CHAPTER 1

1.1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_y + 8\mathbf{a}_z + 16\mathbf{a}_x + 14\mathbf{a}_y - 4\mathbf{a}_z = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26, 10, 4)}{|(26, 10, 4)|} = \underline{(0.92, 0.36, 0.14)}$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5, 0, 0) + (8, 7, -2) - (-30, 12, -24) = (43, -5, 22)$, and $|(43, -5, 22)| = \underline{48.6}$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$:

$|(-10, 4, -8)||(16, 14, -4)|(-2, 11, -10) = (13.4)(21.6)(-2, 11, -10)$

$= \underline{(-580.5, 3193, -2902)}$

1.2. Given three points, $A(4, 3, 2)$, $B(-2, 0, 5)$, and $C(7, -2, 1)$:

a) Specify the vector $\mathbf{A}$ extending from the origin to the point A.

$$\mathbf{A} = (4, 3, 2) = \underline{4\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z}$$

b) Give a unit vector extending from the origin to the midpoint of line AB.

The vector from the origin to the midpoint is given by

$\mathbf{M} = (1/2)(\mathbf{A} + \mathbf{B}) = (1/2)(4 - 2, 3 + 0, 2 + 5) = (1, 1.5, 3.5)$

The unit vector will be

$$\mathbf{m} = \frac{(1, 1.5, 3.5)}{|(1, 1.5, 3.5)|} = \underline{(0.25, 0.38, 0.89)}$$

c) Calculate the length of the perimeter of triangle ABC:

Begin with $\mathbf{AB} = (-6, -3, 3)$, $\mathbf{BC} = (9, -2, -4)$, $\mathbf{CA} = (3, -5, -1)$.

Then

$$|\mathbf{AB}| + |\mathbf{BC}| + |\mathbf{CA}| = 7.35 + 10.05 + 5.91 = \underline{23.32}$$

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or

$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$

Expanding, obtain

$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$

or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64 - 176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. given points $A(8, -5, 4)$ and $B(-2, 3, 2)$, find:

a) the distance from A to B.

$$|\mathbf{B} - \mathbf{A}| = |(-10, 8, -2)| = \underline{12.96}$$

b) a unit vector directed from A towards B. This is found through

$$\mathbf{a}_{AB} = \frac{\mathbf{B} - \mathbf{A}}{|\mathbf{B} - \mathbf{A}|} = \underline{(-0.77, 0.62, -0.15)}$$

c) a unit vector directed from the origin to the midpoint of the line AB.

$$\mathbf{a}_{0M} = \frac{(\mathbf{A} + \mathbf{B})/2}{|(\mathbf{A} + \mathbf{B})/2|} = \frac{(3, -1, 3)}{\sqrt{19}} = \underline{(0.69, -0.23, 0.69)}$$

d) the coordinates of the point on the line connecting A to B at which the line intersects the plane $z = 3$. Note that the midpoint, $(3, -1, 3)$, as determined from part c happens to have z coordinate of 3. This is the point we are looking for.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2 + 2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1, 2, -1) = \underline{(48, 36, 18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \underline{(-0.26, 0.39, 0.88)}$$

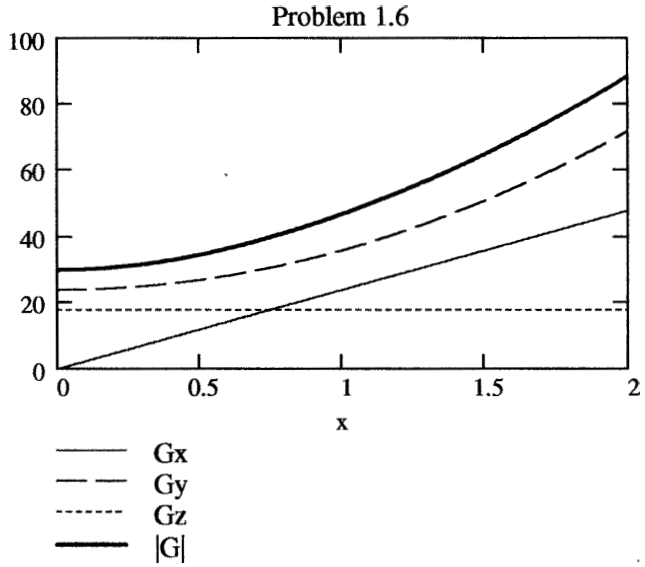
c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \underline{(0.59, 0.20, -0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2 + 2), 18z^2)|$, or $10 = |(4xy, 2x^2 + 4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.6. For the **G** field in Problem 1.5, make sketches of G_x, G_y, G_z and $|\mathbf{G}|$ along the line $y = 1$, $z = 1$, for $0 \le x \le 2$. We find $\mathbf{G}(x, 1, 1) = (24x, 12x^2 + 24, 18)$, from which $G_x = 24x$, $G_y = 12x^2 + 24$, $G_z = 18$, and $|\mathbf{G}| = 6\sqrt{4x^4 + 32x^2 + 25}$. Plots are shown below.



1.7. Given the vector field $\mathbf{E} = 4zy^2 \cos 2x\mathbf{a}_x + 2zy \sin 2x\mathbf{a}_y + y^2 \sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy \sin 2x = 0$, the surfaces are 1) the plane $\underline{z = 0}$, with $|x| < 2$, $|y| < 2$; 2) the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$; 3) the plane $\underline{x = 0}$, with $|y| < 2$, $|z| < 2$; 4) the plane $\underline{x = \pi/2}$, with $|y| < 2$, $|z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy \sin 2x = y^2 \sin 2x$, or on the plane $\underline{2z = y}$, with $|x| < 2$, $|y| < 2$, $|z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2 \cos 2x = zy \sin 2x = y^2 \sin 2x = 0$. This condition is met on the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$.

1.8. Two vector fields are $\mathbf{F} = -10\mathbf{a}_x + 20x(y-1)\mathbf{a}_y$ and $\mathbf{G} = 2x^2y\mathbf{a}_x - 4\mathbf{a}_y + z\mathbf{a}_z$. For the point $P(2, 3, -4)$, find:

a) $|\mathbf{F}|$: **F** at $(2, 3, -4) = (-10, 80, 0)$, so $\underline{|\mathbf{F}| = 80.6}$.

b) $|\mathbf{G}|$: **G** at $(2, 3, -4) = (24, -4, -4)$, so $\underline{|\mathbf{G}| = 24.7}$.

c) a unit vector in the direction of $\mathbf{F} - \mathbf{G}$: $\mathbf{F} - \mathbf{G} = (-10, 80, 0) - (24, -4, -4) = (-34, 84, 4)$. So

$$\mathbf{a} = \frac{\mathbf{F} - \mathbf{G}}{|\mathbf{F} - \mathbf{G}|} = \frac{(-34, 84, 4)}{90.7} = \underline{(-0.37, 0.92, 0.04)}$$

d) a unit vector in the direction of $\mathbf{F} + \mathbf{G}$: $\mathbf{F} + \mathbf{G} = (-10, 80, 0) + (24, -4, -4) = (14, 76, -4)$. So

$$\mathbf{a} = \frac{\mathbf{F} + \mathbf{G}}{|\mathbf{F} + \mathbf{G}|} = \frac{(14, 76, -4)}{77.4} = \underline{(0.18, 0.98, -0.05)}$$

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2+y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53^\circ}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2+y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2+49} \times 7\, dz dx = \int_0^4 \frac{350}{x^2+49} dx$$

$$= 350 \times \frac{1}{7}\left[\tan^{-1}\left(\frac{4}{7}\right) - 0\right] = \underline{26}$$

1.10. Use the definition of the dot product to find the interior angles at A and B of the triangle defined by the three points $A(1, 3, -2)$, $B(-2, 4, 5)$, and $C(0, -2, 1)$:

a) Use $\mathbf{R}_{AB} = (-3, 1, 7)$ and $\mathbf{R}_{AC} = (-1, -5, 3)$ to form $\mathbf{R}_{AB} \cdot \mathbf{R}_{AC} = |\mathbf{R}_{AB}||\mathbf{R}_{AC}| \cos\theta_A$. Obtain $3 + 5 + 21 = \sqrt{59}\sqrt{35}\cos\theta_A$. Solve to find $\underline{\theta_A = 65.3^\circ}$.

b) Use $\mathbf{R}_{BA} = (3, -1, -7)$ and $\mathbf{R}_{BC} = (2, -6, -4)$ to form $\mathbf{R}_{BA} \cdot \mathbf{R}_{BC} = |\mathbf{R}_{BA}||\mathbf{R}_{BC}| \cos\theta_B$. Obtain $6 + 6 + 28 = \sqrt{59}\sqrt{56}\cos\theta_B$. Solve to find $\underline{\theta_B = 45.9^\circ}$.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Given points $A(10, 12, -6)$, $B(16, 8, -2)$, $C(8, 1, -4)$, and $D(-2, -5, 8)$, determine:

a) the vector projection of $\mathbf{R}_{AB} + \mathbf{R}_{BC}$ on $\mathbf{R}_{AD}$: $\mathbf{R}_{AB} + \mathbf{R}_{BC} = \mathbf{R}_{AC} = (8, 1, 4) - (10, 12, -6) = (-2, -11, 10)$ Then $\mathbf{R}_{AD} = (-2, -5, 8) - (10, 12, -6) = (-12, -17, 14)$. So the projection will be:

$$(\mathbf{R}_{AC} \cdot \mathbf{a}_{RAD})\mathbf{a}_{RAD} = \left[(-2, -11, 10) \cdot \frac{(-12, -17, 14)}{\sqrt{629}}\right] \frac{(-12, -17, 14)}{\sqrt{629}} = \underline{(-6.7, -9.5, 7.8)}$$

b) the vector projection of $\mathbf{R}_{AB} + \mathbf{R}_{BC}$ on $\mathbf{R}_{DC}$: $\mathbf{R}_{DC} = (8, -1, 4) - (-2, -5, 8) = (10, 6, -4)$. The projection is:

$$(\mathbf{R}_{AC} \cdot \mathbf{a}_{RDC})\mathbf{a}_{RDC} = \left[(-2, -11, 10) \cdot \frac{(10, 6, -4)}{\sqrt{152}}\right] \frac{(10, 6, -4)}{\sqrt{152}} = \underline{(-8.3, -5.0, 3.3)}$$

c) the angle between $\mathbf{R}_{DA}$ and $\mathbf{R}_{DC}$: Use $\mathbf{R}_{DA} = -\mathbf{R}_{AD} = (12, 17, -14)$ and $\mathbf{R}_{DC} = (10, 6, -4)$. The angle is found through the dot product of the associated unit vectors, or:

$$\theta_D = \cos^{-1}(\mathbf{a}_{RDA} \cdot \mathbf{a}_{RDC}) = \cos^{-1}\left(\frac{(12, 17, -14) \cdot (10, 6, -4)}{\sqrt{629}\sqrt{152}}\right) = \underline{26^\circ}$$

