (see Fig. 2.34). Seepage out the bottom of the lake takes place at a rate f, where f is assumed to be directly proportional to the depth, h, so $f = kh$, and h is also a function of (x, y, t). Rain falls at rate i (units of length/time) and $i = i(x, y, t)$. The lake bottom may be assumed to be flat and horizontal. Derive the two-dimensional continuity equation for this problem.

Problem 2.30 Figure 2.35 shows a section of a two-dimensional channel, with walls that are described by

$$y = \pm 0.5(h + x)$$

where $h = 1$ m is the width of the channel at its entrance, where $x = 0$. A fluid of constant density flows through the channel. The velocity component in the x-direction is solely a function of x. At $x = 0$, the velocity in the x-direction is given by $u = u_0 = 1$ m/s.

(a) Determine the velocity component in the x-direction.
(b) Determine the velocity component in the y-direction.
(c) Calculate the discharge per unit width of the channel.

Problem 2.31 Water is subject to unsteady flow in an open channel, as shown in Fig. 2.36. A discharge per unit area, q, flows into the channel through the

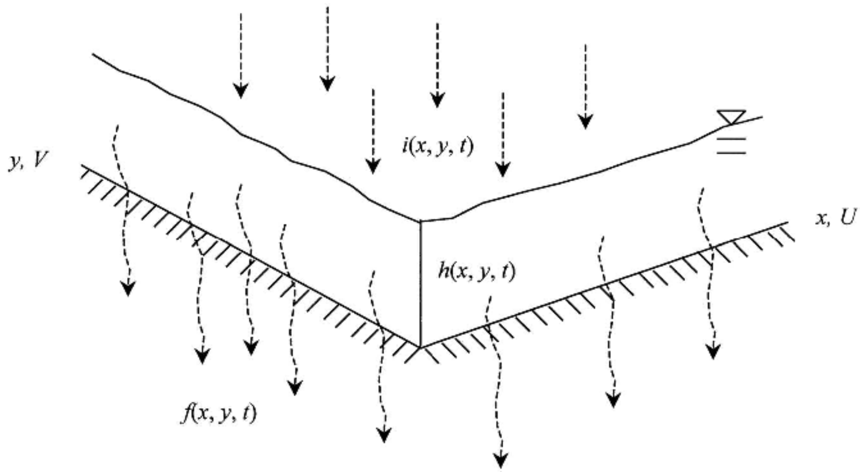

Figure 2.34 Two-dimensional lake schematic, Problem 2.29.

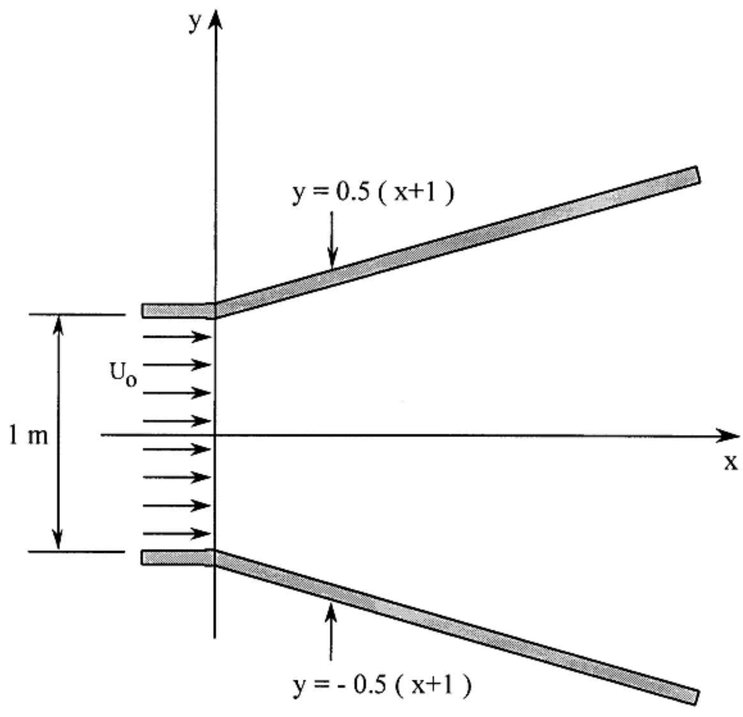

Figure 2.35 Expanding two-dimensional flow, Problem 2.30.

free surface. The water depth in the elementary control volume is h. The width of the channel at the free surface is B.

