3
Viscous Flows

3.1 VARIOUS FORMS OF THE EQUATIONS OF MOTION

Viscous flows are mathematically represented by solutions of the equations of motion, based on momentum transfer in an elementary fluid volume. The equations of motion for viscous flows are the Navier–Stokes equations introduced in the previous chapter. For convenience, we repeat these equations here, for cases in which variations in viscosity are negligible:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla(p + \rho g Z) + \upsilon \nabla^2 \vec{V} \tag{3.1.1a}$$

where V is the velocity, t is time, ∇ is the gradient vector, ρ is the density, p is the pressure, g is gravitational acceleration, Z is the elevation with regard to an arbitrary reference, and υ is the kinematic viscosity. In Appendix 1, tables of Navier–Stokes equations for Cartesian, cylindrical, and spherical coordinate systems are listed. In Appendix 2, relationships are given between stress components and velocity components, as implied by the Navier–Stokes equations.

Using Cartesian tensor notation, Eq. (3.1.1a) is represented as

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - g\frac{\partial Z}{\partial x_i} + \upsilon \frac{\partial^2 u_i}{\partial x_k^2} \tag{3.1.1b}$$

where u_i represents components of the velocity vector and x_i represents the coordinates. This equation incorporates four unknown quantities: three components of the velocity vector and the pressure. Along with the continuity equation, we thus have a system of four differential equations with four unknowns. The solution of this system subject to appropriate initial and boundary conditions provides the required information about the distribution of the unknown quantities in the domain.

The distributions of velocities and pressure depend on the three space coordinates, x, y and z, and the time coordinate, t. It should be noted that the order of the differential equation (3.1.1) varies with regard to the unknown quantities, as well as with regard to the various coordinates. The velocity components contribute terms of first order with regard to time and of both first and second order with regard to the space coordinates. The pressure contributes terms of first order with regard to the space coordinates. The order of the partial derivatives indicates the number of boundary conditions needed for the solution of this system of partial differential equations. The pressure should be given at a certain point in the domain during all times. The velocity distribution at initial conditions should be given for the whole domain. The velocity at a sufficient number of boundaries should be given for the required time period of the simulation. There are several typical boundary conditions for the velocity vector, or its derivatives. The latter are related to shear stresses. Generally, there are four typical boundary conditions for the velocity and the shear stresses:

Boundary between the viscous fluid and a solid boundary — fluid velocity is identical to that of the solid boundary, as the viscous fluid adheres to the solid boundary.

Boundary between two viscous immiscible fluids — velocity and shear stress at both sides of the interface are identical.

Boundary between two immiscible fluids with an extremely large difference of viscosity, e.g., liquid and gas — shear stress vanishes at the interface between the two fluids. (An exception to this rule is with wind-driven flows, where boundary shear stress is significant. Momentum transfer at the air/water interface is discussed in Chap. 12, and a particular application, in a geophysical context, is discussed in Chap. 9.)

Finite domain — the velocity has finite value at every point of the domain.

As viscous fluid flow is basically a rotational flow, the equation of motion (3.1.1) can be represented as an equation of vorticity transport. The rotationality of the flow is represented by the distribution and intensity of the vorticity. The vorticity is a kinematic tensorial characteristic of the flow field. The tensor of vorticity is a second-order asymmetric tensor. Such a tensor has three pairs of components. Each pair incorporates two components of identical absolute value and opposite sign. Therefore the vorticity also can be represented by a vector with three components. Each component of this vector represents one pair of components of the vorticity tensor. By the employment of Cartesian tensor notation, the vorticity vector is defined as

$$\omega_j = \frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \tag{3.1.2}$$

where $i, j, k = 1, 2, 3$. One half of the vorticity represents the angular rotation rate of an elementary fluid volume, as previously noted.

By cross differentiation and subtraction of component equations of Eq. (3.1.1b), the pressure is eliminated from the equation of motion. Then the expression of Eq. (3.1.2) can be introduced to obtain a vorticity equation,

$$\frac{D\omega_j}{Dt} - \omega_k \frac{\partial u_j}{\partial x_k} = v \frac{\partial^2 \omega_j}{\partial x_k^2} \tag{3.1.3}$$

where

$$\frac{D\omega_j}{Dt} = \frac{\partial \omega_j}{\partial t} + u_k \frac{\partial \omega_j}{\partial x_k} \tag{3.1.4}$$

The first term on the LHS of Eq. (3.1.3) represents the total rate of change of vorticity. The second term represents the deformation of a vortex tube. The term on the RHS of Eq. (3.1.3) represents the diffusion of vorticity due to the viscosity of the fluid.

In cases of two-dimensional flow the vorticity vector has a single component, and the term representing the deformation of the vortex tube vanishes. Then, Eq. (3.1.3) yields

$$\frac{\partial \omega}{\partial t} + u_k \frac{\partial \omega}{\partial x_k} = v \frac{\partial^2 \omega}{\partial x_k^2} \tag{3.1.5}$$

Also, for two-dimensional flows, it is possible to apply the expression for the stream function, ψ. The stream function is related to components of the velocity vector according to (see Chap. 2)

$$u = u_1 = \frac{\partial \psi}{\partial y} \qquad v = u_2 = -\frac{\partial \psi}{\partial x} \tag{3.1.6}$$

By using the stream function, the vorticity in a two-dimensional flow field is given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = -\nabla^2 \psi = -\Delta \psi \tag{3.1.7}$$

Introducing Eqs. (3.1.6) and (3.1.7) into Eq. (3.1.5), we obtain

$$\frac{\partial \Delta \psi}{\partial t} + \frac{\partial \psi}{\partial y} \frac{\partial \Delta \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} = v \Delta \Delta \psi \tag{3.1.8}$$

where $\Delta\Delta = \nabla^4$.

In order to obtain the essential parameters governing the physical phenomena described by the Navier–Stokes equations, we nondimensionalize these equations by the employment of characteristic quantities of the flow field (also see Sec. 2.9). As before, these quantities are

$$L, U, \rho, v \tag{3.1.9}$$

where L is a characteristic length of the domain, U is a characteristic velocity of the flow, ρ is the density, and v is the kinematic viscosity of the fluid. The following dimensionless parameters, symbolized with an asterisk, are then obtained:

$$\begin{aligned} t^* &= \frac{tU}{L} \qquad x_i^* = \frac{x_i}{L} \qquad u_i^* = \frac{u_i}{U} \\ p^* &= \frac{p + \rho g Z}{\rho U^2} \qquad \omega^* = \frac{\omega L}{U} \qquad \psi^* = \frac{\psi}{LU} \end{aligned} \tag{3.1.10}$$

By introducing these dimensionless variables into Eqs. (3.1.1), (3.1.5), and (3.1.8), we obtain, respectively,

$$\frac{\partial u_i^*}{\partial t^*} + u_k^* \frac{\partial u_i^*}{\partial x_k^*} = -\frac{\partial p^*}{\partial x_i^*} + \frac{1}{\text{Re}} \frac{\partial^2 u_i^*}{\partial x_k^{*2}} \tag{3.1.11}$$

$$\frac{\partial \omega^*}{\partial t^*} + u_k^* \frac{\partial \omega^*}{\partial x_k^*} = \frac{1}{\text{Re}} \frac{\partial^2 \omega^*}{\partial x_k^{*2}} \tag{3.1.12}$$

$$\frac{\partial \Delta^* \psi^*}{\partial t^*} + \frac{\partial \psi^*}{\partial y^*} \frac{\partial \Delta^* \psi^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial \Delta^* \psi^*}{\partial y^*} = \frac{1}{\text{Re}} \Delta^* \Delta^* \psi^* \tag{3.1.13}$$

where Re is the Reynolds number and Δ^* represents the dimensionless Laplacian operator:

$$\text{Re} = \frac{UL}{v} \qquad \Delta^* = \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial y^{*2}} \tag{3.1.14}$$

The various forms of the equations of motion represented in the preceding paragraphs are used to classify types of solutions of these equations in the following sections. Generally, the Navier–Stokes equations are nonlinear equations with often quite complicated solutions. It is therefore convenient to make some classifications of families of solutions of these equations, as shown below.

3.2 ONE-DIRECTIONAL FLOWS

One-directional flows are characterized by parallel streamlines. For convenience, consider that the flow is along the x coordinate direction. Flow variables may depend on space and time in cases of unsteady flow conditions. They depend only on the space coordinates for steady state conditions. Cartesian coordinate systems are usually applied to describe domains characterized by one- and two-dimensional flows. By applying cylindrical coordinates, we refer either to domains with one-directional axisymmetric flows or to domains with one-directional circulating flows.

3.2.1 Domains Described by Cartesian Coordinates — Steady-State Conditions

At this stage we refer to a two-dimensional domain in which y is the coordinate perpendicular to the flow direction. The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.2.1}$$

where u is the velocity in the x direction, and v is the velocity in the y direction. According to the definition of one-directional flow, the velocity component, v, vanishes in the entire domain. Therefore, Eq. (3.2.1) reduces to

$$v = 0 \qquad \frac{\partial u}{\partial x} = 0 \qquad u = u(y, t) \tag{3.2.2}$$

We now introduce a quantity called piezometric pressure, defined by

$$p' = p + \rho g Z \tag{3.2.3}$$

Substituting Eqs. (3.2.2) and (3.2.3) into Eq. (3.1.1), we obtain the general differential equations representing one-directional flows in a two-dimensional domain,

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + v\frac{\partial^2 u}{\partial y^2} \tag{3.2.4}$$

$$0 = \frac{\partial p'}{\partial y} \Rightarrow p' = p'(x, t) \tag{3.2.5}$$

For steady-state conditions, the LHS of Eq. (3.2.4) vanishes. Then Eqs. (3.2.4) and (3.2.5) yield

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{dp'}{dx} \tag{3.2.6}$$

where μ is the viscosity ($\mu = \rho v$).

Note that in cases of steady state $u = u(y)$ and $p' = p'(x)$ only. Therefore the derivative expressions of Eq. (3.2.6) are not partial derivatives. If a derivative of a function depending on y is identical to the derivative of a function depending on x, then both derivatives must be equal to a constant.

Therefore Eq. (3.2.6) implies that each one of its terms is equal to a constant, and after integrating twice we find

$$\frac{du}{dy} = \frac{y}{\mu}\frac{dp'}{dx} + C_1 \tag{3.2.7}$$

$$u = \frac{y^2}{2\mu}\frac{dp'}{dx} + C_1 y + C_2 \tag{3.2.8}$$

where C_1 and C_2 are integration constants determined by the boundary conditions of the flow domain. Thus two boundary conditions with regard to the velocity field are needed to obtain a complete description of the velocity distribution in the domain. Another set of boundary conditions is needed to obtain the piezometric pressure gradient and the pressure distribution in the domain.