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. The four vertices of a regular tetrahedron are located at $O(0, 0, 0)$, $A(0, 1, 0)$, $B(0.5\sqrt{3}, 0.5, 0)$, and $C(\sqrt{3}/6, 0.5, \sqrt{2/3})$.

a) Find a unit vector perpendicular (outward) to the face ABC: First find

$$\begin{aligned}\mathbf{R}_{BA} \times \mathbf{R}_{BC} &= [(0, 1, 0) - (0.5\sqrt{3}, 0.5, 0)] \times [(\sqrt{3}/6, 0.5, \sqrt{2/3}) - (0.5\sqrt{3}, 0.5, 0)] \\ &= (-0.5\sqrt{3}, 0.5, 0) \times (-\sqrt{3}/3, 0, \sqrt{2/3}) = (0.41, 0.71, 0.29)\end{aligned}$$

The required unit vector will then be:

$$\frac{\mathbf{R}_{BA} \times \mathbf{R}_{BC}}{|\mathbf{R}_{BA} \times \mathbf{R}_{BC}|} = \underline{(0.47, 0.82, 0.33)}$$

b) Find the area of the face ABC:

$$\text{Area} = \frac{1}{2}|\mathbf{R}_{BA} \times \mathbf{R}_{BC}| = \underline{0.43}$$

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1 = (7, 3, -2)$, $\mathbf{r}_2 = (-2, 7, -3)$, and $\mathbf{r}_3 = (0, 2, 3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|} = \frac{(5, 25, 55)}{60.6} = \underline{(0.08, 0.41, 0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{r}_2 - \mathbf{r}_3$: $\mathbf{r}_1 - \mathbf{r}_2 = (9, -4, 1)$ and $\mathbf{r}_2 - \mathbf{r}_3 = (-2, 5, -6)$. So $\mathbf{r}_1 - \mathbf{r}_2 \times \mathbf{r}_2 - \mathbf{r}_3 = (19, 52, 32)$. Then

$$\mathbf{a}_p = \frac{(19, 52, 32)}{|(19, 52, 32)|} = \frac{(19, 52, 32)}{63.95} = \underline{(0.30, 0.81, 0.50)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area} = \frac{1}{2}|\mathbf{r}_1 \times \mathbf{r}_2| = \underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area} = \frac{1}{2}|(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_2 - \mathbf{r}_3)| = \frac{1}{2}|(-9, 4, -1) \times (-2, 5, -6)| = \underline{32.0}$$

1.16. Describe the surfaces defined by the equations:

a) $\mathbf{r} \cdot \mathbf{a}_x = 2$, where $\mathbf{r} = (x, y, z)$: This will be the plane $\underline{x = 2}$.

b) $|\mathbf{r} \times \mathbf{a}_x| = 2$: $\mathbf{r} \times \mathbf{a}_x = (0, z, -y)$, and $|\mathbf{r} \times \mathbf{a}_x| = \underline{\sqrt{z^2 + y^2} = 2}$. This is the equation of a cylinder, centered on the x axis, and of radius 2.

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

1.17c. (continued) Now

$$\frac{1}{2}(\mathbf{a}_{AM}+\mathbf{a}_{AN})=\frac{1}{2}[(0.697,0.627,-0.348)+(-0.507,0.406,0.761)]=(0.095,0.516,0.207)$$

Finally,

$$\mathbf{a}_{bis}=\frac{(0.095,0.516,0.207)}{|(0.095,0.516,0.207)|}=\underline{(0.168,0.915,0.367)}$$

1.18. Given points $A(\rho=5,\phi=70^\circ,z=-3)$ and $B(\rho=2,\phi=-30^\circ,z=1)$, find:

a) unit vector in cartesian coordinates at A toward B: $A(5\cos 70^\circ, 5\sin 70^\circ, -3)=A(1.71,4.70,-3)$, In the same manner, $B(1.73,-1,1)$. So $\mathbf{R}_{AB}=(1.73,-1,1)-(1.71,4.70,-3)=(0.02,-5.70,4)$ and therefore

$$\mathbf{a}_{AB}=\frac{(0.02,-5.70,4)}{|(0.02,-5.70,4)|}=\underline{(0.003,-0.82,0.57)}$$

b) a vector in cylindrical coordinates at A directed toward B: $\mathbf{a}_{AB}\cdot\mathbf{a}_\rho=0.003\cos 70^\circ-0.82\sin 70^\circ=-0.77$. $\mathbf{a}_{AB}\cdot\mathbf{a}_\phi=-0.003\sin 70^\circ-0.82\cos 70^\circ=-0.28$. Thus

$$\mathbf{a}_{AB}=\underline{-0.77\mathbf{a}_\rho-0.28\mathbf{a}_\phi+0.57\mathbf{a}_z}$$

.

c) a unit vector in cylindrical coordinates at B directed toward A:
Use $\mathbf{a}_{BA}=(-0,003,0.82,-0.57)$. Then $\mathbf{a}_{BA}\cdot\mathbf{a}_\rho=-0.003\cos(-30^\circ)+0.82\sin(-30^\circ)=-0.43$, and $\mathbf{a}_{BA}\cdot\mathbf{a}_\phi=0.003\sin(-30^\circ)+0.82\cos(-30^\circ)=0.71$. Finally,

$$\mathbf{a}_{BA}=\underline{-0.43\mathbf{a}_\rho+0.71\mathbf{a}_\phi-0.57\mathbf{a}_z}$$

1.19 a) Express the field $\mathbf{D}=(x^2+y^2)^{-1}(x\mathbf{a}_x+y\mathbf{a}_y)$ in cylindrical components and cylindrical variables: Have $x=\rho\cos\phi$, $y=\rho\sin\phi$, and $x^2+y^2=\rho^2$. Therefore

$$\mathbf{D}=\frac{1}{\rho}(\cos\phi\mathbf{a}_x+\sin\phi\mathbf{a}_y)$$

Then

$$D_\rho=\mathbf{D}\cdot\mathbf{a}_\rho=\frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\rho)+\sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\rho)\right]=\frac{1}{\rho}\left[\cos^2\phi+\sin^2\phi\right]=\frac{1}{\rho}$$

and

$$D_\phi=\mathbf{D}\cdot\mathbf{a}_\phi=\frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi)+\sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)\right]=\frac{1}{\rho}\left[\cos\phi(-\sin\phi)+\sin\phi\cos\phi\right]=0$$

Therefore

$$\underline{\mathbf{D}=\frac{1}{\rho}\mathbf{a}_\rho}$$

1.19b. Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, $\underline{\mathbf{D} = 0.5\mathbf{a}_\rho}$. To express this in cartesian, we use

$$\mathbf{D} = 0.5(\mathbf{a}_\rho \cdot \mathbf{a}_x)\mathbf{a}_x + 0.5(\mathbf{a}_\rho \cdot \mathbf{a}_y)\mathbf{a}_y = 0.5\cos 36^\circ \mathbf{a}_x + 0.5\sin 36^\circ \mathbf{a}_y = \underline{0.41\mathbf{a}_x + 0.29\mathbf{a}_y}$$

1.20. Express in cartesian components:

a) the vector at $A(\rho = 4, \phi = 40^\circ, z = -2)$ that extends to $B(\rho = 5, \phi = -110^\circ, z = 2)$: We have $A(4\cos 40^\circ, 4\sin 40^\circ, -2) = A(3.06, 2.57, -2)$, and $B(5\cos(-110^\circ), 5\sin(-110^\circ), 2) = B(-1.71, -4.70, 2)$ in cartesian. Thus $\mathbf{R}_{AB} = \underline{(-4.77, -7.30, 4)}$.

b) a unit vector at B directed toward A: Have $\mathbf{R}_{BA} = (4.77, 7.30, -4)$, and so

$$\mathbf{a}_{BA} = \frac{(4.77, 7.30, -4)}{|(4.77, 7.30, -4)|} = \underline{(0.50, 0.76, -0.42)}$$

c) a unit vector at B directed toward the origin: Have $\mathbf{r}_B = (-1.71, -4.70, 2)$, and so $-\mathbf{r}_B = (1.71, 4.70, -2)$. Thus

$$\mathbf{a} = \frac{(1.71, 4.70, -2)}{|(1.71, 4.70, -2)|} = \underline{(0.32, 0.87, -0.37)}$$

1.21. Express in cylindrical components:

a) the vector from $C(3, 2, -7)$ to $D(-1, -4, 2)$:
$C(3, 2, -7) \rightarrow C(\rho = 3.61, \phi = 33.7^\circ, z = -7)$ and
$D(-1, -4, 2) \rightarrow D(\rho = 4.12, \phi = -104.0^\circ, z = 2)$.
Now $\mathbf{R}_{CD} = (-4, -6, 9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4, 6, -9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1, -4, 2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1, 4, -2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A field is given in cylindrical coordinates as

$$\mathbf{F} = \left[\frac{40}{\rho^2 + 1} + 3(\cos\phi + \sin\phi)\right]\mathbf{a}_\rho + 3(\cos\phi - \sin\phi)\mathbf{a}_\phi - 2\mathbf{a}_z$$

where the magnitude of $\mathbf{F}$ is found to be:

$$|\mathbf{F}| = \sqrt{\mathbf{F} \cdot \mathbf{F}} = \left[\frac{1600}{(\rho^2 + 1)^2} + \frac{240}{\rho^2 + 1}(\cos\phi + \sin\phi) + 22\right]^{1/2}$$