(a) Refer to the elementary control volume of the open channel shown in part (a) of Fig. 2.36 and develop the differential equation that represents the variation of the water depth along the channel.

(b) Part (b) of Fig. 2.36 indicates that at $x = x_0$, the water depth is h_0. The channel has a rectangular cross section, in which $B = const$. It is found that the water depth downstream of x_0 is represented by $h = h_0 + h_1 \sin(\alpha x + \omega t)$, where, h_0, h_1, α, and ω are constants. It

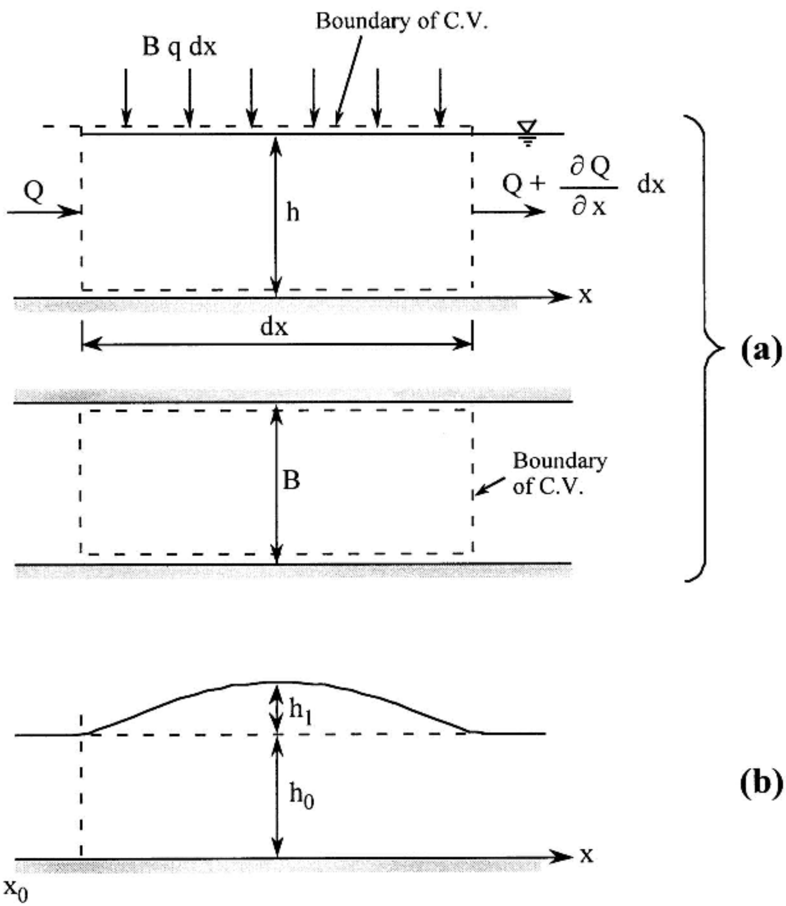

Figure 2.36 Open channel flow, Problem 2.31.

is also given that $q = 0$, and $Q = Q_0$ at x_0. Find the discharge as a function of x and t.

Problem 2.32 Figure 2.37 shows two containers that contain fluids. The volume of container 1 is $U_1 = 30\ \text{m}^3$ and it contains at time $t < 0$ pure water, with density $\rho_1 = 1{,}000\ \text{kg/m}^3$. The volume of container 2 is $U_2 = 20\ \text{m}^3$ and it includes for time $t < 0$ salt water, with density $\rho_2 = 1{,}020\ \text{kg/m}^3$. At time $t = 0$ two pumps start to circulate water between the two containers. Each pump delivers a volumetric discharge of $Q = 10\ \text{m}^3/\text{s}$. A mixer is submerged in each container to insure well-mixed conditions.

(a) What is the final density of the water in both containers?
(b) Develop expressions for the variation of the water density in each container as functions of t.
(c) Show that the expressions that you developed in part (b) converge to the result of part (a) when $t \to \infty$.
(d) Calculate the value of the time t at which the density of the water in container 1 is equal to 99% of the density when $t \to \infty$. What is the density of the water in container 2? By how many percent is it larger than the density at $t \to \infty$?