Multiplying Eq. (3.2.7) by the viscosity, we obtain the expression for the shear stress distribution. Integrating Eq. (3.2.8) between y_1 and y_2, which represent locations of two different streamlines, we obtain the expression for the discharge per unit width flowing between these two streamlines. The expressions for the shear stress (τ) and the discharge per unit width (q) are given, respectively, by

$$\tau = y\frac{dp^*}{dx} + \mu C_1 \tag{3.2.9}$$

and

$$q = \frac{1}{6\mu}\frac{dp^*}{dx}(y_2^3 - y_1^3) + \frac{C_1}{2}(y_2^2 - y_1^2) + C_2(y_2 - y_1) \tag{3.2.10}$$

Now, instead of piezometric pressure, we may refer to the following quantities:

$$h = \frac{p'}{\rho g} \qquad J = -\frac{dh}{dx} \tag{3.2.11}$$

where h is the *piezometric head*, and J is the *hydraulic gradient*. With regard to pressure distribution in the domain, Eq. (3.2.5) yields

$$\frac{\partial p}{\partial y} + \rho g\frac{\partial Z}{\partial y} = 0 \tag{3.2.12}$$

Direct integration of this expression and the use of Eq. (3.2.6) gives

$$p = p_0 - \rho g(Z - Z_0) + \mu\frac{d^2u}{dy^2}(x - x_0) \tag{3.2.13}$$

where subscript 0 is associated with a point of reference, representing the boundary condition for pressure.

In summary, the family of steady-state one-directional flows is well represented by simple analytical solutions. Differences between solutions, or members of this family, originate from the different boundary conditions that determine the values of the integration constants C_1 and C_2. The special case of laminar flow between parallel flat plates, called *plane Poiseuille flow*, is often used to approximate flow through porous media. Physical models, called Hele–Shaw models, have been used extensively to simulate flow in aquifers. Such a model consists of parallel vertical plates, separated by a small gap within which a viscous liquid flows. Although this is viscous laminar flow, namely rotational flow, the average velocity in the cross section of the gap is closely represented as if it originated from a potential function given by the piezometric head. Such a presentation is consistent with basic modeling of homogeneous flow through porous media. It also is interesting to note that flows through fractures in geological formations are usually considered in terms of flow between parallel flat plates.

3.2.2 Domains Described by Cylindrical Coordinates — Steady-State Conditions

With regard to cylindrical coordinate systems, two types of flow with parallel streamlines can be identified. One type incorporates axial flows and the other incorporates circulating flows. For axial one-directional flow in the x direction, the Navier–Stokes equations are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p'}{\partial x} + v\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{3.2.14}$$

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \tag{3.2.15}$$

where x is the axial coordinate, r is the radial coordinate, and u is the axial flow velocity.

In cases of steady-state conditions, Eq. (3.2.14) simplifies to

$$\frac{d}{dr}\left(r\frac{du}{dr}\right) = \frac{r}{\mu}\frac{dp'}{dx} \tag{3.2.16}$$

The LHS of this equation is a function of r, and the RHS is a function of x. Therefore each side of this equation must be a constant, and after integrating twice we find

$$\frac{du}{dr} = \frac{r}{2\mu}\frac{dp'}{dx} + \frac{C_1}{r} \tag{3.2.17}$$

$$u = \frac{r^2}{4\mu}\frac{dp'}{dx} + C_1 \ln r + C_2 \tag{3.2.18}$$

where C_1 and C_2 are integration constants determined by the boundary conditions of the problem.

In the case of viscous pipe flow, termed *Poiseuille flow*, C_1 should vanish, to allow finite values of the velocity in the entire cross-sectional area of the pipe (i.e., when r approaches 0), and the value of C_2 is determined by the vanishing value of the velocity at the wall of the pipe. Therefore, for viscous pipe flow, Eq. (3.2.18) yields

$$u = -\frac{R^2}{4\mu}\frac{dp'}{dx}\left[1-\left(\frac{r}{R}\right)^2\right] \tag{3.2.19}$$

where R is the pipe radius. Integrating this result over the pipe cross section, we obtain the discharge flowing through the pipe,

$$Q = -\frac{\pi R^4}{8\mu}\frac{dp'}{dx} \tag{3.2.20}$$

This equation is called the *Poiseuille–Hagen law*. It was derived by Poiseuille from experiments with small glass tubes that were designed to simulate blood flow through blood vessels. Ironically, Poiseuille flow is very different from real blood flow, which is subject to strong pressure variations (pulsating flow) and flows through flexible tubes. Nonetheless, experiments of Reynolds, Stanton, and others have indicated that Eq. (3.2.20) is applicable as long as the Reynolds number ($\mathrm{Re} = VD/v$) is smaller than about 2000. In addition, flow through porous media is often simulated as a flow through stochastic bundles of capillaries. Such a simulation has been shown to provide an adequate characterization of flow and transport processes in porous matrices.

By dividing Eq. (3.2.20) by the cross-sectional area and applying Eq. (3.2.11), the average velocity is obtained as

$$V = \frac{D^2 gJ}{32v} \tag{3.2.21}$$

where D is the pipe diameter. This expression can be represented in the form of the *Darcy–Weissbach equation* as

$$J = \frac{64}{\mathrm{Re}}\frac{1}{D}\frac{V^2}{2g} \tag{3.2.22}$$

The term (64/Re) represents the Darcy–Weissbach friction coefficient for laminar pipe flow.

In the case of annular flow, the velocity vanishes at the inner tube (where $r = r_1$), as well as at the outer tube (where $r = r_2$). Introducing these boundary conditions into Eq. (3.2.18), we obtain the following expressions for the constants of Eq. (3.2.18):

$$C_1 = \frac{r_2^2 - r_1^2}{4\mu \ln(r_2/r_1)} \frac{dp^*}{dx} \tag{3.2.23}$$

$$C_2 = \frac{dp^*}{dx}\left[-\frac{r_2^2 + r_1^2}{8\mu} + \frac{r_2^2 - r_1^2}{8\mu}\frac{\ln(r_2 r_1)}{\ln(r_2/r_1)}\right] \tag{3.2.24}$$

For two-dimensional circulating flow, there is only a single component of the velocity in the θ-direction. The Navier–Stokes equations yield, when there is no pressure gradient in the flow direction,

$$-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \tag{3.2.25}$$

$$0 = \mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right) = \mu\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv)\right] \tag{3.2.26}$$

$$0 = -\frac{\partial p'}{\partial z} \tag{3.2.27}$$

where v is the rotation velocity (velocity in the θ direction), r is the radial coordinate, and z is the vertical coordinate. Equations (3.2.25) and (3.2.27) indicate that p' is a function only of r. Integration of Eq. (3.2.26) provides the velocity distribution,

$$v = Ar + \frac{B}{r} \tag{3.2.28}$$

where A and B are constants that must be determined by the boundary conditions.

If the fluid occupies the space between two coaxial rotating cylinders, whose angular velocities are Ω_1 and Ω_2, respectively, then the values of A and B are given by

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \tag{3.2.29}$$

$$B = \frac{(\Omega_1 - \Omega_2) r_1^2 r_2^2}{(r_2^2 - r_1^2)} \tag{3.2.30}$$

(recall that r_1 and r_2 are the radii of the inner and outer cylinders, respectively).

In the limiting case of $r_2 = \infty$, Eqs. (3.2.28)–(3.2.30) refer to steady flow in an infinite domain around a rotating cylinder whose radius and angular velocity are r_1 and Ω_1, respectively. In such a case, these equations yield

$$v = \frac{\Omega_1 r_1^2}{r} \tag{3.2.31}$$

This expression is identical to the velocity distribution in a potential (irrotational) vortex with circulation Γ, given by

$$\Gamma = 2\pi\Omega_1 r_1^2 \tag{3.2.32}$$

The solution of the Navier–Stokes equations given by Eq. (3.2.31) is an interesting case in which the potential flow solution is identical to that of the viscous flow solution.

In the limiting case of $\Omega_1 = r_1 = 0$, Eqs. (3.2.28)–(3.2.30) represent steady flow inside a cylindrical rotating tank, whose radius and angular velocity are r_2 and Ω_2, respectively. In this case, the result is

$$v = \Omega_2 r \tag{3.2.33}$$

This expression represents a rotational vortex.

3.3 CREEPING FLOWS

For very small Reynolds number, namely with small flow velocities and small size of the body, or with large viscosity of the fluid, the nonlinear inertial terms of the Navier–Stokes equations are much smaller than the viscous friction terms. Such flows are called creeping flows. In these flows, the Navier–Stokes equations can be approximated by the Stokes equations,

$$\rho \frac{\partial u_i}{\partial t} = -\frac{\partial p'}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_k^2} \tag{3.3.1}$$

These equations (for each component), along with the equation of continuity, represent the basic equations for creeping flows. Considering a solid body subject to slow movement in the domain, or slow movement of fluid around a stationary solid body, the fluid velocity at the body surface is equal to that of the solid surface. This provides a convenient boundary condition. Also, by taking the divergence of Eq. (3.3.1), we obtain

$$\frac{\partial^2 p}{\partial x_k^2} = 0 \tag{3.3.2}$$

This indicates that the pressure is a harmonic function in creeping flows.

In two-dimensional, steady creeping flow, Eq. (3.3.1) becomes

$$\nabla^4 \psi = 0 \tag{3.3.3}$$

indicating that the stream function is a biharmonic function (for the assumed conditions).

Considering a very slow motion of a sphere of radius r_0, with velocity U in the x direction, the pressure function is given by

$$p = -\frac{3}{2}\frac{\mu U r_0 x}{r^3} \tag{3.3.4}$$

where the center of the sphere represents the origin of the coordinate system and $p \to 0$ for $r \to \infty$ has been assumed. Incorporating both the net pressure force implied by Eq. (3.3.4) and skin friction drag, the *drag coefficient* for the sphere is

$$C_D = \frac{F_D}{(\rho/2)\pi r_0^2 U^2} = \frac{24}{\text{Re}} \tag{3.3.5}$$

where F_D is the total drag force applied to the moving sphere. Equation (3.3.5) can be used to measure the viscosity of fluids. It is useful with regard to settling of solid particles in a fluid medium (see Chap. 15).