Sketch $|\mathbf{F}|$:

a) vs. ϕ with $\rho = 3$: in this case the above simplifies to

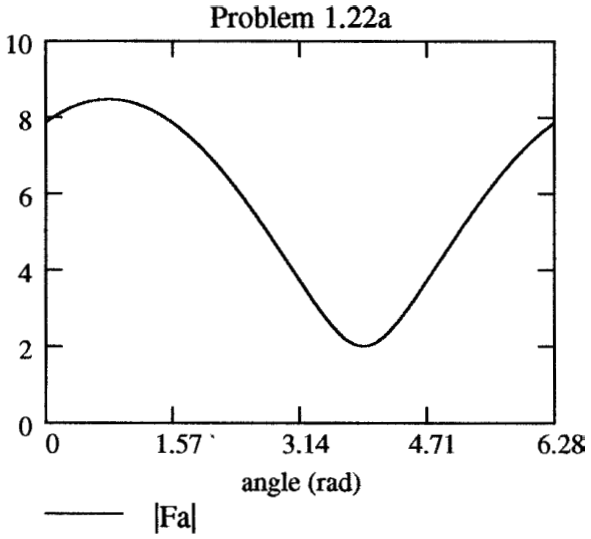
$$|\mathbf{F}(\rho = 3)| = |Fa| = [38 + 24(\cos\phi + \sin\phi)]^{1/2}$$

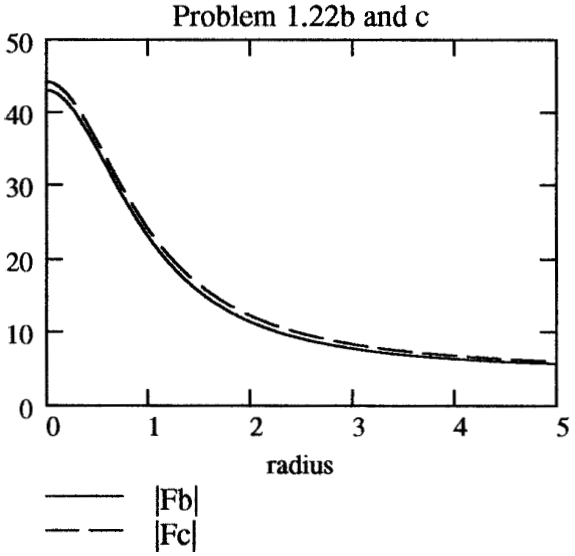
b) vs. ρ with $\phi = 0$, in which:

$$|\mathbf{F}(\phi = 0)| = |Fb| = \left[\frac{1600}{(\rho^2+1)^2} + \frac{240}{\rho^2+1} + 22\right]^{1/2}$$

c) vs. ρ with $\phi = 45^\circ$, in which

$$|\mathbf{F}(\phi = 45^\circ)| = |Fc| = \left[\frac{1600}{(\rho^2+1)^2} + \frac{240\sqrt{2}}{\rho^2+1} + 22\right]^{1/2}$$





1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100^\circ$, $\phi = 130^\circ$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2\int_{100^\circ}^{130^\circ} \int_3^5 \rho \, d\rho \, d\phi \; + \; \int_3^{4.5} \int_{100^\circ}^{130^\circ} 3 \, d\phi \, dz$$
$$+ \; \int_3^{4.5} \int_{100^\circ}^{130^\circ} 5 \, d\phi \, dz \; + \; 2\int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \; + \; 4 \times 2 \; + \; 2 \times \left[\frac{30^\circ}{360^\circ} \times 2\pi \times 3 \; + \; \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100^\circ$, $z = 3$) and B($\rho = 5$, $\phi = 130^\circ$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. At point $P(-3, 4, 5)$, express the vector that extends from P to $Q(2, 0, -1)$ in:

a) rectangular coordinates.

$$\mathbf{R}_{PQ} = \mathbf{Q} - \mathbf{P} = \underline{5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z}$$

Then $|\mathbf{R}_{PQ}| = \sqrt{25 + 16 + 36} = 8.8$

b) cylindrical coordinates. At P, $\rho = 5$, $\phi = \tan^{-1}(4/-3) = -53.1^\circ$, and $z = 5$. Now,

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\rho = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\rho = 5\cos\phi - 4\sin\phi = 6.20$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\phi = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\phi = -5\sin\phi - 4\cos\phi = 1.60$$

Thus

$$\mathbf{R}_{PQ} = \underline{6.20\mathbf{a}_\rho + 1.60\mathbf{a}_\phi - 6\mathbf{a}_z}$$

and $|\mathbf{R}_{PQ}| = \sqrt{6.20^2 + 1.60^2 + 6^2} = 8.8$

c) spherical coordinates. At P, $r = \sqrt{9 + 16 + 25} = \sqrt{50} = 7.07$, $\theta = \cos^{-1}(5/7.07) = 45^\circ$, and $\phi = \tan^{-1}(4/-3) = -53.1^\circ$.

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_r = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_r = 5\sin\theta\cos\phi - 4\sin\theta\sin\phi - 6\cos\theta = 0.14$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\theta = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi - 4\cos\theta\sin\phi - (-6)\sin\theta = 8.62$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\phi = (5\mathbf{a}_x - 4\mathbf{a}_y - 6\mathbf{a}_z) \cdot \mathbf{a}_\phi = -5\sin\phi - 4\cos\phi = 1.60$$

1.24. (continued)

Thus

$$\mathbf{R}_{PQ} = \underline{0.14\mathbf{a}_r + 8.62\mathbf{a}_\theta + 1.60\mathbf{a}_\phi}$$

and $|\mathbf{R}_{PQ}| = \sqrt{0.14^2 + 8.62^2 + 1.60^2} = 8.8$

d) Show that each of these vectors has the same magnitude. Each does, as shown above.

1.25. Given point $P(r = 0.8,\ \theta = 30^\circ,\ \phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2}\left(\cos\phi\,\mathbf{a}_r + \frac{\sin\phi}{\sin\theta}\,\mathbf{a}_\phi\right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.

b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.

c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. a) Determine an expression for $\mathbf{a}_y$ in spherical coordinates at $P(r = 4,\ \theta = 0.2\pi,\ \phi = 0.8\pi)$: Use $\mathbf{a}_y \cdot \mathbf{a}_r = \sin\theta\sin\phi = 0.35$, $\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta\sin\phi = 0.48$, and $\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos\phi = -0.81$ to obtain

$$\mathbf{a}_y = \underline{0.35\mathbf{a}_r + 0.48\mathbf{a}_\theta - 0.81\mathbf{a}_\phi}$$

b) Express $\mathbf{a}_r$ in cartesian components at P: Find $x = r\sin\theta\cos\phi = -1.90$, $y = r\sin\theta\sin\phi = 1.38$, and $z = r\cos\theta = -3.24$. Then use $\mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta\cos\phi = -0.48$, $\mathbf{a}_r \cdot \mathbf{a}_y = \sin\theta\sin\phi = 0.35$, and $\mathbf{a}_r \cdot \mathbf{a}_z = \cos\theta = 0.81$ to obtain

$$\mathbf{a}_r = \underline{-0.48\mathbf{a}_x + 0.35\mathbf{a}_y + 0.81\mathbf{a}_z}$$

1.27. The surfaces $r = 2$ and 4, $\theta = 30^\circ$ and 50°, and $\phi = 20^\circ$ and 60° identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}\int_2^4 r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20^\circ}^{60^\circ}\int_{30^\circ}^{50^\circ}(4^2 + 2^2)\sin\theta\,d\theta\,d\phi + \int_2^4\int_{20^\circ}^{60^\circ} r(\sin 30^\circ + \sin 50^\circ)\,dr\,d\phi + 2\int_{30^\circ}^{50^\circ}\int_2^4 r\,dr\,d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_2^4 dr + 2\int_{30^\circ}^{50^\circ}(4+2)\,d\theta + \int_{20^\circ}^{60^\circ}(4\sin 50^\circ + 4\sin 30^\circ + 2\sin 50^\circ + 2\sin 30^\circ)\,d\phi = \underline{17.49}$$

1.27. (continued)

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50^\circ, \phi = 20^\circ)$ to $B(r = 4, \theta = 30^\circ, \phi = 60^\circ)$ or

$$A(x = 2\sin 50^\circ \cos 20^\circ,\ y = 2\sin 50^\circ \sin 20^\circ,\ z = 2\cos 50^\circ)$$

to

$$B(x = 4\sin 30^\circ \cos 60^\circ,\ y = 4\sin 30^\circ \sin 60^\circ,\ z = 4\cos 30^\circ)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

1.28. a) Determine the cartesian components of the vector from $A(r = 5, \theta = 110^\circ, \phi = 200^\circ)$ to $B(r = 7, \theta = 30^\circ, \phi = 70^\circ)$: First transform the points to cartesian: $x_A = 5\sin 110^\circ \cos 200^\circ = -4.42$, $y_A = 5\sin 110^\circ \sin 200^\circ = -1.61$, and $z_A = 5\cos 110^\circ = -1.71$; $x_B = 7\sin 30^\circ \cos 70^\circ = 1.20$, $y_B = 7\sin 30^\circ \sin 70^\circ = 3.29$, and $z_B = 7\cos 30^\circ = 6.06$. Now

$$\mathbf{R}_{AB} = \mathbf{B} - \mathbf{A} = \underline{5.62\mathbf{a}_\mathbf{x} + 4.90\mathbf{a}_y + 7.77\mathbf{a}_z}$$

b) Find the spherical components of the vector at $P(2, -3, 4)$ extending to $Q(-3, 2, 5)$: First, $\mathbf{R}_{PQ} = \mathbf{Q} - \mathbf{P} = (-5, 5, 1)$. Then at P, $r = \sqrt{4 + 9 + 16} = 5.39$, $\theta = \cos^{-1}(4/\sqrt{29}) = 42.0^\circ$, and $\phi = \tan^{-1}(-3/2) = -56.3^\circ$. Now