Problem 2.33 Figure 2.38 shows a two-dimensional incompressible flow between two long plates. Plate (a) is stagnant. Plate (b) rotates around the origin with constant angular velocity Ω. The radial flow in the domain is not a function of the angular coordinate θ. At time $t = 0$, the angle between the two plates is π.

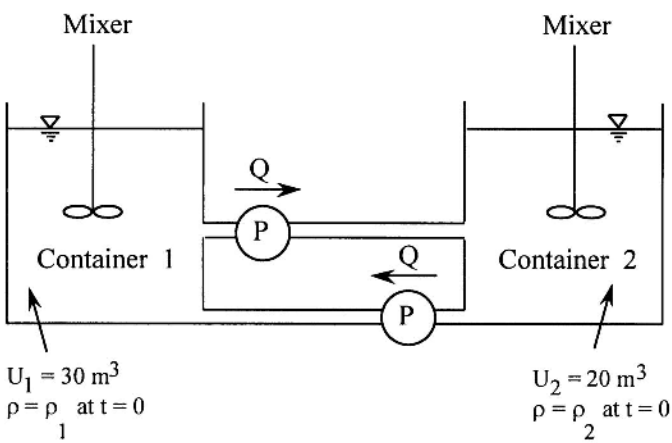

Figure 2.37 Definition sketch, Problem 2.32.

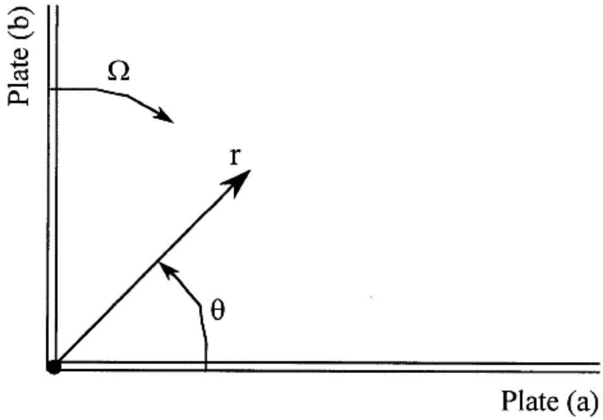

Figure 2.38 Definition sketch, Problem 2.33.

(a) Determine the velocity distribution in the flow domain at time t_1, where $t_1 < \pi/\Omega$.
(b) Determine the expression for the stream function.
(c) Is the value of the stream function at the two plates subject to variation with time? Explain.

Problem 2.34 Figure 2.39 shows viscous incompressible fluid between three plates. Plates (a) and (b) are stagnant, while plate (c) moves downward with constant velocity V. Due to the movement of plate (c), the fluid is subject to flow. There is a parabolic distribution of the velocity component in the x-direction, and it vanishes at plates (a) and (c).

(a) Determine the expressions for the velocity components in the flow domain when the distance between plates (a) and (c) is h.
(b) Determine the expression for the stream function in the domain.
(c) Calculate the variation of the discharge flowing between plates (a) and (c) as a function of time and x-coordinate.

Problem 2.35 A two-dimensional velocity field (u, v) may be defined in terms of a stream function, Ψ, where

$$\vec{V} = \vec{\nabla} \times \Psi(\hat{k})$$

Calculate $\vec{\nabla} \times \vec{V}$, $\nabla^2\vec{V}$, and $\vec{V} \cdot \vec{\nabla}\vec{V}$ in terms of Ψ.

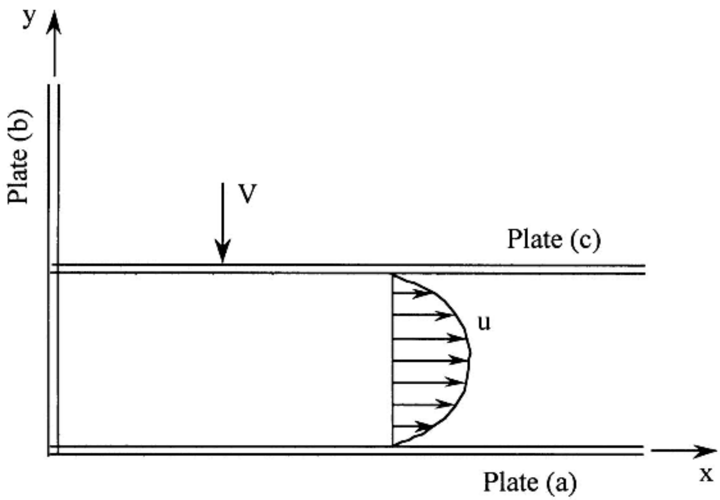

Figure 2.39 Definition sketch, Problem 2.34.