Experimental results indicate that expression (3.3.5) is accurate for extremely small values of Reynolds number. However, the velocity distribution obtained using the Stokes equation (3.3.1) is not usually very accurate, particularly at larger distances from the sphere. This is because of the formation of a wake region behind the sphere. The solution of the Stokes equation yields a velocity distribution that is symmetrical with regard to a plane perpendicular to the flow direction and passing through the center of the sphere. In other words, it does not incorporate a wake region. This result is also seen by considering the orders of magnitude of the inertial and viscous terms of the Navier–Stokes equations,

$$\rho u_k \frac{\partial u_i}{\partial x_k} = O\left(\rho \frac{U^2}{r}\right) \qquad \mu \frac{\partial^2 u_i}{\partial x_k^2} = O\left(\mu \frac{U}{r^2}\right) \tag{3.3.6}$$

These expressions indicate that the ratio between the inertial and viscous terms is proportional to r. Therefore for distances much greater than r_0 the viscous terms become relatively unimportant, and it may be concluded that the solution of the Stokes equation is not applicable at large distances from the sphere.

An improvement of Stokes' analysis was provided by Oseen, who considered the deviation imposed on the uniform flow U by the presence of the sphere. Therefore he considered a velocity distribution,

$$u = U + u' \qquad v = v' \qquad w = w' \tag{3.3.7}$$

where u', v', and w' are the velocity deviations in the x, y, and z directions, respectively. By introducing Eq. (3.3.7) into the Navier–Stokes equations and neglecting the second-order terms with regard to the velocity deviations, Oseen obtained

$$\frac{\partial u_i'}{\partial t} + U\frac{\partial u_i'}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + v\frac{\partial^2 u_i'}{\partial x_k^2} \tag{3.3.8}$$

Here, x represents the direction of the uniform flow U, and x_i represents each of the coordinates. The terms of Eq. (3.3.8) which were added to Eq. (3.3.1) have been shown to improve the calculation of creeping flow at large distances from the center of the sphere.

Applying the divergence operation on Eq. (3.3.7), the continuity equation is written as

$$\frac{\partial u_k'}{\partial x_k} = 0 \tag{3.3.9}$$

(since the uniform flow also must follow continuity). For steady flows, it is possible to consider that each component of the velocity deviation from the uniform flow velocity, U consists of two parts, given by

$$u_i = u_{1i}' + u_{2i}' \tag{3.3.10}$$

where u_{1i}' is a potential flow component, originating from a potential function ϕ. Therefore

$$u_{1i} = -\frac{\partial \phi}{\partial x_i} \qquad \frac{\partial^2 \phi}{\partial x_i^2} = 0 \tag{3.3.11}$$

It is considered that u_{1i}' is associated with the balance of the pressure gradient term of Eq. (3.3.8), whereas u_{2i}' is associated with the frictional force. By applying these assumptions, and introducing Eq. (3.3.11) into Eq. (3.3.8), we obtain

$$p = \rho U \frac{\partial \phi}{\partial x} \tag{3.3.12}$$

The components u_{2i}' are represented by

$$u_{2i} = \frac{\partial W}{\partial x_i} - \delta_i W \frac{U}{v} \tag{3.3.13}$$

where $\delta_1 = 1$, and $\delta_2 = \delta_3 = 0$. The function W must satisfy

$$\frac{\partial W}{\partial x} = \frac{v}{U} \frac{\partial^2 W}{\partial x_k^2} \tag{3.3.14}$$

The appropriate solution of Eqs. (3.3.11) and (3.3.14) represents the essence of Oseen's analysis. Such solutions were obtained for a sphere moving at a uniform speed U. In this case the drag coefficient is

$$C_{\mathrm{D}} = \frac{24}{\mathrm{Re}} \left(1 + \frac{3}{16} \mathrm{Re}\right) \tag{3.3.15}$$

Generally, the drag coefficient can be expressed in terms of a series expansion of the Reynolds number. Equation (3.3.15) represents the first and second terms of such a series. Additional terms have been developed in more recent studies. Stokes' solution of Eq. (3.3.5) is considered to be applicable in cases of Reynolds numbers smaller than one. Oseen's solution given in Eq. (3.3.15) is applicable up to Reynolds numbers equal to 2. For higher Reynolds numbers more terms should be added to the power series given by Eq. (3.3.15). Flow through porous media can be considered as creeping flow around the solid particles that comprise the porous matrix. When the Reynolds number of the flow, based on a characteristic size of the matrix particle, is smaller than unity, then *Darcy's law* is useful (see Sec. 4.4), and the gradient of the piezometric head is proportional to the average interstitial flow velocity, as well as the specific discharge.

3.4 UNSTEADY FLOWS

There are several exact solutions of the Navier–Stokes equations for unsteady flows. Examples of such flows in the present section also are used to visualize the basic concept of the *boundary layer*.

3.4.1 Quasi-Steady-State Oscillations of a Flat Plate

Consider a flat plate subject to cosinusoidal oscillations. The domain is subject to a uniform pressure distribution. Therefore the Navier–Stokes equations (3.1.1) reduce to

$$\frac{\partial u}{\partial t} = v\frac{\partial^2 u}{\partial y^2} \qquad p' = \text{constant} \tag{3.4.1}$$

It should be noted that Eq. (3.4.1) is identical to the diffusion equation, which is applicable in problems of heat conduction or mass diffusion. The exact solution of Eq. (3.4.1) given in the following paragraphs is similar to some particular solutions of heat conduction in solids. Further discussion of diffusion is presented in Chap. 10.

The differential Eq. (3.4.1) is subject to the boundary conditions,

$$\begin{aligned} u &= U_0 \cos(\omega t) && \text{at} \quad y = 0 \\ u &= 0 && \text{at} \quad y \to \infty \end{aligned} \tag{3.4.2}$$

Noting that we are looking for a quasi-steady-state solution, only two spatial boundary conditions are required to solve this equation. We assume that the solution is of the form

$$u = \text{Re}[U(y)\exp(i\omega t)] \tag{3.4.3}$$

Here, Re represents the real part of the complex quantity. We introduce Eq. (3.4.3) into Eq. (3.4.1) to obtain

$$\frac{d^2U}{dy^2} - \frac{i\omega}{v}U = 0 \tag{3.4.4}$$

By solving this differential equation and presenting the boundary conditions for U, which are implied by Eq. (3.4.2), we obtain

$$U = U_0 \exp\left[-y\sqrt{\frac{\omega}{2v}}(1+i)\right] \tag{3.4.5}$$

Finally, introducing Eq. (3.4.5) into Eq. (3.4.3), the complete solution is obtained,

$$u = U_0 \exp\left(-y\sqrt{\frac{\omega}{2v}}\right)\cos\left(\omega t - y\sqrt{\frac{\omega}{2v}}\right) \tag{3.4.6}$$

Equation (3.4.6) indicates that the amplitude of the velocity oscillations is subject to exponential decrease with the coordinate y. The practical outcome of this expression may be evaluated by considering the value of $y = \delta$, where the amplitude is 1 percent of its value at the flat plate. From Eq. (3.4.6),

$$\delta = \sqrt{\frac{v}{\pi f}}\ln(100) \tag{3.4.7}$$

where f is the frequency of the plate oscillations ($\omega = 2\pi f$). For water, with kinematic viscosity $v = 10^{-6}$ m^2/s, and assuming a frequency $f = 1$ s^{-1}, we obtain $\delta = 2.6 \times 10^{-3}$ m. This result indicates that only a very thin layer of fluid adjacent to the flat plate is subject to oscillations induced by the flat plate motion. The layer in which the oscillation amplitude is larger than 1 percent of the flat plate amplitude can be termed as a boundary layer. The phenomena of boundary layers is typical of regions close to solid boundaries of flow domains occupied by fluid with low viscosity. Boundary layers are discussed in more detail in Chap. 6.

3.4.2 Unsteady Motion of a Flat Plate

Consider a flat plate at rest at time $t \leq 0$ but moving at constant velocity U for $t > 0$. The basic differential Eq. (3.4.1) also is applicable in this case, but the boundary conditions are different. In this case

$$\begin{aligned} &u = 0 \quad &&\text{at} \quad t \leq 0 \quad &&\text{for all values of } y \\ &u = U \quad &&\text{at} \quad t > 0 \quad &&\text{for } y = 0 \\ &u = 0 \quad && && \text{for } y \to \infty \end{aligned} \tag{3.4.8}$$

It is convenient to define a new dimensionless coordinate,

$$\eta = \frac{y}{2\sqrt{\nu t}} \tag{3.4.9}$$

The modified boundary conditions, in terms of η, are

$$\begin{aligned} u &= U \qquad \text{at} \qquad \eta = 0 \\ u &= 0 \qquad \text{at} \qquad \eta \to \infty \end{aligned} \tag{3.4.10}$$

The second boundary condition of Eq. (3.4.10) incorporates both the first and the third boundary conditions of Eq. (3.4.8).

Using the definition (3.4.9), it is easy to find

$$\frac{\partial u}{\partial y} = \frac{du}{d\eta}\frac{\eta}{y} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{d^2 u}{d\eta^2}\left(\frac{\eta}{y}\right)^2 \qquad \frac{\partial u}{\partial t} = \frac{du}{d\eta}\left(-\frac{\eta}{2t}\right) \tag{3.4.11}$$

Introducing Eq. (3.4.11) into Eq. (3.4.1), integrating twice, and introducing the boundary conditions of Eq. (3.4.10), we obtain

$$u = U\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\xi^2} d\xi\right) = U(1 - \operatorname{erf}(\eta)) = U \operatorname{erfc}(\eta) \tag{3.4.12}$$

where erf and erfc are the error and complementary error functions, respectively, and ξ is a dummy variable of integration. Again referring to water, as an example, we find that only a thin layer adjacent to the flat plate takes part in the flow, even up to extremely large times.

3.5 NUMERICAL SIMULATION CONSIDERATIONS

Numerical schemes aiming at the solution of the mass conservation and Navier–Stokes equations are usually based on finite difference or finite element methods. By these methods the numerical grid and the basic equations of mass and momentum conservation are used to create a set of approximately linear equations, which incorporate the unknown values of various variables at all grid points. The basic four equations of mass and momentum conservation incorporate four unknown variables for each grid point. These unknown values, for the three-dimensional domain, are the three components of the velocity vector and the pressure. If the domain is two-dimensional, or axisymmetrical, then the two components of the velocity vector can be replaced by the stream function.