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_r = -5\sin(42^\circ)\cos(-56.3^\circ) + 5\sin(42^\circ)\sin(-56.3^\circ) + 1\cos(42^\circ) = -3.90$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\theta = -5\cos(42^\circ)\cos(-56.3^\circ) + 5\cos(42^\circ)\sin(-56.3^\circ) - 1\sin(42^\circ) = -5.82$$

$$\mathbf{R}_{PQ} \cdot \mathbf{a}_\phi = -(-5)\sin(-56.3^\circ) + 5\cos(-56.3^\circ) = -1.39$$

So finally,

$$\mathbf{R}_{PQ} = \underline{-3.90\mathbf{a}_r - 5.82\mathbf{a}_\theta - 1.39\mathbf{a}_\phi}$$

c) If $\mathbf{D} = 5\mathbf{a}_r - 3\mathbf{a}_\theta + 4\mathbf{a}_\phi$, find $\mathbf{D} \cdot \mathbf{a}_\rho$ at $M(1, 2, 3)$: First convert $\mathbf{a}_\rho$ to cartesian coordinates at the specified point. Use $\mathbf{a}_\rho = (\mathbf{a}_\rho \cdot \mathbf{a}_x)\mathbf{a}_x + (\mathbf{a}_\rho \cdot \mathbf{a}_y)\mathbf{a}_y$. At $A(1, 2, 3)$, $\rho = \sqrt{5}$, $\phi = \tan^{-1}(2) = 63.4^\circ$, $r = \sqrt{14}$, and $\theta = \cos^{-1}(3/\sqrt{14}) = 36.7^\circ$. So $\mathbf{a}_\rho = \cos(63.4^\circ)\mathbf{a}_x + \sin(63.4^\circ)\mathbf{a}_y = 0.45\mathbf{a}_x + 0.89\mathbf{a}_y$. Then

$$(5\mathbf{a}_r - 3\mathbf{a}_\theta + 4\mathbf{a}_\phi) \cdot (0.45\mathbf{a}_x + 0.89\mathbf{a}_y) =$$
$$5(0.45)\sin\theta\cos\phi \ + \ 5(0.89)\sin\theta\sin\phi \ - \ 3(0.45)\cos\theta\cos\phi$$
$$- \ 3(0.89)\cos\theta\sin\phi \ + \ 4(0.45)(-\sin\phi) \ + \ 4(0.89)\cos\phi = \underline{0.59}$$

1.29. Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

a) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad: Use

$$\mathbf{a}_x = (\mathbf{a}_x \cdot \mathbf{a}_r)\mathbf{a}_r + (\mathbf{a}_x \cdot \mathbf{a}_\theta)\mathbf{a}_\theta + (\mathbf{a}_x \cdot \mathbf{a}_\phi)\mathbf{a}_\phi =$$
$$\sin(1)\cos(0.8)\mathbf{a}_r + \cos(1)\cos(0.8)\mathbf{a}_\theta + (-\sin(0.8))\mathbf{a}_\phi = \underline{0.59\mathbf{a}_r + 0.38\mathbf{a}_\theta - 0.72\mathbf{a}_\phi}$$

1.29 (continued) Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

b) $x = 3$, $y = 2$, $z = -1$: First, transform the point to spherical coordinates. Have $r = \sqrt{14}$, $\theta = \cos^{-1}(-1/\sqrt{14}) = 105.5^\circ$, and $\phi = \tan^{-1}(2/3) = 33.7^\circ$. Then

$$\begin{aligned}\mathbf{a}_x &= \sin(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_r + \cos(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_\theta + (-\sin(33.7^\circ))\mathbf{a}_\phi \\ &= \underline{0.80\mathbf{a}_r - 0.22\mathbf{a}_\theta - 0.55\mathbf{a}_\phi}\end{aligned}$$

c) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$: Again, convert the point to spherical coordinates. $r = \sqrt{\rho^2 + z^2} = \sqrt{8.5}$, $\theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59.0^\circ$, and $\phi = 0.7\,\text{rad} = 40.1^\circ$. Now

$$\begin{aligned}\mathbf{a}_x &= \sin(59^\circ)\cos(40.1^\circ)\mathbf{a}_r + \cos(59^\circ)\cos(40.1^\circ)\mathbf{a}_\theta + (-\sin(40.1^\circ))\mathbf{a}_\phi \\ &= \underline{0.66\mathbf{a}_r + 0.39\mathbf{a}_\theta - 0.64\mathbf{a}_\phi}\end{aligned}$$

1.30. Given $A(r = 20, \theta = 30^\circ, \phi = 45^\circ)$ and $B(r = 30, \theta = 115^\circ, \phi = 160^\circ)$, find:

a) $|\mathbf{R}_{AB}|$: First convert A and B to cartesian: Have $x_A = 20\sin(30^\circ)\cos(45^\circ) = 7.07$, $y_A = 20\sin(30^\circ)\sin(45^\circ) = 7.07$, and $z_A = 20\cos(30^\circ) = 17.3$. $x_B = 30\sin(115^\circ)\cos(160^\circ) = -25.6$, $y_B = 30\sin(115^\circ)\sin(160^\circ) = 9.3$, and $z_B = 30\cos(115^\circ) = -12.7$. Now $\mathbf{R}_{AB} = \mathbf{R}_B - \mathbf{R}_A = (-32.6, 2.2, -30.0)$, and so $|\mathbf{R}_{AB}| = \underline{44.4}$.

b) $|\mathbf{R}_{AC}|$, given $C(r = 20, \theta = 90^\circ, \phi = 45^\circ)$. Again, converting C to cartesian, obtain $x_C = 20\sin(90^\circ)\cos(45^\circ) = 14.14$, $y_C = 20\sin(90^\circ)\sin(45^\circ) = 14.14$, and $z_C = 20\cos(90^\circ) = 0$. So $\mathbf{R}_{AC} = \mathbf{R}_C - \mathbf{R}_A = (7.07, 7.07, -17.3)$, and $|\mathbf{R}_{AC}| = \underline{20.0}$.

c) the distance from A to C on a great circle path: Note that A and C share the same r and ϕ coordinates; thus moving from A to C involves only a change in θ of 60°. The requested arc length is then

$$\text{distance} = 20 \times \left[60\left(\frac{2\pi}{360}\right)\right] = \underline{20.9}$$

CHAPTER 2

2.1. Four 10nC positive charges are located in the $z = 0$ plane at the corners of a square 8cm on a side. A fifth 10nC positive charge is located at a point 8cm distant from the other charges. Calculate the magnitude of the total force on this fifth charge for $\epsilon = \epsilon_0$:

Arrange the charges in the xy plane at locations (4,4), (4,-4), (-4,4), and (-4,-4). Then the fifth charge will be on the z axis at location $z = 4\sqrt{2}$, which puts it at 8cm distance from the other four. By symmetry, the force on the fifth charge will be z-directed, and will be four times the z component of force produced by each of the four other charges.

$$F = \frac{4}{\sqrt{2}} \times \frac{q^2}{4\pi\epsilon_0 d^2} = \frac{4}{\sqrt{2}} \times \frac{(10^{-8})^2}{4\pi(8.85 \times 10^{-12})(0.08)^2} = \underline{4.0 \times 10^{-4} \text{ N}}$$

2.2. A charge $Q_1 = 0.1\ \mu$C is located at the origin, while $Q_2 = 0.2\ \mu$C is at $A(0.8, -0.6, 0)$. Find the locus of points in the $z = 0$ plane at which the x component of the force on a third positive charge is zero.

To solve this problem, the z coordinate of the third charge is immaterial, so we can place it in the xy plane at coordinates $(x, y, 0)$. We take its magnitude to be Q_3. The vector directed from the first charge to the third is $\mathbf{R}_{13} = x\mathbf{a}_x + y\mathbf{a}_y$; the vector directed from the second charge to the third is $\mathbf{R}_{23} = (x - 0.8)\mathbf{a}_x + (y + 0.6)\mathbf{a}_y$. The force on the third charge is now

$$\begin{aligned}\mathbf{F}_3 &= \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{Q_2\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right] \\ &= \frac{Q_3 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{0.1(x\mathbf{a}_x + y\mathbf{a}_y)}{(x^2 + y^2)^{1.5}} + \frac{0.2[(x - 0.8)\mathbf{a}_x + (y + 0.6)\mathbf{a}_y]}{[(x - 0.8)^2 + (y + 0.6)^2]^{1.5}}\right]\end{aligned}$$

We desire the x component to be zero. Thus,

$$0 = \left[\frac{0.1x\mathbf{a}_x}{(x^2 + y^2)^{1.5}} + \frac{0.2(x - 0.8)\mathbf{a}_x}{[(x - 0.8)^2 + (y + 0.6)^2]^{1.5}}\right]$$

or

$$\underline{x[(x - 0.8)^2 + (y + 0.6)^2]^{1.5} = 2(0.8 - x)(x^2 + y^2)^{1.5}}$$

2.3. Point charges of 50nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3}\right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8}\right]\mathbf{a}_x = \underline{21.5\mathbf{a}_x\ \mu\text{N}}$$

where distances are in meters.