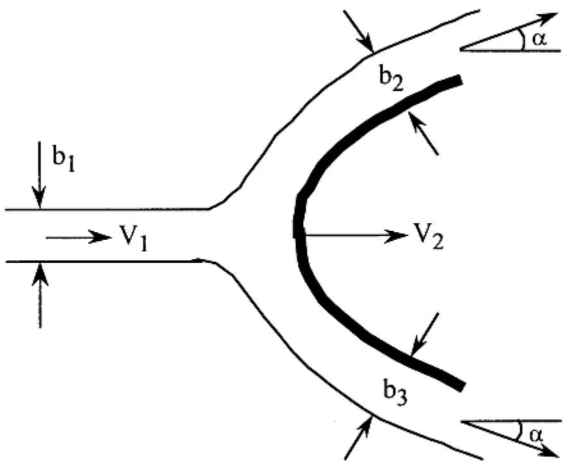

Figure 2.40 Definition sketch, Problem 2.36.

Problem 2.36 A fluid two-dimensional jet of width b_1 and velocity V_1 is directed at a concave plate, which moves with velocity V_2, as shown in Fig. 2.40. Due to the impact with the plate, the fluid jet is divided into two identical jets, which are oriented with angle α to the longitudinal x-direction at the edges of the plate. The fluid density is ρ.

(a) Calculate the thickness of the two jets created by the impact of the jet with the concave plate.
(b) Determine the velocity of the two jets at the edges of the plate.
(c) Determine the power delivered from the jet to the plate.
(d) What should be the relationship between V_1 and V_2 to deliver maximum power?

Problem 2.37 Repeat Problem 2.36 for jet impact with a convex plate, as shown in Fig. 2.41.

Problem 2.38 Consider plane Couette flow, with one wall ($y = 0$) fixed and a second rigid wall ($y = h$) moving at constant speed U in its own plane. Sketch the flow and solve the Navier–Stokes equations for the case of constant density (also no rotation), to show that a possible flow is $\vec{u} = (Uy/h)(\hat{i})$. Also calculate the shear stress on each wall.

Problem 2.39 A *line sink* (large width-to-height ratio) drains a large water reservoir by a rectangular conduit as shown in Fig. 2.42. Assuming the flow is fully developed in the conduit (i.e., at some distance downstream of the reservoir), calculate the following:

(a) Velocity distribution (neglect side wall effects).
(b) Magnitude of shear stress at upper and lower surfaces and at middle of conduit.
(c) Considering the entire length (L) as a control volume, verify that there is zero net force acting on the fluid in the direction along the pipe.

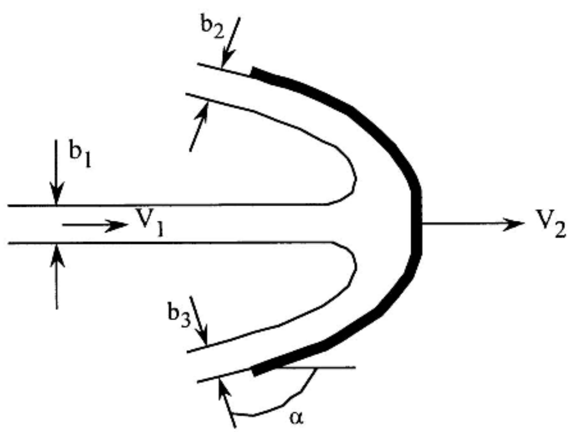

Figure 2.41 Definition sketch, Problem 2.37.