As previously noted, the number of boundary conditions needed to solve a differential equation is determined by its order and the dimensions of the domain. With regard to the spatial derivatives of the velocity components, the Navier–Stokes equations are second-order partial differential equations. Therefore two boundary conditions are needed for each velocity component, with regard to each relevant coordinate. Velocity components also are subject to the first derivative in time. Therefore the initial distribution of all velocity components in the entire domain is needed. The pressure is subject to the first spatial derivative. Therefore boundary conditions also are required for the pressure, with regard to each relevant coordinate. If the stream function is applied, in a two-dimensional or axisymmetrical domain, then the basic set of four differential equations can be replaced by the fourth-order differential equation, which is given by Eq. (3.1.8). The solution of this equation requires four boundary conditions for the stream function with regard to each relevant coordinate, and initial distribution of the stream function in the domain.

For numerical simulation of the Navier–Stokes equations, it is common to consider applying the vorticity tensor, as shown in Eq. (3.1.3), or the vorticity vector, as given by Eq. (3.1.5). However, boundary conditions for vorticity are derived from appropriate considerations based on values of the velocity components.

Typical boundary conditions for the solution of the Navier–Stokes equations have been considered in Sec. 3.1. However, at this point it is appropriate to review the various types of boundary conditions, useful for the numerical solution of the various forms of these equations.

3.5.1 Basic Presentation

The solution of Eq. (3.1.1) is based on the following considerations:

At a solid surface, all velocity components are identical to those of the solid surface; if the solid surface is at rest then all velocity components vanish.

At the interface between two immiscible fluids, pressure and components of the velocity and shear stress are identical at both sides of the interface; shear stress components are proportional to the gradients of the velocity components.

At the interface between two immiscible fluids with large differences in viscosity, e.g., liquid and gas, the shear stress vanishes (except for the case of wind-driven flows).

At the entrance of the domain and/or exit cross sections the distribution of the velocity components is prescribed.

At the entrance or exit cross section of the domain the pressure distribution is prescribed.

The initial distribution of velocity components should be given.

3.5.2 Presentation with the Stream Function

For the solution of Eq. (3.1.8), the following considerations hold:

At a solid surface, spatial derivatives of the stream function are identical to velocity components of the solid surface; if the solid surface is at rest, spatial derivatives of the stream function vanish. The solid boundary represents a streamline at which the stream function has a constant value.

At the interface between two immiscible fluids, the first and second gradients of the stream function are identical on both sides of the interface. The interface represents a streamline, at which the stream function has a constant value.

At the interface between two immiscible fluids with large viscosity difference, e.g., liquid and gas (the interface is considered as the free surface of the liquid), the second gradient of the stream function vanishes. The free surface of the fluid is a streamline.

The initial distribution of the stream function in the domain should be given.

It should be noted that interfaces and free surfaces usually represent a sort of nonlinear boundary condition with regard to the velocity components, since the position of the boundary itself (where the boundary condition is to be applied) is part of the solution to the problem. Furthermore, determination of the exact location of free surfaces is very complicated.

Difficulties in solving the Navier–Stokes equations are very often associated with the nonlinear second term of Eq. (3.1.1), or the second and third terms of Eq. (3.1.8). If the flow is dominated by the nonlinear terms, then the numerical simulation is extremely complex, and some methods should be used to obtain a convergent numerical scheme. Furthermore, if boundary conditions are nonlinear, then the numerical solution may require significant approximations to assure convergence of the simulation process. The topic of "computational fluid mechanics" refers to different methods of solving these differential equations. For the present section, we consider only the numerical solution of creeping flows. In such flows the right-hand side terms of Eq. (3.1.8) are very small. Therefore the Navier–Stokes equations are approximated by

$$\Delta\Delta\Psi = 0 \qquad \frac{\partial^4 \Psi}{\partial x^4} + 2\frac{\partial^2 \Psi}{\partial x^2}\frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^4 \Psi}{\partial y^4} = 0 \tag{3.5.1}$$

This is an *elliptic* differential equation (see Sec. 1.3.3).

As an example, consider a domain bounded on a square, where

$$\begin{aligned}
\Psi &= 0 \quad \text{at} \quad x = 0, 1 \quad y = 0, 1 \\
\frac{\partial \Psi}{\partial n} &= 0 \quad \text{at} \quad x = 0, 1 \quad y = 0 \\
\frac{\partial \Psi}{\partial n} &= 1 \quad \text{at} \quad y = 1
\end{aligned} \tag{3.5.2}$$

and a derivative with regard to n is the normal derivative. We introduce a new variable $w(x, y)$, which is defined by

$$\begin{aligned}
\Delta \Psi &= \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = w \\
\Delta w &= \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0
\end{aligned} \tag{3.5.3}$$

The terms of these expressions can be approximated using the following finite difference approximations:

$$\left(\frac{\partial \Omega}{\partial x}\right)_{i,j} \approx \frac{\Omega_{i+1/2,j} - \Omega_{i-1/2,j}}{\Delta x} \tag{3.5.4}$$

$$\left(\frac{\partial \Omega}{\partial y}\right)_{i,j} \approx \frac{\Omega_{i,j+1/2} - \Omega_{i,j-1/2}}{\Delta y} \tag{3.5.5}$$

$$\begin{aligned}
\left(\frac{\partial^2 \Omega}{\partial x^2}\right)_{i,j} &\approx \frac{1}{\Delta x}\left[\left(\frac{\partial \Omega}{\partial x}\right)_{i+1/2,j} - \left(\frac{\partial \Omega}{\partial x}\right)_{i-1/2,j}\right] \\
&\approx \frac{\Omega_{i+1/2,j} - 2\Omega_{i,j} + \Omega_{i-1/2,j}}{(\Delta x)^2}
\end{aligned} \tag{3.5.6}$$

$$\begin{aligned}
\left(\frac{\partial^2 \Omega}{\partial y^2}\right)_{i,j} &\approx \frac{1}{\Delta y}\left[\left(\frac{\partial \Omega}{\partial y}\right)_{,j+1/2} - \left(\frac{\partial \Omega}{\partial y}\right)_{i,j+1/2}\right] \\
&\approx \frac{\Omega_{i,j+1/2} - 2\Omega_{i,j} + \Omega_{i,j+1/2}}{(\Delta y)^2}
\end{aligned} \tag{3.5.7}$$

where Ω is a dummy variable representing Ψ or w. Subscripts i, j refer to the point i, j of the finite difference grid shown in Fig. 3.1.

Since the numerical grid shown in Fig. 3.1 consists of small squares, for simplicity we assume that $\Delta x = \Delta y = k$. Therefore by introducing these values and Eqs. (3.5.6) and (3.5.7) into Eq. (3.5.3), we obtain

$$\begin{aligned}
\Psi_{i+1,j} + \Psi_{i-1,j} + \Psi_{i,j+1} + \Psi_{i,j-1} - 4\Psi_{i,j} &= k^2 w \\
w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j} &= 0
\end{aligned} \tag{3.5.8}$$

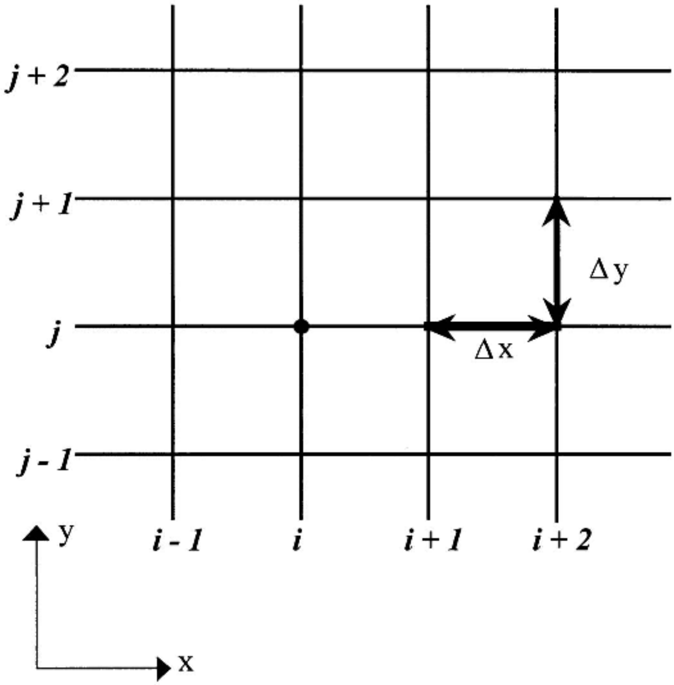

Figure 3.1 The finite difference grid.

Also, the boundary conditions of Eq. (3.5.2) become

$$\begin{aligned} &\Psi = 0 \qquad \text{on all boundaries} \\ &w = \frac{\partial^2 \Psi}{\partial n^2} \quad \text{on all boundaries} \end{aligned} \tag{3.5.9}$$

The set of linear equations obtained by considering all grid points and using Eqs. (3.5.8) and (3.5.9) can be solved by an appropriate iterative procedure. Basically the set of two differential equations given by Eq. (3.5.3) is solved very similarly to the solution of the Laplace equation, which is discussed in greater detail in the following chapter.

PROBLEMS

Solved Problems

Problem 3.1 Introduce the expression for the vorticity vector into Eq. (3.1.5), to obtain an equation of motion based on the velocity components.

Solution

The vorticity vector in a two-dimensional flow field is given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Introducing this expression into Eq. (3.1.5), we obtain

$$\frac{\partial^2 v}{\partial x \partial t} - \frac{\partial^2 u}{\partial y \partial t} + u\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial y \partial x}\right) + v\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right)$$
$$= \nu\left(\frac{\partial^3 v}{\partial x^3} - \frac{\partial^3 u}{\partial y \partial x^2} + \frac{\partial^3 v}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3}\right)$$

Problem 3.2 Figure 3.2 shows a plate with an orientation angle α, on which a fluid layer with thickness b is subject to flow with a free surface. The viscosity and density of the fluid are μ and ρ, respectively.

(a) Determine the value of the gradient of the piezometric head in the x-direction.
(b) Determine the value of the pressure gradient in the y-direction. What is the value of the pressure at the channel bottom?
(c) Determine the velocity and shear stress distributions.
(d) Determine the discharge per unit width and the average velocity.

Solution

(a) From Fig. 3.2,

$$\frac{\partial Z}{\partial x} = -\sin\alpha : \frac{\partial Z}{\partial y} = \cos\alpha$$

The gradient of the piezometric pressure in the x-direction is given by

$$\frac{dp^*}{dx} = \frac{\partial p}{\partial x} + \frac{\partial Z}{\partial x} = \frac{\partial p}{\partial x} - \rho g \sin\alpha = -\rho g J$$

Along the streamline representing the free surface of the fluid, the pressure vanishes. Therefore the pressure gradient in the x-direction is zero along that streamline, as well as along other streamlines, and the piezometric head gradient in the x-direction is given by $J = \sin\alpha$.