2.4. Let $Q_1 = 8\ \mu$C be located at $P_1(2, 5, 8)$ while $Q_2 = -5\ \mu$C is at $P_2(6, 15, 8)$. Let $\epsilon = \epsilon_0$.

a) Find $\mathbf{F}_2$, the force on Q_2: This force will be

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0}\frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|^3} = \frac{(8\times 10^{-6})(-5\times 10^{-6})}{4\pi\epsilon_0}\frac{(4\mathbf{a}_x + 10\mathbf{a}_y)}{(116)^{1.5}} = \underline{(-1.15\mathbf{a}_x - 2.88\mathbf{a}_y)\ \text{mN}}$$

b) Find the coordinates of P_3 if a charge Q_3 experiences a total force $\mathbf{F}_3 = 0$ at P_3: This force in general will be:

$$\mathbf{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{Q_2\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = (x-2)\mathbf{a}_x + (y-5)\mathbf{a}_y$ and $\mathbf{R}_{23} = (x-6)\mathbf{a}_x + (y-15)\mathbf{a}_y$. Note, however, that all three charges must lie in a straight line, and the location of Q_3 will be along the vector $\mathbf{R}_{12}$ extended past Q_2. The slope of this vector is $(15-5)/(6-2) = 2.5$. Therefore, we look for P_3 at coordinates $(x, 2.5x, 8)$. With this restriction, the force becomes:

$$\mathbf{F}_3 = \frac{Q_3}{4\pi\epsilon_0}\left[\frac{8[(x-2)\mathbf{a}_x + 2.5(x-2)\mathbf{a}_y]}{[(x-2)^2 + (2.5)^2(x-2)^2]^{1.5}} - \frac{5[(x-6)\mathbf{a}_x + 2.5(x-6)\mathbf{a}_y]}{[(x-6)^2 + (2.5)^2(x-6)^2]^{1.5}}\right]$$

where we require the term in large brackets to be zero. This leads to

$$8(x-2)[((2.5)^2+1)(x-6)^2]^{1.5} - 5(x-6)[((2.5)^2+1)(x-2)^2]^{1.5} = 0$$

which reduces to

$$8(x-6)^2 - 5(x-2)^2 = 0$$

or

$$x = \frac{6\sqrt{8} - 2\sqrt{5}}{\sqrt{8} - \sqrt{5}} = \underline{21.1}$$

The coordinates of P_3 are thus $\underline{P_3(21.1, 52.8, 8)}$

2.5. Let a point charge $Q_1 25$ nC be located at $P_1(4, -2, 7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3, 4, -2)$.

a) If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at $P_3(1, 2, 3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\times(-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60\times(4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}}\right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0, y, 0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of $\mathbf{E}$ at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\times(-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60\times 3}{[13 + (y-4)^2]^{1.5}}\right]$$

To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.6. Point charges of 120 nC are located at $A(0, 0, 1)$ and $B(0, 0, -1)$ in free space.

a) Find $\mathbf{E}$ at $P(0.5, 0, 0)$: This will be

$$\mathbf{E}_P = \frac{120 \times 10^{-9}}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} + \frac{\mathbf{R}_{BP}}{|\mathbf{R}_{BP}|^3}\right]$$

where $\mathbf{R}_{AP} = 0.5\mathbf{a}_x - \mathbf{a}_z$ and $\mathbf{R}_{BP} = 0.5\mathbf{a}_x + \mathbf{a}_z$. Also, $|\mathbf{R}_{AP}| = |\mathbf{R}_{BP}| = \sqrt{1.25}$. Thus:

$$\mathbf{E}_P = \frac{120 \times 10^{-9}\mathbf{a}_x}{4\pi(1.25)^{1.5}\epsilon_0} = \underline{772 \text{ V/m}}$$

b) What single charge at the origin would provide the identical field strength? We require

$$\frac{Q_0}{4\pi\epsilon_0(0.5)^2} = 772$$

from which we find $Q_0 = \underline{21.5 \text{ nC}}$.

2.7. A 2 μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$. Have

$$\mathbf{E}_P = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} = \frac{2 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{4\mathbf{a}_x + 9\mathbf{a}_y - 3\mathbf{a}_z}{(106)^{1.5}}\right] = 65.9\mathbf{a}_x + 148.3\mathbf{a}_y - 49.4\mathbf{a}_z$$

Then, at point P, $\rho = \sqrt{8^2 + 12^2} = 14.4$, $\phi = \tan^{-1}(12/8) = 56.3^\circ$, and $z = z$. Now,

$$E_\rho = \mathbf{E}_p \cdot \mathbf{a}_\rho = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\rho) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\rho) = 65.9\cos(56.3^\circ) + 148.3\sin(56.3^\circ) = \underline{159.7}$$

and

$$E_\phi = \mathbf{E}_p \cdot \mathbf{a}_\phi = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\phi) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\phi) = -65.9\sin(56.3^\circ) + 148.3\cos(56.3^\circ) = \underline{27.4}$$

Finally, $E_z = \underline{-49.4}$

2.8. Given point charges of -1 μC at $P_1(0, 0, 0.5)$ and $P_2(0, 0, -0.5)$, and a charge of 2 μC at the origin, find $\mathbf{E}$ at $P(0, 2, 1)$ in spherical components, assuming $\epsilon = \epsilon_0$.

The field will take the general form:

$$\mathbf{E}_P = \frac{10^{-6}}{4\pi\epsilon_0}\left[-\frac{\mathbf{R}_1}{|\mathbf{R}_1|^3} + \frac{2\mathbf{R}_2}{|\mathbf{R}_2|^3} - \frac{\mathbf{R}_3}{|\mathbf{R}_3|^3}\right]$$

where $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3$ are the vectors to P from each of the charges in their original listed order. Specifically, $\mathbf{R}_1 = (0, 2, 0.5)$, $\mathbf{R}_2 = (0, 2, 1)$, and $\mathbf{R}_3 = (0, 2, 1.5)$. The magnitudes are $|\mathbf{R}_1| = 2.06$, $|\mathbf{R}_2| = 2.24$, and $|\mathbf{R}_3| = 2.50$. Thus

$$\mathbf{E}_P = \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{-(0, 2, 0.5)}{(2.06)^3} + \frac{2(0, 2, 1)}{(2.24)^3} - \frac{(0, 2, 1.5)}{(2.50)^3}\right] = 89.9\mathbf{a}_y + 179.8\mathbf{a}_z$$

Now, at P, $r = \sqrt{5}$, $\theta = \cos^{-1}(1/\sqrt{5}) = 63.4^\circ$, and $\phi = 90^\circ$. So

$$E_r = \mathbf{E}_P \cdot \mathbf{a}_r = 89.9(\mathbf{a}_y \cdot \mathbf{a}_r) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_r) = 89.9\sin\theta\sin\phi + 179.8\cos\theta = \underline{160.9}$$

$$E_\theta = \mathbf{E}_P \cdot \mathbf{a}_\theta = 89.9(\mathbf{a}_y \cdot \mathbf{a}_\theta) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_\theta) = 89.9\cos\theta\sin\phi + 179.8(-\sin\theta) = \underline{-120.5}$$

$$E_\phi = \mathbf{E}_P \cdot \mathbf{a}_\phi = 89.9(\mathbf{a}_y \cdot \mathbf{a}_\phi) + 179.8(\mathbf{a}_z \cdot \mathbf{a}_\phi) = 89.9\cos\phi = \underline{0}$$

2.9. A 100 nC point charge is located at $A(-1, 1, 3)$ in free space.

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x+1)}{[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x+1) = 0.56[(x+1)^2 + (y-1)^2 + (z-3)^2]^{1.5}}$$

b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.10. Charges of 20 and -20 nC are located at $(3, 0, 0)$ and $(-3, 0, 0)$, respectively. Let $\epsilon = \epsilon_0$. Determine $|\mathbf{E}|$ at $P(0, y, 0)$: The field will be

$$\mathbf{E}_P = \frac{20 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{\mathbf{R}_1}{|\mathbf{R}_1|^3} - \frac{\mathbf{R}_2}{|\mathbf{R}_2|^3} \right]$$

where $\mathbf{R}_1$, the vector from the positive charge to point P is $(-3, y, 0)$, and $\mathbf{R}_2$, the vector from the negative charge to point P, is $(3, y, 0)$. The magnitudes of these vectors are $|\mathbf{R}_1| = |\mathbf{R}_2| = \sqrt{9 + y^2}$. Substituting these into the expression for $\mathbf{E}_P$ produces

$$\mathbf{E}_P = \frac{20 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{-6\mathbf{a}_x}{(9 + y^2)^{1.5}} \right]$$

from which

$$\underline{|\mathbf{E}_P| = \frac{1079}{(9 + y^2)^{1.5}} \text{ V/m}}$$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0} \left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}} \right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

2.11. (continued)

b) Find **E** at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1+36+25]^{1.5}}\right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find **E** at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1+36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54^\circ$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11\cos\phi - 180.63\sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63\cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find **E** at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1+36+25} = 7.87$, $\phi = 80.54^\circ$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58^\circ$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11\sin\theta\cos\phi - 180.63\sin\theta\sin\phi - 150.53\cos\theta = \underline{-237.1}$$

2.12. The volume charge density $\rho_v = \rho_0 e^{-|x|-|y|-|z|}$ exists over all free space. Calculate the total charge present: This will be 8 times the integral of ρ_v over the first octant, or

$$Q = 8\int_0^\infty \int_0^\infty \int_0^\infty \rho_0 e^{-x-y-z}\, dx\, dy\, dz = \underline{8\rho_0}$$

2.13. A uniform volume charge density of 0.2 μC/m^3 (note typo in book) is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{.03}^{.05} 0.2\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{.05} = 8.21 \times 10^{-5}\ \mu\text{C} = \underline{82.1\ \text{pC}}$$

b) find r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3}\right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3\right]^{1/3} = \underline{4.24\ \text{cm}}$$

2.14. Let

$$\rho_v = 5e^{-0.1\rho}\ (\pi - |\phi|)\ \frac{1}{z^2+10}\ \ \mu\text{C/m}^3$$

in the region $0 \le \rho \le 10$, $-\pi < \phi < \pi$, all z, and $\rho_v = 0$ elsewhere.