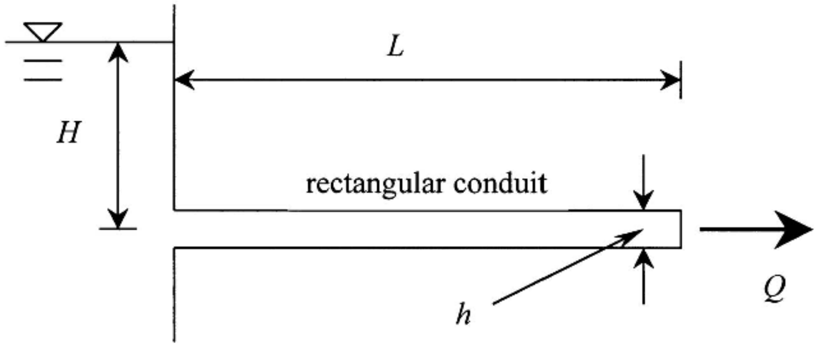

Figure 2.42 Flow through long rectangular conduit, Problem 2.39.

Problem 2.40 Figure 2.43 shows a hose of diameter D_1, which is connected to a nozzle by a flange. The diameter of the nozzle exit is D_2. Water (density ρ) flows through the hose and nozzle with discharge Q. F_H and F_V represent the horizontal and vertical forces applied by the fireman, to keep the hose and nozzle in the appropriate position.

(a) Determine the force needed to hold the two parts of the flange together.
(b) Determine the horizontal, vertical, and total forces applied by the fireman.

Problem 2.41 Water flows in an open rectangular channel with a constriction, as shown in Fig. 2.44. The water depth and channel width before the

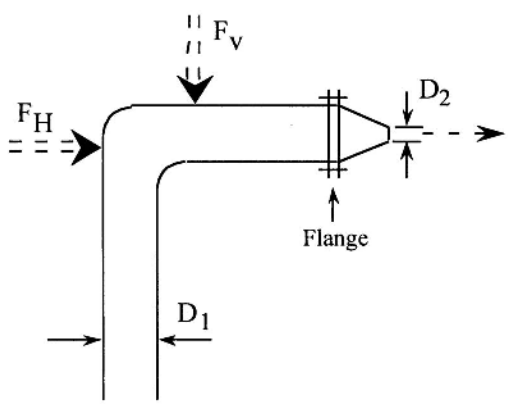

Figure 2.43 Flow around a bend, Problem 2.40.

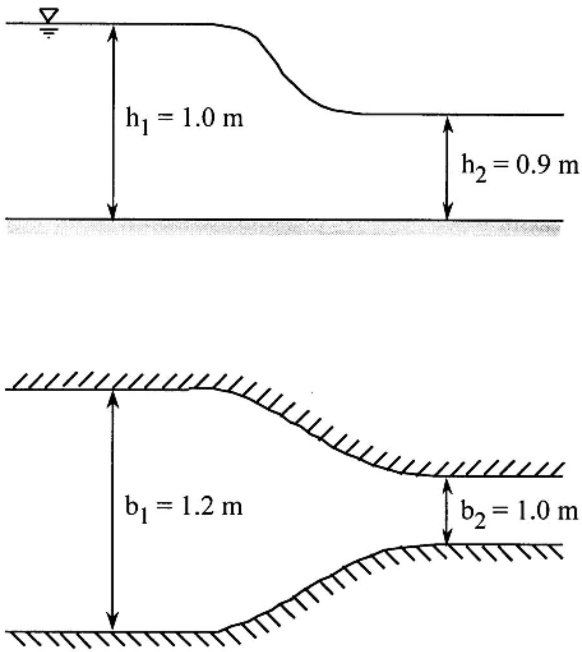

Figure 2.44 Open channel flow constriction, Problem 2.41.

constriction are $h_1 = 1.0$ m and $b_1 = 1.2$ m, respectively. At the constriction, the water depth and channel width are $h_2 = 0.9$ m and $b_2 = 1.0$ m, respectively. The pressure distribution in each vertical cross section is hydrostatic.

(a) Determine the discharge flowing through the channel.
(b) Determine the force applied by the water on the constriction.

Problem 2.42 Water flows through a gate as shown in Fig. 2.45. The channel has a rectangular cross section and its width is 1 m. The water depth upstream of the gate is $h_1 = 0.8$ m. The water depth downstream of the gate is $h_2 = 0.2$ m. At that location, the flow velocity is $V_2 = 3$ m/s.

(a) Determine the discharge in the channel.
(b) Determine the force of the water on the gate.