(b) According to Eq. (3.2.12) and the value of the partial derivatives of Z, as given in the previous part of this solution, we obtain

$$\frac{\partial p}{\partial y} + \rho g \cos\alpha = 0 \qquad \Rightarrow \qquad \frac{\partial p}{\partial y} = -\rho g \cos\alpha$$

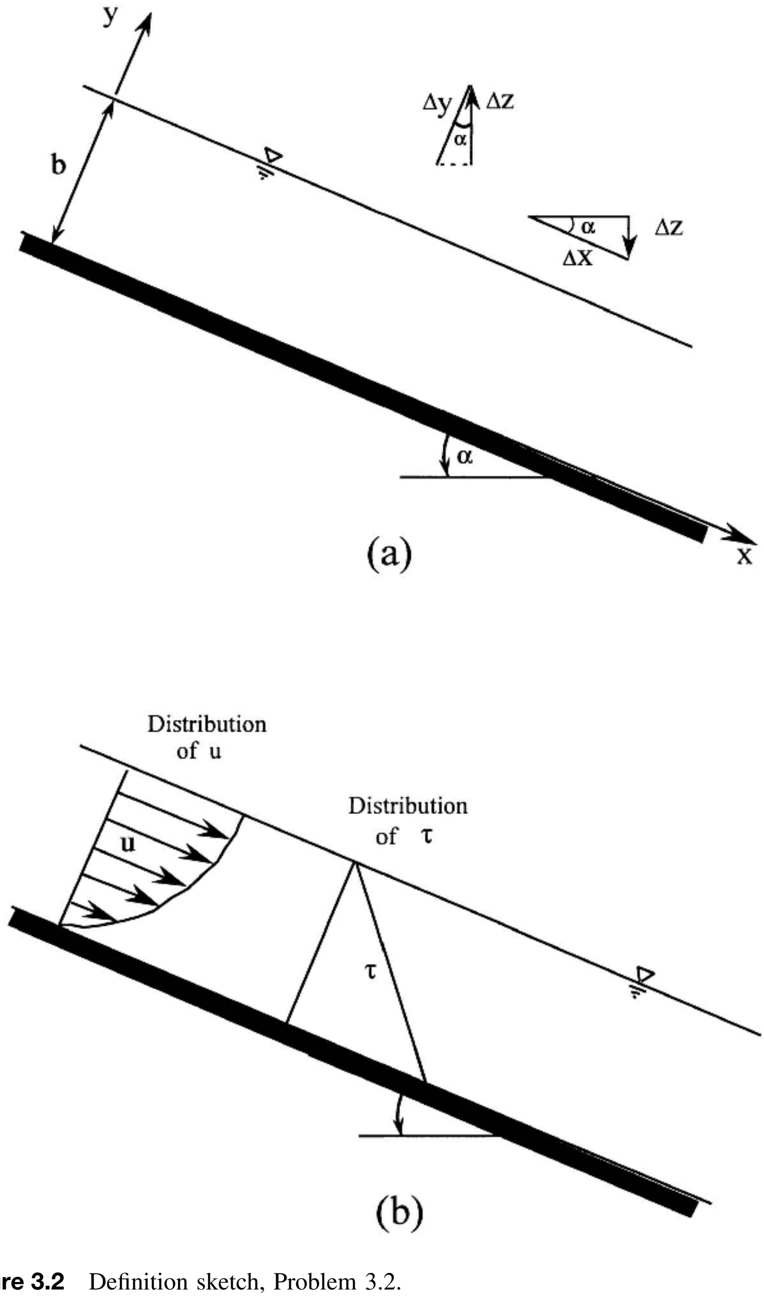

Figure 3.2 Definition sketch, Problem 3.2.

Direct integration of this expression, while considering that the pressure vanishes at the free surface of the fluid layer (at $y = b$), results in

$$p = \rho g(b - y)\cos\alpha$$

This expression indicates that the pressure at the fluid layer bottom is $(p)_{y=0} = \rho g b \cos\alpha$.

(c) Due to the very low viscosity of air, the shear stress vanishes at the free surface of the fluid layer. Therefore according to Eq. (3.2.9), we obtain

$$0 = -b\rho g \sin\alpha + \mu C_1 \qquad \Rightarrow \qquad C_1 = \frac{b\rho g}{\mu}\sin\alpha = \frac{bg}{\nu}\sin\alpha$$

At the bottom of the fluid layer ($y = 0$), the velocity vanishes. Therefore Eq. (3.2.8) yields $C_2 = 0$. By introducing values of the piezometric head gradient and those of C_1 and C_2 into Eqs. (3.2.8) and (3.2.9), we obtain the following expressions for the velocity and shear stress distributions, respectively:

$$u = \frac{g\sin\alpha}{\nu}\left(by - \frac{y^2}{2}\right) \qquad \tau = (b - y)\rho g \sin\alpha$$

(d) While referring to Eq. (3.2.10), we may consider that $y_1 = 0$, and $y_2 = b$. By introducing values of the piezometric head gradient and those of C_1 and C_2 into Eqs. (3.2.10), we obtain the following expression for the discharge per unit width and the average flow velocity, respectively:

$$q = \frac{gb^3\sin\alpha}{3\nu} \qquad V = \frac{q}{b} = \frac{gb^2\sin\alpha}{3\nu}$$

Problem 3.3 A fluid layer flows between two plates, with orientation angle α with respect to horizontal. The thickness of the fluid layer is b. The lower plate is stationary. The upper plate moves upward with velocity U. The pressure at the bottom of the fluid layer is given at two points: at $x = 0$ the pressure is p_0, and at $x = L$ the pressure is p_L. The viscosity and density of the fluid are μ and ρ, respectively.

(a) Determine the value of the gradient of the piezometric head in the x-direction.
(b) Determine the pressure distribution in the entire domain.
(c) Determine the velocity and shear stress distributions.
(d) Determine the discharge per unit width and the average velocity.
(e) Determine the power per unit area that is needed to move the upper plate.

Solution

(a) From geometrical considerations,

$$\frac{\partial Z}{\partial x} = -\sin\alpha : \frac{\partial Z}{\partial y} = \cos\alpha$$

The gradient of the piezometric pressure in the x-direction is then given by

$$\frac{dp^*}{dx} = \frac{\partial p}{\partial x} + \frac{\partial Z}{\partial x} = \frac{p_L - p_o}{L} - \rho g \sin\alpha = -\rho g J$$

$$\Rightarrow \qquad J = \frac{p_o - p_L}{\rho g L} + \sin\alpha$$

(b) From part (a),

$$\frac{\partial p}{\partial x} = \frac{p_L - p_0}{L} : \quad \Rightarrow \qquad p = p_0 + \frac{p_L - p_0}{L} x + f(y)$$

where $f(y)$ is a function of y that vanishes at $y = 0$. Differentiation of the last expression yields

$$\frac{\partial p}{\partial y} = f'(y)$$

According to Eq. (3.2.12) and the value of the partial derivatives of Z, as given in part (a) of this solution, we obtain

$$\frac{\partial p}{\partial y} + \rho g \cos\alpha = 0 \qquad \Rightarrow \qquad \frac{\partial p}{\partial y} = -\rho g \cos\alpha = f'(y)$$

Direct integration of this expression yields

$$f(y) = -\rho g y \cos\alpha \qquad \Rightarrow \qquad p = p_0 + \frac{p_L - p_0}{L} x - \rho g y \cos\alpha$$

This expression indicates that the pressure at $x = 0$ at the top of the fluid layer is

$$(p)_{y=b} = p_0 - \rho g b \cos\alpha$$

(c) At the fluid layer bottom ($y = 0$), the velocity vanishes. Therefore by using Eq. (3.2.8), we find $C_2 = 0$. At the upper plate the fluid velocity is identical to that of the moving plate. Therefore Eq. (3.2.8) yields for $y = b$,

$$-U = \frac{b^2}{2\mu}\left(\frac{p_L - p_0}{L} - \rho g \sin\alpha\right) + C_1 b$$

$$\Rightarrow \qquad C_1 = \frac{b}{2\mu}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right) - \frac{U}{b}$$

By introducing values of the piezometric pressure gradient and those of C_1 and C_2 into Eqs. (3.2.8) and (3.2.9), we obtain the following expressions for the velocity and shear stress distributions, respectively:

$$u = \frac{b}{2\mu}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right)(by - y^2) - \frac{U}{b}y$$

$$\tau = \frac{b}{2}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right)(b - 2y) - \mu\frac{U}{b}$$

(d) While referring to Eq. (3.2.10) we consider that $y_1 = 0$ and $y_2 = b$. By introducing values of the piezometric pressure gradient and those of C_1 and C_2 into Eqs. (3.2.10), we obtain the following expressions for the discharge per unit width and the average flow velocity, respectively:

$$q = \frac{b^3}{12\mu}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right) - \frac{Ub}{2}$$

$$\Rightarrow \qquad V = \frac{b^2}{12\mu}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right) - \frac{U}{2}$$

(e) The power per unit width that is needed to move the upper plate is given by

$$N = (\tau u)_{y=b} = \frac{b^2 U}{2}\left(\frac{p_0 - p_L}{L} + \rho g \sin\alpha\right) + \frac{U^2 b}{2}$$

Problem 3.4 Determine the settling velocity of a sand particle in water. The particle may be assumed to be approximately spherical, with a diameter $d =$ 0.2 mm. Its density is $\rho_s = 2{,}400\ \mathrm{kg/m^3}$. The density and kinematic viscosity of the water are $\rho_w = 1{,}000\ \mathrm{kg/m^3}$ and $\nu = 10^{-6}\ \mathrm{m^2/s}$, respectively.

Solution

The settling velocity is found by setting up an equilibrium force balance. First, the submerged weight of the sand particle is

$$W = \frac{4}{3}\pi r_0^3(\rho_s - \rho_w)g = \frac{4}{3}\pi(0.1\times 10^{-3})^3(2{,}400 - 1{,}000)$$
$$= 5.86 \times 10^{-9}\ \mathrm{N}$$

where $r_0 = d/2$ is the radius of the particle. This expression is equal to the drag force during steady-state settling of the sand particle. According to Eq. (3.3.5),

$$W = \frac{24\nu}{Ud}\frac{\rho_w}{2}\pi r_0^2 U^2$$

$$\Rightarrow \qquad U = \frac{W}{6\pi\rho_w \nu r_0}$$

$$= \frac{5.86\times 10^{-9}}{6\pi \times 1{,}000 \times 10^{-6} \times 0.1 \times 10^{-3}} = 3.1 \times 10^{-3}\ \mathrm{m/s}$$

However, in order to use this equation, the Reynolds number must be checked. The value of the Reynolds number is

$$\text{Re} = \frac{Ud}{\nu} = \frac{3.1 \times 10^{-3} \times 0.2 \times 10^{-3}}{10^{-6}} = 0.62$$

which is less than 1. Therefore, use of the Stokes approximation was appropriate.