a) Determine the total charge present: This will be the integral of ρ_v over the region where it exists; specifically,

$$Q = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\int_{0}^{10} 5e^{-0.1\rho}\ (\pi - |\phi|)\ \frac{1}{z^2+10}\rho\, d\rho\, d\phi\, dz$$

which becomes

$$Q = 5\left[\frac{e^{-0.1\rho}}{(0.1)^2}(-0.1-1)\right]_0^{10}\int_{-\infty}^{\infty} 2\int_0^{\pi}\ (\pi-\phi)\ \frac{1}{z^2+10}d\phi\, dz$$

or

$$Q = 5\times 26.4\int_{-\infty}^{\infty}\pi^2\frac{1}{z^2+10}\,dz$$

Finally,

$$Q = 5\times 26.4\times\pi^2\left[\frac{1}{\sqrt{10}}\tan^{-1}\left(\frac{z}{\sqrt{10}}\right)\right]_{-\infty}^{\infty} = \frac{5(26.4)\pi^3}{\sqrt{10}} = 1.29\times 10^3\ \mu\text{C} = \underline{1.29\ \text{mC}}$$

b) Calculate the charge within the region $0 \le \rho \le 4$, $-\pi/2 < \phi < \pi/2$, $-10 < z < 10$: With the limits thus changed, the integral for the charge becomes:

$$Q' = \int_{-10}^{10} 2\int_0^{\pi/2}\int_0^4\ 5e^{-0.1\rho}\ (\pi-\phi)\ \frac{1}{z^2+10}\rho\, d\rho\, d\phi\, dz$$

Following the same evaulation procedure as in part a, we obtain $Q' = \underline{0.182\ \text{mC}}$.

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^{15} C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2\times 10^{-6})^3\times 10^{15} = \underline{3.35\times 10^{-2}\ \text{C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35\times 10^{-2}}{(0.003)^3} = \underline{1.24\times 10^6\ \text{C/m}^3}$$

2.16. The region in which $4 < r < 5$, $0 < \theta < 25^\circ$, and $0.9\pi < \phi < 1.1\pi$ contains the volume charge density of $\rho_v = 10(r-4)(r-5)\sin\theta\sin(\phi/2)$. Outside the region, $\rho_v = 0$. Find the charge within the region: The integral that gives the charge will be

$$Q = 10\int_{.9\pi}^{1.1\pi}\int_0^{25^\circ}\int_4^5 (r-4)(r-5)\sin\theta\sin(\phi/2)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

2.16. (continued) Carrying out the integral, we obtain

$$Q = 10\left[\frac{r^5}{5} - 9\frac{r^4}{4} + 20\frac{r^3}{3}\right]_4^5 \left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{25^\circ} \left[-2\cos\left(\frac{\theta}{2}\right)\right]_{.9\pi}^{1.1\pi}$$
$$= 10(-3.39)(.0266)(.626) = \underline{0.57}\ \text{C}$$

2.17. A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$:

a) Find $\mathbf{E}$ at $P(1, 2, 3)$: This will be

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_P}{|\mathbf{R}_P|^2}$$

where $\mathbf{R}_P = (1, 2, 3) - (1, -2, 5) = (0, 4, -2)$, and $|\mathbf{R}_P|^2 = 20$. So

$$\mathbf{E}_P = \frac{16 \times 10^{-9}}{2\pi\epsilon_0}\left[\frac{4\mathbf{a}_y - 2\mathbf{a}_z}{20}\right] = \underline{57.5\mathbf{a}_y - 28.8\mathbf{a}_z\ \text{V/m}}$$

b) Find $\mathbf{E}$ at that point in the $z = 0$ plane where the direction of $\mathbf{E}$ is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$: With $z = 0$, the general field will be

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(y+2)\mathbf{a}_y - 5\mathbf{a}_z}{(y+2)^2 + 25}\right]$$

We require $|E_z| = -|2E_y|$, so $2(y + 2) = 5$. Thus $y = 1/2$, and the field becomes:

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2.5\mathbf{a}_y - 5\mathbf{a}_z}{(2.5)^2 + 25}\right] = \underline{23\mathbf{a}_y - 46\mathbf{a}_z}$$

2.18. Uniform line charges of 0.4 μC/m and -0.4 μC/m are located in the $x = 0$ plane at $y = -0.6$ and $y = 0.6$ m respectively. Let $\epsilon = \epsilon_0$.

a) Find $\mathbf{E}$ at $P(x, 0, z)$: In general, we have

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{+P}}{|\mathbf{R}_{+P}|} - \frac{\mathbf{R}_{-P}}{|\mathbf{R}_{-P}|}\right]$$

where $\mathbf{R}_{+P}$ and $\mathbf{R}_{-P}$ are, respectively, the vectors directed from the positive and negative line charges to the point P, and these are normal to the z axis. We thus have $\mathbf{R}_{+P} = (x, 0, z) - (0, -.6, z) = (x, .6, 0)$, and $\mathbf{R}_{-P} = (x, 0, z) - (0, .6, z) = (x, -.6, 0)$. So

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{x\mathbf{a}_x + 0.6\mathbf{a}_y}{x^2 + (0.6)^2} - \frac{x\mathbf{a}_x - 0.6\mathbf{a}_y}{x^2 + (0.6)^2}\right] = \frac{0.4 \times 10^{-6}}{2\pi\epsilon_0}\left[\frac{1.2\mathbf{a}_y}{x^2 + 0.36}\right] = \underline{\frac{8.63\mathbf{a}_y}{x^2 + 0.36}\ \text{kV/m}}$$

2.18. (continued)

b) Find $\mathbf{E}$ at $Q(2, 3, 4)$: This field will in general be:

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{+Q}}{|\mathbf{R}_{+Q}|} - \frac{\mathbf{R}_{-Q}}{|\mathbf{R}_{-Q}|}\right]$$

where $\mathbf{R}_{+Q} = (2, 3, 4) - (0, -.6, 4) = (2, 3.6, 0)$, and $\mathbf{R}_{-Q} = (2, 3, 4) - (0, .6, 4) = (2, 2.4, 0)$. Thus

$$\mathbf{E}_Q = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2\mathbf{a}_x + 3.6\mathbf{a}_y}{2^2 + (3.6)^2} - \frac{2\mathbf{a}_x + 2.4\mathbf{a}_y}{2^2 + (2.4)^2}\right] = \underline{-625.8\mathbf{a}_x - 241.6\mathbf{a}_y \text{ V/m}}$$

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find $\mathbf{E}$ in cartesian coordinates at $P(1, 2, 3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2\times10^{-6})}{2\pi\epsilon_0}\,\frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y \text{ kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1, 2, 3) - (0, 0, 3) = (1, 2, 0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2\times10^{-6})}{4\pi\epsilon_0}\int_{-4}^{4}\frac{\mathbf{a}_x + 2\mathbf{a}_y + (3-z)\mathbf{a}_z}{[5 + (3-z)^2]^{1.5}}\,dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597\left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z-3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)}\right]_{-4}^{4} \text{ V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \text{ kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.20. Uniform line charges of 120 nC/m lie along the entire extent of the three coordinate axes. Assuming free space conditions, find $\mathbf{E}$ at $P(-3, 2, -1)$: Since all line charges are infinitely-long, we can write:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{\mathbf{R}_{xP}}{|\mathbf{R}_{xP}|^2} + \frac{\mathbf{R}_{yP}}{|\mathbf{R}_{yP}|^2} + \frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2}\right]$$

where $\mathbf{R}_{xP}$, $\mathbf{R}_{yP}$, and $\mathbf{R}_{zP}$ are the normal vectors from each of the three axes that terminate on point P. Specifically, $\mathbf{R}_{xP} = (-3, 2, -1) - (-3, 0, 0) = (0, 2, -1)$, $\mathbf{R}_{yP} = (-3, 2, -1) - (0, 2, 0) = (-3, 0, -1)$, and $\mathbf{R}_{zP} = (-3, 2, -1) - (0, 0, -1) = (-3, 2, 0)$. Substituting these into the expression for $\mathbf{E}_P$ gives

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{2\mathbf{a}_y - \mathbf{a}_z}{5} + \frac{-3\mathbf{a}_x - \mathbf{a}_z}{10} + \frac{-3\mathbf{a}_x + 2\mathbf{a}_y}{13}\right] = \underline{-1.15\mathbf{a}_x + 1.20\mathbf{a}_y - 0.65\mathbf{a}_z \text{ kV/m}}$$

2.21. Two identical uniform line charges with $\rho_l = 75$ nC/m are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other? The charges are parallel to the z axis and are separated by 0.8 m. Thus the field from the charge at $y = -0.4$ evaluated at the location of the charge at $y = +0.4$ will be $\mathbf{E} = [\rho_l/(2\pi\epsilon_0(0.8))]\mathbf{a}_y$. The force on a differential length of the line at the positive y location is $d\mathbf{F} = dq\mathbf{E} = \rho_l dz\mathbf{E}$. Thus the force per unit length acting on the line at postive y arising from the charge at negative y is

$$\mathbf{F} = \int_0^1 \frac{\rho_l^2\, dz}{2\pi\epsilon_0(0.8)}\,\mathbf{a}_y = 1.26\times 10^{-4}\,\mathbf{a}_y \text{ N/m} = \underline{126\,\mathbf{a}_y\ \mu\text{N/m}}$$

The force on the line at negative y is of course the same, but with $-\mathbf{a}_y$.