Problem 2.43 A cart carries a container with water. It moves freely on an inclined area, whose slope is $\alpha = 30°$, as shown in Fig. 2.46. The width of the container is $b = 2$ m, and its length is $L = 2$ m. The top of the container is

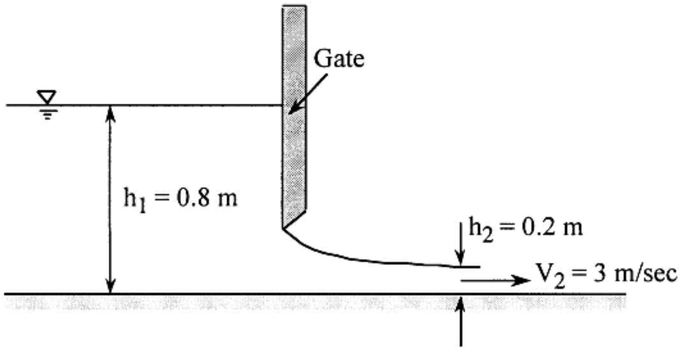

Figure 2.45 Flow under a sluice gate, Problem 2.42.

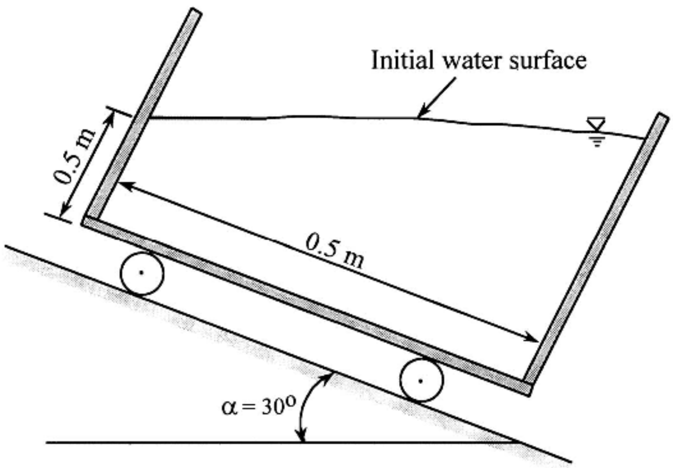

Figure 2.46 Water containing cart on a sloping surface, Problem 2.43.

open, and its side walls are very tall. The initial water depth, measured along the upper wall of the container, is 0.5 m.

(a) Determine the orientation angle between the free surface of the water with respect to horizontal.
(b) Determine the horizontal and vertical components of the pressure gradient in the water.
(c) Determine the total force applied on the front wall, back wall, and bottom of the container.

Problem 2.44 Figure 2.47 shows fluid with density ρ flowing through a two-dimensional conduit, whose width and length are b and h, respectively. At the entrance of the conduit, the velocity is u_0 and is uniformly distributed. The pressure at the entrance is p_A. At the exit of the conduit, the velocity profile is a parabola, given by

$$u = U\left[1 - \left(\frac{2y}{b}\right)^2\right]$$

where U is the maximum value of the velocity at the exit cross section and $y = 0$ represents the centerline. The pressure at the exit cross section is given by

$$p_B = p_A - 2.25\rho\frac{U^2}{2}$$

(a) Determine the relationship between u_0 and U.
(b) Determine the force applied per unit width of the conduit.

Problem 2.45 A jet aircraft flies at a constant speed V. The jet engine pumps air with volumetric discharge Q_0 and density ρ_0. The mixture of fuel and air has a density almost identical to that of the air. After the burning of the mixture, it flows out with the volumetric discharge $Q_1 = (2/3)Q_0$ and unknown density ρ_1. The inlet cross section area is A_0. The outlet cross section area is $A_1 = 0.1A_0$. The flow velocity through the inlet cross section is identical to that of the outlet cross section. The volumetric discharges Q_0 and Q_1 are independent of the flow velocity V.

(a) Determine the fluid density ρ_1 at the outlet cross section.
(b) Determine the drag force that is overcome by the jet engine.
(c) What is the power of the jet engine?