Problem 3.5 A flat plate is subject to oscillatory motions, with velocity given by

$$U_0 \sin(\omega t)$$

On top of the plate there is a semi-infinite fluid domain with uniform pressure distribution. The density and kinematic viscosity of the fluid are ρ and ν, respectively.

(a) Determine the velocity distribution in the domain.
(b) Determine the shear stress distribution. What is the phase lag between the maximum values of the shear stress and that of the velocity?
(c) What are the force and power per unit area needed to move the plate? What are the maximum values of these parameters?

Solution

(a) This problem is represented by the differential Eq. (3.5.1), subject to the following boundary conditions:

$$u = U_0 \sin(\omega t) \qquad \text{at} \qquad y = 0$$

$$u = 0 \qquad \text{at} \qquad y \to \infty$$

These boundary conditions suggest consideration of the following expression for the velocity:

$$u = \text{Im}\left[U(y)\exp(i\omega t)\right]$$

Similarly as in Eqs. (3.5.4)–(3.5.6), the velocity distribution is found as

$$u = U_0 \exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right) \sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)$$

(b) The shear stress is given by

$$\tau = \rho\nu\frac{\partial u}{\partial y} = -\rho\nu\sqrt{\frac{\omega}{2\nu}}U_0\left[\exp\left(-y\sqrt{\frac{\omega}{2\nu}}\right)\right]$$
$$\left[\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) + \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)\right]$$

The maximum value of μ is obtained when

$$\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) = 1 \qquad \Rightarrow \qquad \omega t - y\sqrt{\frac{\omega}{2\nu}} = \frac{\pi}{2}$$

The maximum value of τ is obtained when

$$\sin\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) + \cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right) \rightarrow \max$$
$$\Rightarrow \qquad \omega t - y\sqrt{\frac{\omega}{2\nu}} = \frac{\pi}{4}$$

Therefore the phase difference between u_{max} and τ_{max} is $\pi/4$.

(c) The force per unit area needed to move the plate is equal to the negative value of the shear stress at $y = 0$, namely

$$\frac{F}{A} = -(\tau)_{y=0} = \rho\sqrt{\frac{\omega\nu}{2}}U_0[\sin(\omega t) + \cos(\omega t)]$$

where F is the force and A is the area of the plate. The power needed to move the plate is equal to the product of that force with the velocity of the plate, or

$$\frac{N}{A} = \frac{F}{A}(u)_{y=0} = \rho\sqrt{\frac{\omega\nu}{2}}U_0^2[\sin(\omega t)][\sin(\omega t) + \cos(\omega t)]$$

The maximum value of this parameter is obtained when $\{\sin(\omega t)\,[\sin(\omega t)+ \cos(\omega t)] \rightarrow \max\}$. Differentiation of this expression indicates that the maximum value of the power is obtained when

$$\omega t = \frac{\pi}{2}\left(n - \frac{1}{4}\right) \qquad \text{where} \qquad n = 1, 2 \ldots$$

Unsolved Problems

Problem 3.6 The velocity distribution for flow between two plates is given by

$$u = U\left[1 - \left(\frac{2y}{b}\right)^2\right]$$

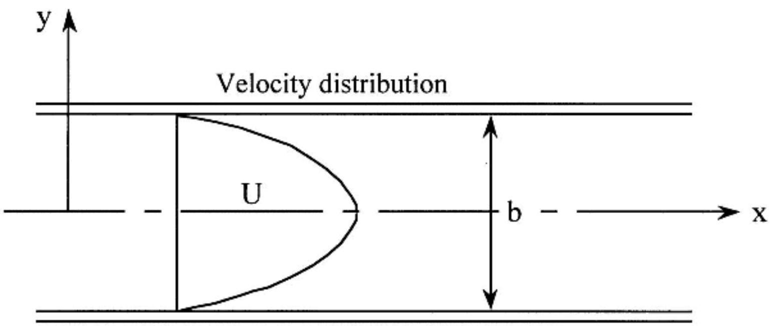

Figure 3.3 Flow between two plates, Problem 3.6.

where b is the gap between the two plates, U is the velocity at the centerline of the fluid layer, and y is the distance from the centerline (see Fig. 3.3).

(a) Show that the flow is a rotational flow. What is the vorticity distribution in the fluid layer?
(b) What boundary conditions are satisfied by the velocity distribution?
(c) Considering that the characteristic length and velocity are the gap between the plate and the average flow velocity, respectively, what is the expression for the dimensionless velocity distribution?
(d) What is the expression for the Reynolds number?

Problem 3.7 Water flows on an oblique plate forming a roof, as shown in Fig. 3.4. The water flows as a fluid layer with thickness $b = 10^{-3}$ m. The water density is $\rho = 1{,}000\ \mathrm{kg/m^3}$. Its kinematic viscosity is $\nu = 10^{-6}\ \mathrm{m^2/s}$. The slope of the roof is $\alpha = 30°$. Due to wind gusts, the surface of the flowing

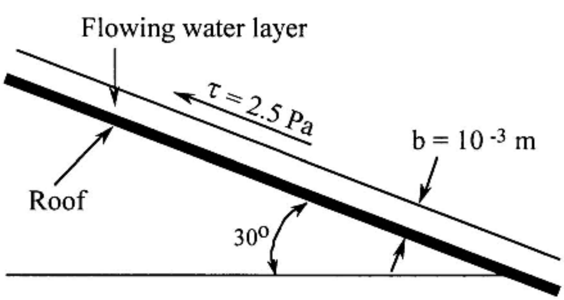

Figure 3.4 Flow along a sloping surface, Problem 3.7.

water layer is subject to a shear stress $\tau = 2.5$ Pa, in the upward direction of the roof.

(a) What is the water discharge per unit width of the roof?
(b) Prove that the flow is laminar.
(c) What is the shear stress applied on the roof?

Problem 3.8 A gap of thickness $b = 5 \times 10^{-4}$ m separates two vertical belts and is occupied by viscous oil, whose density is $\rho = 800\ \mathrm{kg/m^3}$, as shown in Fig. 3.5. The viscosity of the oil is $\mu = 8 \times 10^{-2}$ Pa s. One belt moves upward with a velocity of $V_1 = 2$ m/s. The other belt moves downward. The gravitational forces and the movement of the belts only affect the flow of the oil layer. The net discharge of the oil is zero.

(a) What is the velocity of the second belt?
(b) Draw a schematic of the velocity and shear stress distributions in the oil layer.

Problem 3.9 Figure 3.6 shows a "belt pump", which diverts oil from a lower tank to an upper one. The density of the oil is $\rho = 800\ \mathrm{kg/m^3}$ and its viscosity is $\mu = 8 \times 10^{-2}$ Pa s. The belt moves with velocity $U = 0.2$ m/s. The thickness of the oil layer is $b = 2 \times 10^{-3}$ m. The orientation angle of the belt is $\theta = 45°$. The horizontal distance between the two tanks is $L = 5$ m.

(a) What is the discharge delivered by the pump?
(b) What is the power needed to operate the pump?

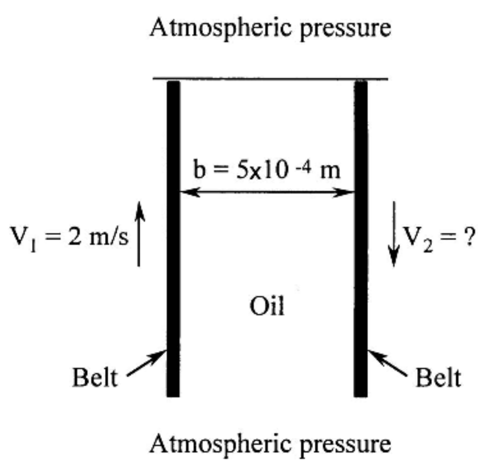

Figure 3.5 Flow of oil between two belts, Problem 3.8.

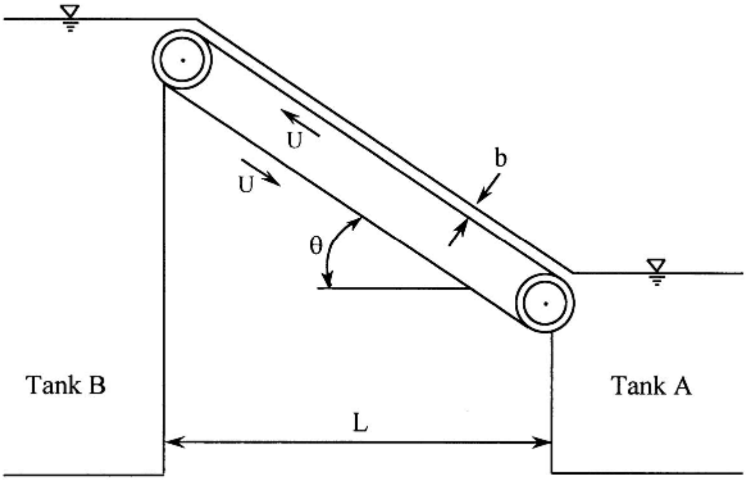

Figure 3.6 Belt pump, Problem 3.9.

(c) What is the efficiency of the pump?
(d) What should be the thickness of the fluid layer which maximizes the discharge?

Problem 3.10 The domain for a flow of oil is defined by the following stream function:

$$\Psi = U\left(y - \frac{y^3}{3b^2}\right)$$

where $U = 0.1$ m/s and $b = 0.05$ m. The density of the oil is $\rho = 800\ \text{kg/m}^3$, and its kinematic viscosity is $\nu = 8 \times 10^{-5}\ \text{m}^2/\text{s}$.

(a) Prove that the flow domain is the gap between two parallel plates, where the size of that gap is $2b$.
(b) What are the velocity and shear stress distributions in the flow domain?
(c) What is the gradient of the piezometric head?
(d) What is the power loss along a unit length of the flow domain?

Problem 3.11 Figure 3.7 shows oil flowing steadily along a vertical wall in a thin layer of thickness $b = 3 \times 10^{-3}$ m, with a discharge per unit width $q = 3 \times 10^{-3}\ \text{m}^2/\text{s}$. The density of the oil is $\rho = 800\ \text{kg/m}^3$.