2.22. A uniform surface charge density of 5 nC/m^2 is present in the region $x = 0$, $-2 < y < 2$, and all z. If $\epsilon = \epsilon_0$, find $\mathbf{E}$ at:

a) $P_A(3, 0, 0)$: We use the superposition integral:

$$\mathbf{E} = \int\int \frac{\rho_s da}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

where $\mathbf{r} = 3\mathbf{a}_x$ and $\mathbf{r}' = y\mathbf{a}_y + z\mathbf{a}_z$. The integral becomes:

$$\mathbf{E}_{PA} = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-2}^{2}\frac{3\mathbf{a}_x - y\mathbf{a}_y - z\mathbf{a}_z}{[9+y^2+z^2]^{1.5}}\,dy\,dz$$

Since the integration limits are symmetric about the origin, and since the y and z components of the integrand exhibit odd parity (change sign when crossing the origin, but otherwise symmetric), these will integrate to zero, leaving only the x component. This is evident just from the symmetry of the problem. Performing the z integration first on the x component, we obtain (using tables):

$$E_{x,PA} = \frac{3\rho_s}{4\pi\epsilon_0}\int_{-2}^{2}\frac{dy}{(9+y^2)}\left[\frac{z}{\sqrt{9+y^2+z^2}}\right]_{-\infty}^{\infty} = \frac{3\rho_s}{2\pi\epsilon_0}\int_{-2}^{2}\frac{dy}{(9+y^2)}$$
$$= \frac{3\rho_s}{2\pi\epsilon_0}\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{y}{3}\right)\Big|_{-2}^{2} = \underline{106\text{ V/m}}$$

The student is encouraged to verify that if the y limits were $-\infty$ to ∞, the result would be that of the infinite charged plane, or $E_x = \rho_s/(2\epsilon_0)$.

b) $P_B(0, 3, 0)$: In this case, $\mathbf{r} = 3\mathbf{a}_y$, and symmetry indicates that only a y component will exist. The integral becomes

$$E_{y,PB} = \frac{\rho_s}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\int_{-2}^{2}\frac{(3-y)\,dy\,dz}{[(z^2+9)-6y+y^2]^{1.5}} = \frac{\rho_s}{2\pi\epsilon_0}\int_{-2}^{2}\frac{(3-y)\,dy}{(3-y)^2}$$
$$= -\frac{\rho_s}{2\pi\epsilon_0}\ln(3-y)\Big|_{-2}^{2} = \underline{145\text{ V/m}}$$

2.23. Given the surface charge density, $\rho_s = 2\,\mu\text{C/m}^2$, in the region $\rho < 0.2$ m, $z = 0$, and is zero elsewhere, find **E** at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of **E** will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of **E** will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.4}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1\text{ kV/m}}$.

b) With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1\text{ kV/m}}$.

2.24. Surface charge density is positioned in free space as follows: $20\,\text{nC/m}^2$ at $x = -3$, $-30\,\text{nC/m}^2$ at $y = 4$, and $40\,\text{nC/m}^2$ at $z = 2$. Find the magnitude of **E** at the three points, $(4, 3, -2)$, $(-2, 5, -1)$, and $(0, 0, 0)$. Since all three sheets are infinite, the field magnitude associated with each one will be $\rho_s/(2\epsilon_0)$, which is position-independent. For this reason, the *net* field magnitude will be the same everywhere, whereas the field direction will depend on which side of a given sheet one is positioned. We take the first point, for example, and find

$$\mathbf{E}_A = \frac{20\times10^{-9}}{2\epsilon_0}\mathbf{a}_x + \frac{30\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - \frac{40\times10^{-9}}{2\epsilon_0}\mathbf{a}_z = 1130\mathbf{a}_x + 1695\mathbf{a}_y - 2260\mathbf{a}_z\text{ V/m}$$

The magnitude of $\mathbf{E}_A$ is thus $\underline{3.04\,\text{kV/m}}$. This will be the magnitude at the other two points as well.

2.25. Find **E** at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2, 0, 6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, $0.2\,\text{nC/m}^2$ at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12\times10^{-9})}{4\pi\epsilon_0}\,\frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4+36)^{1.5}}\right] + \left[\frac{(3\times10^{-9})}{2\pi\epsilon_0}\,\frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4+9)}\right] - \left[\frac{(0.2\times10^{-9})\mathbf{a}_x}{2\epsilon_0}\right]$$
$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z\text{ V/m}}$$

2.26. A uniform line charge density of 5 nC/m is at $y = 0$, $z = 2$ m in free space, while -5 nC/m is located at $y = 0$, $z = -2$ m. A uniform surface charge density of $0.3\,\text{nC/m}^2$ is at $y = 0.2$ m, and $-0.3\,\text{nC/m}^2$ is at $y = -0.2$ m. Find $|\mathbf{E}|$ at the origin: Since each pair consists of equal and opposite charges, the effect at the origin is to double the field produce by one of each type. Taking the sum of the fields at the origin from the surface and line charges, respectively, we find:

$$\mathbf{E}(0,0,0) = -2\times\frac{0.3\times10^{-9}}{2\epsilon_0}\mathbf{a}_y - 2\times\frac{5\times10^{-9}}{2\pi\epsilon_0(2)}\mathbf{a}_z = -33.9\mathbf{a}_y - 89.9\mathbf{a}_z$$

so that $|\mathbf{E}| = \underline{96.1\,\text{V/m}}$.

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2, 3, -4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x + 4y)}{(4x - 2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2, 3, -4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28. Let $\mathbf{E} = 5x^3\,\mathbf{a}_x - 15x^2y\,\mathbf{a}_y$, and find:

a) the equation of the streamline that passes through $P(4, 2, 1)$: Write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-15x^2y}{5x^3} = \frac{-3y}{x}$$

So

$$\frac{dy}{y} = -3\,\frac{dx}{x} \quad \Rightarrow \quad \ln y = -3\ln x + \ln C$$

Thus

$$y = e^{-3\ln x}e^{\ln C} = \frac{C}{x^3}$$

At P, have $2 = C/(4)^3 \Rightarrow C = 128$. Finally, at P,

$$\underline{y = \frac{128}{x^3}}$$

2.28. (continued)

b) a unit vector $\mathbf{a}_E$ specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: At Q, $\mathbf{E}_Q = 135\mathbf{a}_x + 270\mathbf{a}_y$, and $|\mathbf{E}_Q| = 301.9$. Thus $\mathbf{a}_E = \underline{0.45\mathbf{a}_x + 0.89\mathbf{a}_y}$.

c) a unit vector $\mathbf{a}_N = (l, m, 0)$ that is perpendicular to $\mathbf{a}_E$ at Q: Since this vector is to have no z component, we can find it through $\mathbf{a}_N = \pm(\mathbf{a}_E \times \mathbf{a}_z)$. Performing this, we find $\mathbf{a}_N = \underline{\pm(0.89\mathbf{a}_x - 0.45\mathbf{a}_y)}$.

2.29. If $\mathbf{E} = 20e^{-5y}\left(\cos 5x\,\mathbf{a}_x - \sin 5x\,\mathbf{a}_y\right)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is just $\left(\cos 5x\,\mathbf{a}_x - \sin 5x\,\mathbf{a}_y\right)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad \Rightarrow \quad dy = -\tan 5x\, dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. Given the electric field intensity $\mathbf{E} = 400y\mathbf{a}_x + 400x\mathbf{a}_y$ V/m, find:

a) the equation of the streamline passing through the point $A(2, 1, -2)$: Write:

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{x}{y} \quad \Rightarrow \quad x\,dx = y\,dy$$

Thus $x^2 = y^2 + C$. Evaluating at A yields $C = 3$, so the equation becomes

$$\underline{\frac{x^2}{3} - \frac{y^2}{3} = 1}$$

b) the equation of the surface on which $|\mathbf{E}| = 800$ V/m: Have $|\mathbf{E}| = 400\sqrt{x^2 + y^2} = 800$. Thus $\underline{x^2 + y^2 = 4}$, or we have a circular-cylindrical surface, centered on the z axis, and of radius 2.

c) A sketch of the part a equation would yield a parabola, centered at the origin, whose axis is the positive x axis, and for which the slopes of the asymptotes are ± 1.

d) A sketch of the trace produced by the intersection of the surface of part b with the $z = 0$ plane would yield a circle centered at the origin, of radius 2.

2.31. In cylindrical coordinates with $\mathbf{E}(\rho, \phi) = E_\rho(\rho, \phi)\mathbf{a}_\rho + E_\phi(\rho, \phi)\mathbf{a}_\phi$, the differential equation describing the direction lines is $E_\rho/E_\phi = d\rho/(\rho d\phi)$ in any constant-z plane. Derive the equation of the line passing through the point $P(\rho = 4, \phi = 10^\circ, z = 2)$ in the field $\mathbf{E} = 2\rho^2 \cos 3\phi \mathbf{a}_\rho + 2\rho^2 \sin 3\phi \mathbf{a}_\phi$: Using the given information, we write

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \cot 3\phi$$

Thus

$$\frac{d\rho}{\rho} = \cot 3\phi \, d\phi \quad \Rightarrow \quad \ln \rho = \frac{1}{3} \ln \sin 3\phi + \ln C$$

or $\rho = C(\sin 3\phi)^{1/3}$. Evaluate this at P to obtain $C = 7.14$. Finally,

$$\underline{\rho^3 = 364 \sin 3\phi}$$

CHAPTER 3

3.1. An empty metal paint can is placed on a marble table, the lid is removed, and both parts are discharged (honorably) by touching them to ground. An insulating nylon thread is glued to the center of the lid, and a penny, a nickel, and a dime are glued to the thread so that they are not touching each other. The penny is given a charge of +5 nC, and the nickel and dime are discharged. The assembly is lowered into the can so that the coins hang clear of all walls, and the lid is secured. The outside of the can is again touched momentarily to ground. The device is carefully disassembled with insulating gloves and tools.

a) What charges are found on each of the five metallic pieces? All coins were insulated during the entire procedure, so they will retain their original charges: Penny: $\underline{+5}$ nC; nickel: $\underline{0}$; dime: $\underline{0}$. The penny's charge will have induced an equal and opposite negative charge (-5 nC) on the inside wall of the can and lid. This left a charge layer of +5 nC on the outside surface which was neutralized by the ground connection. Therefore, the can retained a net charge of $\underline{-5\,\text{nC}}$ after disassembly.

b) If the penny had been given a charge of $+5$ nC, the dime a charge of -2 nC, and the nickel a charge of -1 nC, what would the final charge arrangement have been? Again, since the coins are insulated, they retain their original charges. The charge induced on the inside wall of the can and lid is equal to negative the sum of the coin charges, or -2 nC. This is the charge that the can/lid contraption retains after grounding and disassembly.