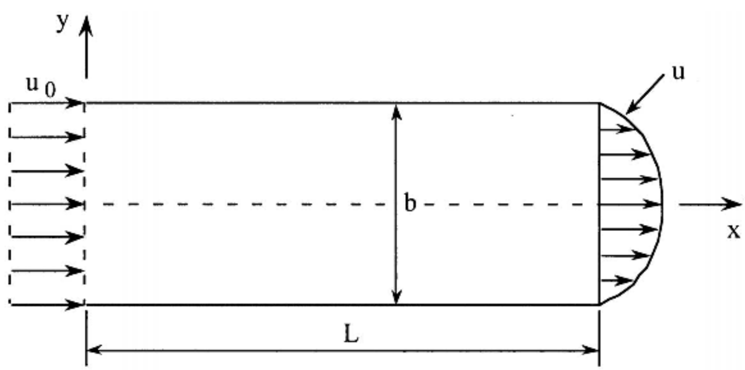

Figure 2.47 Definition sketch, Problem 2.44.

Problem 2.46 Consider that the length of the equator is 40,000 km, and that the earth makes a complete rotation in 24 hours.

(a) Calculate the value of the effective gravitational acceleration at a point on the earth's surface, whose inclination angle (latitude) is 30°.
(b) Provide the two equations of motion, based on Eq. (2.7.20), for a two-dimensional horizontal flow at a point on the ocean with an inclination angle of 30°.

Problem 2.47 A mass discharge of dry steam with $Q_m = 1$ kg/s flows through a turbine and delivers a power $N = 1{,}000$ W through the shaft of the turbine. The entrance and exit flow velocities are $V_1 = 20$ m/s and $V_2 = 10$ m/s, respectively. The entrance and exit specific enthalpy values are $h_1 = 80\ \mathrm{m^2/s^2}$ and $h_2 = 100\ \mathrm{m^2/s^2}$, respectively. The entrance elevation is higher than that of the exit of the turbine by 1 m.

(a) What are the values of ρA (where ρ is the density and A is the cross-sectional area) at the entrance and exit of the turbine?
(b) Determine the net heat transferred from the turbine into the environment per unit mass of flow.
(c) Determine the rate of heat transferred from the turbine into the environment.

Problem 2.48 A 3 m diameter tank is filled with water to a depth $h = 10$ m. A value on a 30 cm pipe at the bottom of the tank is opened suddenly and water is allowed to drain as shown in Fig. 2.48. Estimate the time needed for the tank to drain halfway (until $h = 5$ m). State all assumptions.

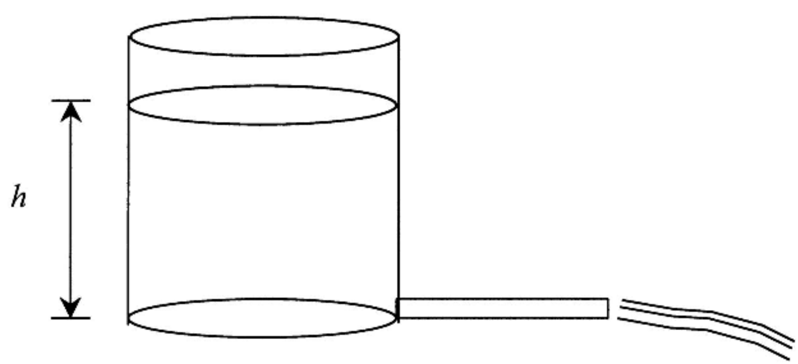

Figure 2.48 Water drainage from tank, Problem 2.48.

Problem 2.49 The pressure at the water surface of a container is 4×10^4 Pa. The water is pumped from the container through a pipe that ends with a nozzle with exit diameter $D = 100$ mm. The water flows as a free jet through the nozzle. As shown in Fig. 2.49, the elevation of the water surface in the container is higher by 1.5 m than the pump. Also, the exit nozzle is elevated by 1.5 m above the pump. The free water jet leaves the nozzle with an angle of 45° and it reaches its maximum elevation 3 m above the nozzle exit. Effects of friction between the air and the free jet are negligible.

(a) Determine the velocity of the water jet at the exit of the nozzle.
(b) Determine the distance between the exit of the nozzle and the point at the same elevation, through which the water jet passes.
(c) Assuming that the efficiency of the water pump is 0.8, determine the power needed to operate the pump.
(d) Draw a schematic of the total and piezometric heads between the container and the exit of the nozzle.