(a) What are the viscosity and kinematic viscosity of the oil?
(b) What is the shear stress applied on the wall by the flowing oil?

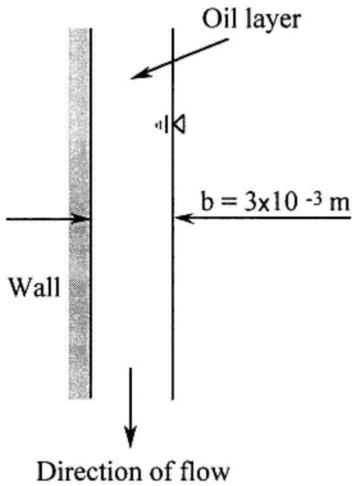

Figure 3.7 Flow of oil along a vertical wall, Problem 3.11.

Problem 3.12 Oil flows due to gravity on an oblique plate in a layer of thickness $b = 3 \times 10^{-3}$ m, as shown in Fig. 3.8. The angle of orientation of the plate is $\theta = 30^\circ$ with respect to horizontal. The plate moves upward with velocity $V = 0.1$ m/s. The kinematic viscosity of the oil is $\nu = 8 \times 10^{-5}$ m^2/s, and its density is $\rho = 800$ kg/m^3.

(a) Calculate and draw a schematic of the velocity and shear stress distributions.
(b) What is the direction and value of the discharge per unit width?

Problem 3.13 Oil is located between two flat plates, as shown in Fig. 3.9. The kinematic viscosity of the oil is $\nu = 8 \times 10^{-5}$ m^2/s, and its density is $\rho =$

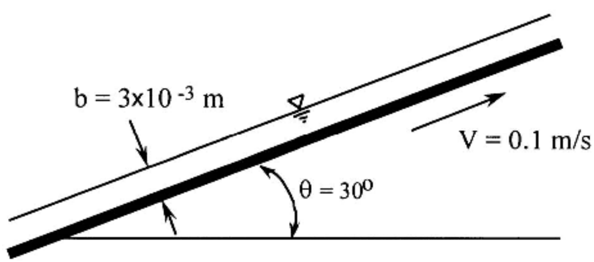

Figure 3.8 Flow of oil on a sloping surface, Problem 3.12.

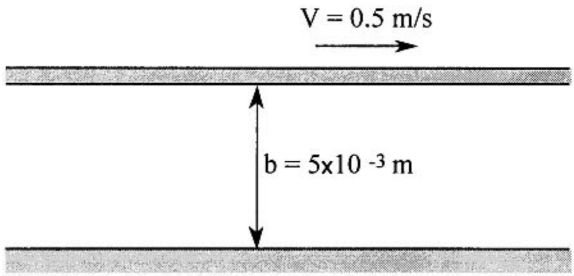

Figure 3.9 Viscous flow between two plates, Problem 3.13.

800 kg/m^3. The upper plate moves to the right with a velocity $V = 0.5$ m/s. The lower plate is stationary. The gap between the plates has thickness $b = 5 \times 10^{-3}$ m. The net discharge of the oil is zero.

(a) What is the pressure gradient between the plates?
(b) What is the shear stress at each one of the plates?
(c) Where does the shear stress obtain its maximum and minimum absolute values?
(d) Draw a schematic of the velocity and shear stress distributions.

Problem 3.14 Oil flows out of a tank, as shown in Fig. 3.10. The oil density is $\rho = 800$ kg/m^3 and its viscosity is $\mu = 8 \times 10^{-2}$ Pa s. The difference in elevation between the oil-free surface in the tank and the outlet is $\Delta h = 12.2$ m. The oil flows out through a pipe whose diameter and length are

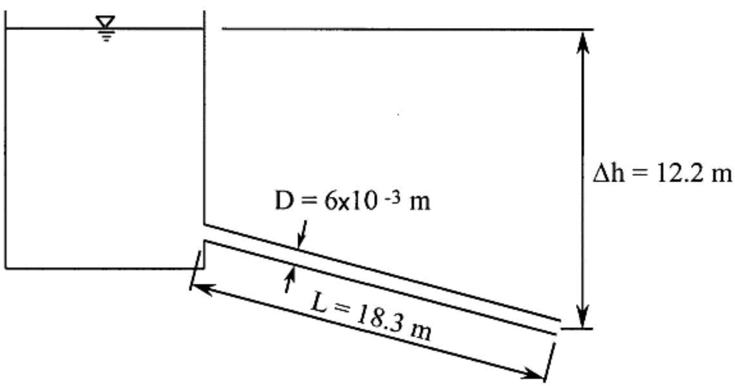

Figure 3.10 Definition sketch, Problem 3.14.

$D = 6 \times 10^{-3}$ m and $L = 18.3$ m. Piezometric head loss at the pipe entrance is negligible.

(a) What is the gradient of the piezometric head along the pipe?
(b) What is the oil discharge?
(c) Is the flow laminar? Why or why not?
(d) What is the power loss due to the flow through the pipe?

Problem 3.15 A motor shaft, with diameter $D_1 = 5 \times 10^{-2}$ m, rotates at a rate of $n = 1{,}200$ rpm, inside a bearing, as shown in Fig. 3.11. The internal diameter of the bearing is $D_2 = 5.02 \times 10^{-2}$ m. Its length is $L = 0.1$ m. The viscosity of the oil is $\mu = 10^{-2}$ Pa s. It occupies the gap between the bearing and the shaft. The shaft and the bearing form a system of coaxial cylinders.

(a) What is the shear stress applied on the oil?
(b) What is the power loss in the bearing?

Problem 3.16 Helium flows through a pipe of diameter D. The flow of helium is different from that of other fluids in that the usual no-slip condition at a solid boundary does not apply. There is some sliding at the pipe wall, and the helium has some velocity at that location. The boundary condition at the pipe wall is

$$(u^3)_{r=R} = \left(K\frac{du}{dr}\right)_{r=R}$$

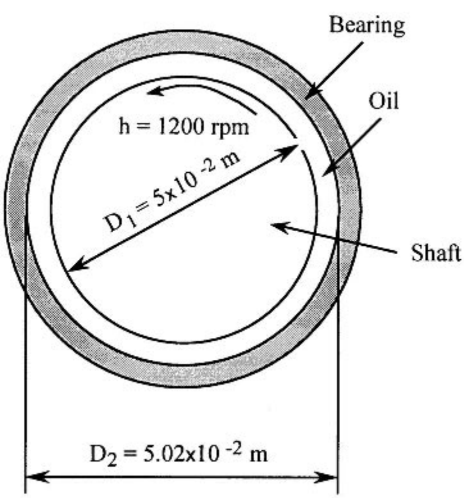

Figure 3.11 Definition sketch, Problem 3.15.

where r is the radial coordinate, R is the pipe radius, and K is a constant.

(a) Determine the velocity profile of the helium pipe flow.
(b) Determine the shear stress distribution.
(c) Determine the relationship between the discharge and the gradient of the piezometric pressure.

Problem 3.17 Oil flows from container A to container B through a pipe of length $L = 10$ m and diameter $d = 10^{-2}$ m, as shown in Fig. 3.12. The kinematic viscosity of the oil is $\nu = 2 \times 10^{-4}$ m^2/s. The diameter of each container is $D = 1$ m. When the flow starts, at $t = 0$, the oil level in container A is $H_1 = 1$ m, and in container B it is $H_2 = 0.1$ m. Steady flow conditions may be assumed.

(a) Calculate the initial discharge and average flow velocity. Prove that the flow is laminar.
(b) Develop the expression for the variation of oil levels in the containers. Find at what time the oil level in container B is equal to 0.5 m.

Problem 3.18 Figure 3.13 indicates two containers holding oil, with kinematic viscosity $\nu = 8 \times 10^{-5}$ m^2/s and density $\rho = 800$ kg/m^3. The containers are connected by a pipe with length $L = 500$ m and diameter $d = 5 \times 10^{-2}$ m. The oil level in container A is $H_1 = 55$ m, and in container B the oil level is $H_2 = 50$ m. Assume that oil levels in both containers are kept constant. Local head losses and the velocity head loss at the pipe exit may be neglected.

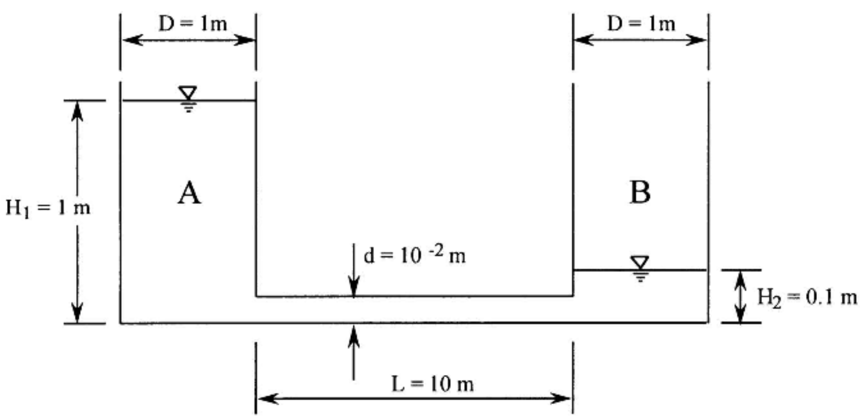

Figure 3.12 Definition sketch, Problem 3.17.

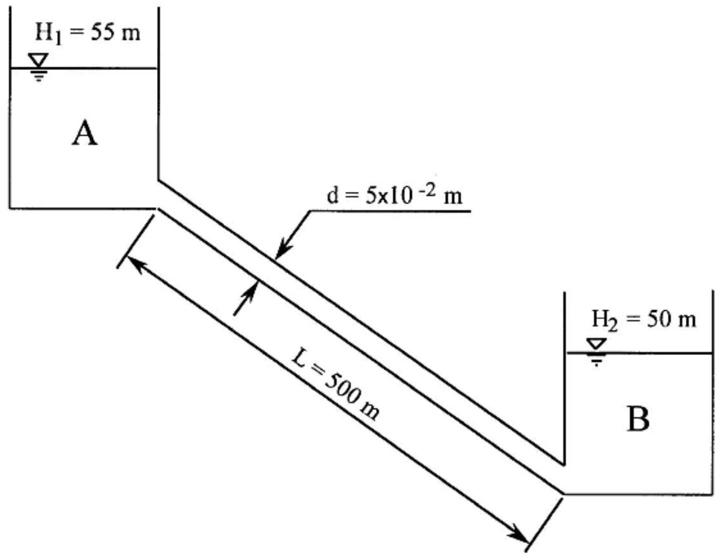

Figure 3.13 Definition sketch, Problem 3.18.

(a) What are the oil discharge and the average flow velocity?
(b) What is the Reynolds number of the flow?
(c) What is the shear stress at the pipe wall?
(d) What is the power loss due to the oil flow?

Problem 3.19 Figure 3.14 shows a laboratory system similar to an infusion system. At time $t = 0$, the fluid level is at point A. The initial fluid volume in the container is $U = 10^{-3}$ m^3. The container is a top open cylinder, with initial fluid depth $h_0 = 0.1$ m. The kinematic viscosity of the fluid is $\nu = 10^{-5}$ m^2/s, and its density is $\rho = 1{,}020$ kg/m^3. The fluid flows out of the container through a tube whose length is $L = 2$ m and whose diameter is $d = 10^{-3}$ m. At the exit of the pipe the pressure is kept constant, at $p = 10^4$ Pa. The bottom of the container is elevated to a level of $H = 1.5$ m above the pipe exit. It may be assumed that the flow is steady, with variable head loss.

(a) What is the initial, maximum fluid discharge (at time $t = 0$, when the fluid level is at point A)?
(b) What is the final, minimum discharge (when the fluid level is at point B)?
(c) How much time is required to empty the container?

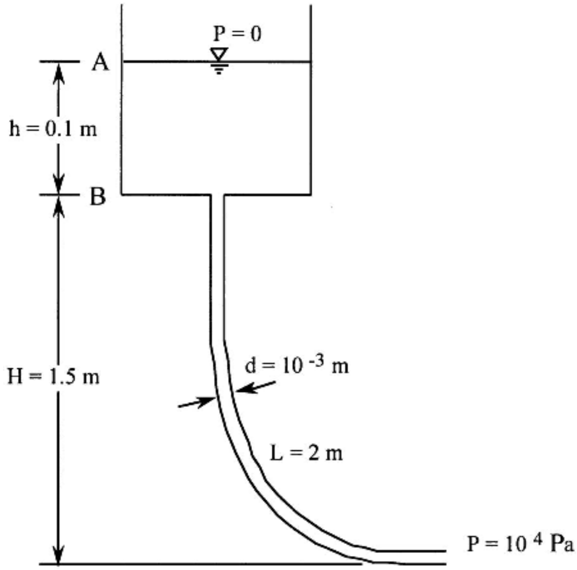

Figure 3.14 Fluid drainage by gravity, Problem 3.19.

Problem 3.20 Two types of fluids occupy the gap between two parallel horizontal flat plates, as shown in Fig. 3.15. There is no pressure gradient along the flow direction. The width of the gap between the plates is $b = 10^{-2}$ m. The lower half is occupied by a fluid whose density and viscosity are $\rho = 900$ kg/m^3 and $\mu = 0.1$ Pa s, respectively. The upper half of the gap is occupied by a second fluid, whose density and viscosity are $\rho = 700$ kg/m^3

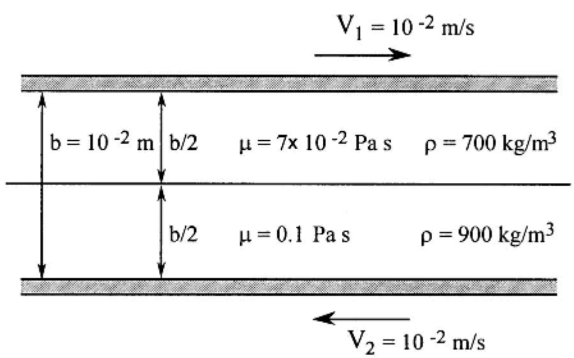

Figure 3.15 Flow of two fluids between plates, Problem 3.20.

and $\mu = 7 \times 10^{-2}$ Pa s, respectively. The upper plate moves to the right with a velocity of $V_1 = 10^{-2}$ m/s. The lower plate moves to the left with a velocity $V_2 = 10^{-2}$ m/s.

(a) Calculate and draw a schematic of the velocity and shear stress distributions in the fluid layers.
(b) Calculate the net discharge of each fluid.

Problem 3.21 Figure 3.16 shows oil flowing from container (1) to container (2) through a tube whose length and diameter are $L = 1.2$ m and $D = 4 \times 10^{-3}$ m, respectively. The oil flow is driven by a constant pressure $p = 10^4$ Pa, maintained in the free space of container (1), as well as the difference between the elevations of the oil free surfaces in both containers. Initially, that difference of elevation is $H_0 = 1$ m. Container (2) is open to the atmosphere.

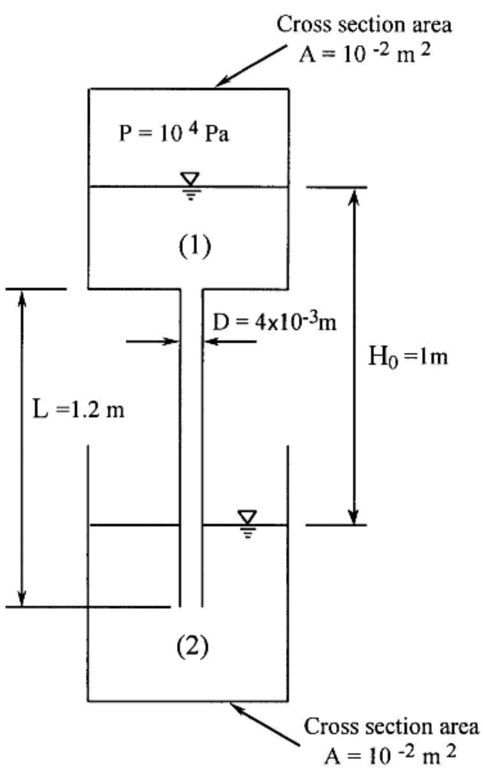

Figure 3.16 Definition sketch, Problem 3.21.

The cross-sectional area of each container is $A = 10^{-2}$ m^2. The density and viscosity of the oil are $\rho = 900$ kg/m^3 and $\mu = 0.1$ Pa s, respectively.

(a) Determine the general expression representing the relationship between the discharge Q, which flows from container (1) to container (2), and the parameters H, p, D, L, ρ, and μ.
(b) Determine the maximum, initial value of the discharge.
(c) Determine the time T, during which H will reduce from $H_0 = 1$ m to $H = 0.5$ m.

Problem 3.22 A viscous fluid flows from container A to container B through an annulus, as shown in Fig. 3.17. The annulus consists of a steel member, whose diameter is $D_1 = 0.02$ m, and a pipe, whose internal diameter is $D_2 = 0.022$ m. The difference between fluid levels in containers A and B is kept constant, at $H = 5$ m. The length of the annulus is $L = 100$ m. The fluid density and viscosity are $\rho = 900$ kg/m^3 and $\mu = 5 \times 10^{-3}$ Pa s, respectively.

(a) Determine the distributions of the velocity and shear stress in the annulus cross section.
(b) Determine the discharge, which flows through the annulus.

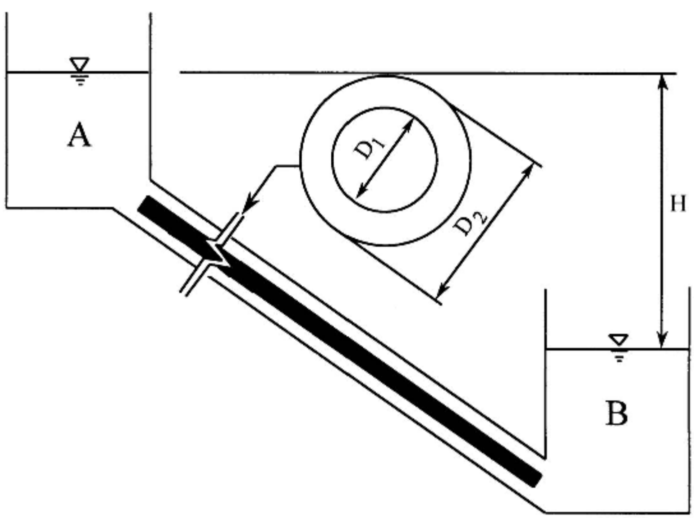

Figure 3.17 Definition sketch, Problem 3.22.

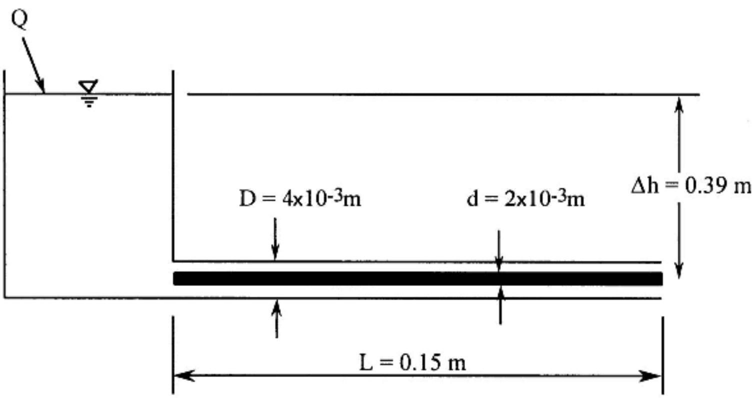

Figure 3.18 Definition sketch, Problem 3.23.

Problem 3.23 In Fig. 3.18, fluid flows out of the container through a horizontal pipe, whose length is $L = 0.15$ m and inner diameter is $D = 4 \times 10^{-3}$ m. Inside the pipe a metal member is inserted. The diameter of the metal member is $d = 2 \times 10^{-3}$ m, and it is coaxial with the pipe. The density and viscosity of the fluid are $\rho = 800$ kg/m^3 and $\mu = 5 \times 10^{-3}$ Pa s, respectively. The difference between the elevation of the free surface of the container fluid and the pipe exit is $\Delta h = 0.39$ m. A discharge Q flows into the container, to maintain the free surface of the container fluid at a constant level.

(a) What is the value of the discharge Q, flowing into the container?
(b) By how much should Q be increased, if the metal member is taken out of the pipe?
(c) What is the total shear force applied on the metal member?

SUPPLEMENTAL READING

Batchelor, G. K., 1967. *An Introduction to Fluid Dynamics*, Cambridge University Press, London.

Streeter, V. L., 1961. *Handbook of Fluid Dynamics*, McGraw-Hill, New York. (Summarizes the different presentations and uses of the viscous flow equations.)

Wendt, J. F., ed., 1996. *Computational Fluid Dynamics*, Springer Verlag, Berlin. (Includes an introduction and survey of different approaches aimed at appropriate use of numerical approaches for the solution of the equations of motion.)