Problem 2.50 Show that for a steady one-directional flow field ($u_1 = u$) of an incompressible fluid with no horizontal variations (i.e., in the x_1 or x_2 directions) of any property, the energy equation can be simplified to

$$\mu \left(\frac{\partial u}{\partial z} \right)^2 = \frac{\partial \varphi_z}{\partial z}$$

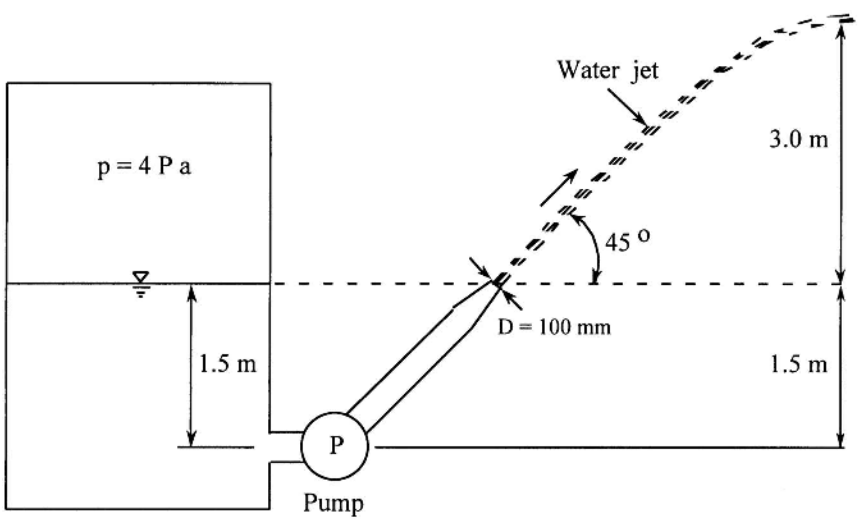

Figure 2.49 Definition sketch, Problem 2.49.

(i.e., viscous dissipation is balanced by radiative heating). Note that the result could be written in terms of ordinary derivatives, since variations occur only in the $x_3 = z$ direction.

Problem 2.51 A horizontal circular pipe 1 m in diameter carries water at a flow rate of 10 m^3/s. Neglecting heat transfer through the walls, find the temperature increase for the water traveling a length of pipe corresponding to a pressure drop of 5 atm. (about 500 kPa). Hint: apply the integral energy equation to a control volume bounded by the pipe walls and sections separated by the distance indicated above. Use $c_v = 4200$ J/kg-°C.

Problem 2.52

(a) Write the conservation equations in rectangular coordinate form for mass, momentum, and energy for an incompressible fluid with no motion and no horizontal variation of any quantity. Also assume an inertial reference system.

(b) Repeat part (a), but for conditions of steady motion in one horizontal direction (x, or x_1) only and, like all other quantities, uniform in horizontal directions.

Problem 2.53 Show that heat energy changes in a fixed volume $d\forall$ are given, for a temperature change of dT, by $(\rho c_v dT d\forall)$. You may assume that (ρc_v) is constant. Following the basic procedures in deriving the basic conservation equations, develop an equation for temperature in a fluid at rest. Although there is no advective flow, assume that there is an average molecular velocity U that must be considered in the balance. Also assume there is a source of heat $Q(x_i, t)$ per unit volume, per unit time at each point of the fluid. Your final result should look like

$$\frac{\partial \theta}{\partial t} = -\nabla(U\theta) + \frac{Q}{\rho c_v}$$

Problem 2.54 A rotating table is built for testing a scale model of a large lake. If the horizontal length scale ratio is $1 : 10^5$, the vertical length scale ration is $1 : 800$ (this is a *distorted* scale model), and the lake is at latitude 44° (N), how fast (in rpm) should the table be rotated in order to simulate Coriolis effects? (Hint: first decide which are the important dimensionless numbers for this problem, arising from scaling of the momentum equations.)

Problem 2.55 Show that in a natural water body with characteristic horizontal dimension L and vertical dimension H, with $H \ll L$, the characteristic vertical velocity W should be much less than the characteristic horizontal velocity U.

SUPPLEMENTAL READING

Aris, R., 1962. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Englewood Cliffs, New Jersey. (Provides a comprehensive description of differences between streamlines, pathlines, etc., as well as derivation of the basic equations of motion and the Reynolds transport theorem.)

Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics*, Cambridge University Press, London. (Provides a comprehensive presentation of the basic equations of conservation.)

Pedlosky, J., 1987. *Geophysical Fluid Dynamics*, Springer-Verlag, New York. (A very good intermediate level text for geostrophic flows.)