3.2. A point charge of 12 nC is located at the origin. four uniform line charges are located in the $x = 0$ plane as follows: 80 nC/m at $y = -1$ and -5 m, -50 nC/m at $y = -2$ and -4 m.

a) Find **D** at $P(0, -3, 2)$: Note that this point lies in the center of a symmetric arrangement of line charges, whose fields will all cancel at that point. Thus **D** arise from the point charge alone, and will be

$$\mathbf{D} = \frac{12 \times 10^{-9}(-3\mathbf{a}_y + 2\mathbf{a}_z)}{4\pi(3^2 + 2^2)^{1.5}} = -6.11 \times 10^{-11}\mathbf{a}_y + 4.07 \times 10^{-11}\mathbf{a}_z \ \text{C/m}^2$$
$$= \underline{-61.1\mathbf{a}_y + 40.7\mathbf{a}_z \ \text{pC/m}^2}$$

b) How much electric flux crosses the plane $y = -3$ and in what direction? The plane intercepts all flux that enters the $-y$ half-space, or exactly half the total flux of 12 nC. The answer is thus $\underline{6\ \text{nC}}$ and in the $-\mathbf{a}_y$ direction.

c) How much electric flux leaves the surface of a sphere, 4m in radius, centered at $C(0, -3, 0)$? This sphere encloses the point charge, so its flux of 12 nC is included. The line charge contributions are most easily found by translating the whole assembly (sphere and line charges) such that the sphere is centered at the origin, with line charges now at $y = \pm 1$ and ± 2. The flux from the line charges will equal the total line charge that lies within the sphere. The length of each of the inner two line charges (at $y = \pm 1$) will be

$$h_1 = 2r\cos\theta_1 = 2(4)\cos\left[\sin^{-1}\left(\frac{1}{4}\right)\right] = 1.94\,\text{m}$$

That of each of the outer two line charges (at $y = \pm 2$) will be

$$h_2 = 2r\cos\theta_2 = 2(4)\cos\left[\sin^{-1}\left(\frac{2}{4}\right)\right] = 1.73\,\text{m}$$

3.2c. (continued) The total charge enclosed in the sphere (and the outward flux from it) is now

$$Q_l + Q_p = 2(1.94)(-50 \times 10^{-9}) + 2(1.73)(80 \times 10^{-9}) + 12 \times 10^{-9} = \underline{348\,\text{nC}}$$

3.3. The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m^2.

a) What is the total amount of charge present? We integrate over the surface to find:

$$Q = 2\int_0^{\infty}\int_0^{2\pi} 5e^{-20z}(.08)d\phi\,dz\ \text{nC} = 20\pi(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_0^{\infty} = \underline{0.25\,\text{nC}}$$

b) How much flux leaves the surface $\rho = 8$ cm, 1 cm $< z <$ 5cm, $30^\circ < \phi < 90^\circ$? We just integrate the charge density on that surface to find the flux that leaves it.

$$\Phi = Q' = \int_{.01}^{.05}\int_{30^\circ}^{90^\circ} 5e^{-20z}(.08)\,d\phi\,dz\ \text{nC} = \left(\frac{90-30}{360}\right)2\pi(5)(.08)\left(\frac{-1}{20}\right)e^{-20z}\Big|_{.01}^{.05}$$
$$= 9.45 \times 10^{-3}\,\text{nC} = \underline{9.45\,\text{pC}}$$

3.4. The cylindrical surfaces $\rho = 1, 2,$ and 3 cm carry uniform surface charge densities of 20, -8, and 5 nC/m^2, respectively.

a) How much electric flux passes through the closed surface $\rho = 5$ cm, $0 < z < 1$ m? Since the densities are uniform, the flux will be

$$\Phi = 2\pi(a\rho_{s1} + b\rho_{s2} + c\rho_{s3})(1\,\text{m}) = 2\pi\,[(.01)(20) - (.02)(8) + (.03)(5)] \times 10^{-9} = \underline{1.2\,\text{nC}}$$

b) Find $\mathbf{D}$ at $P(1\,\text{cm}, 2\,\text{cm}, 3\,\text{cm})$: This point lies at radius $\sqrt{5}$ cm, and is thus inside the outermost charge layer. This layer, being of uniform density, will not contribute to $\mathbf{D}$ at P. We know that in cylindrical coordinates, the layers at 1 and 2 cm will produce the flux density:

$$\mathbf{D} = D_\rho \mathbf{a}_\rho = \frac{a\rho_{s1} + b\rho_{s2}}{\rho}\mathbf{a}_\rho$$

or

$$D_\rho = \frac{(.01)(20) + (.02)(-8)}{\sqrt{.05}} = 1.8\,\text{nC/m}^2$$

At P, $\phi = \tan^{-1}(2/1) = 63.4^\circ$. Thus $D_x = 1.8\cos\phi = 0.8$ and $D_y = 1.8\sin\phi = 1.6$. Finally,

$$\mathbf{D}_P = \underline{(0.8\mathbf{a}_x + 1.6\mathbf{a}_y)\ \text{nC/m}^2}$$

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2+z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\Phi = \int_0^5 \int_0^3 \mathbf{D}\big|_{x=2} \cdot \mathbf{a}_x \, dy \, dz + \int_0^3 \int_0^2 \mathbf{D}\big|_{z=5} \cdot \mathbf{a}_z \, dx \, dy$$
$$= 5\int_0^3 4(2)y \, dy \; + \; 2\int_0^3 4(5)y \, dy = \underline{360 \text{ C}}$$

3.6. Two uniform line charges, each 20 nC/m, are located at $y = 1$, $z = \pm 1$ m. Find the total flux leaving a sphere of radius 2 m if it is centered at

a) $A(3, 1, 0)$: The result will be the same if we move the sphere to the origin and the line charges to $(0, 0, \pm 1)$. The length of the line charge within the sphere is given by $l = 4\sin[\cos^{-1}(1/2)] = 3.46$. With two line charges, symmetrically arranged, the total charge enclosed is given by $Q = 2(3.46)(20\,\text{nC/m}) = \underline{139\,\text{nC}}$

b) $B(3, 2, 0)$: In this case the result will be the same if we move the sphere to the origin and keep the charges where they were. The length of the line joining the origin to the midpoint of the line charge (in the yz plane) is $l_1 = \sqrt{2}$. The length of the line joining the origin to either endpoint of the line charge is then just the sphere radius, or 2. The half-angle subtended at the origin by the line charge is then $\psi = \cos^{-1}(\sqrt{2}/2) = 45^\circ$. The length of each line charge in the sphere is then $l_2 = 2 \times 2\sin\psi = 2\sqrt{2}$. The total charge enclosed (with two line charges) is now $Q' = 2(2\sqrt{2})(20\,\text{nC/m}) = \underline{113\,\text{nC}}$

3.7. Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere.

a) Find the total charge enclosed by the spherical surface $r = 1$ mm: To find the charge we integrate:

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.001} 2e^{-1000r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Integration over the angles gives a factor of 4π. The radial integration we evaluate using tables; we obtain

$$Q = 8\pi \left[\frac{-r^2 e^{-1000r}}{1000}\Big|_0^{.001} + \frac{2}{1000}\frac{e^{-1000r}}{(1000)^2}(-1000r - 1)\Big|_0^{.001} \right] = \underline{4.0 \times 10^{-9}\,\text{nC}}$$

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm: The gaussian surface is a spherical shell of radius 1 mm. The enclosed charge is the result of part a. We thus write $4\pi r^2 D_r = Q$, or

$$D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(.001)^2} = \underline{3.2 \times 10^{-4}\,\text{nC/m}^2}$$

3.8. Uniform line charges of 5 nC/m ar located in free space at $x = 1$, $z = 1$, and at $y = 1$, $z = 0$.

a) Obtain an expression for $\mathbf{D}$ in cartesian coordinates at $P(0, 0, z)$. In general, we have

$$\mathbf{D}(z) = \frac{\rho_s}{2\pi}\left[\frac{\mathbf{r}_1 - \mathbf{r}_1'}{|\mathbf{r}_1 - \mathbf{r}_1'|^2} + \frac{\mathbf{r}_2 - \mathbf{r}_2'}{|\mathbf{r}_2 - \mathbf{r}_2'|^2}\right]$$

where $\mathbf{r}_1 = \mathbf{r}_2 = z\mathbf{a}_z$, $\mathbf{r}_1' = \mathbf{a}_y$, and $\mathbf{r}_2' = \mathbf{a}_x + \mathbf{a}_z$. Thus

$$\begin{aligned}\mathbf{D}(z) &= \frac{\rho_s}{2\pi}\left[\frac{[z\mathbf{a}_z - \mathbf{a}_y]}{[1+z^2]} + \frac{[(z-1)\mathbf{a}_z - \mathbf{a}_x]}{[1+(z-1)^2]}\right] \\ &= \underline{\frac{\rho_s}{2\pi}\left[\frac{-\mathbf{a}_x}{[1+(z-1)^2]} - \frac{\mathbf{a}_y}{[1+z^2]} + \left(\frac{(z-1)}{[1+(z-1)^2]} + \frac{z}{[1+z^2]}\right)\mathbf{a}_z\right]}\end{aligned}$$

b) Plot $|\mathbf{D}|$ vs. z at P, $-3 < z < 10$: Using part a, we find the magnitude of $\mathbf{D}$ to be

$$|\mathbf{D}| = \frac{\rho_s}{2\pi}\left[\frac{1}{[1+(z-1)^2]^2} + \frac{1}{[1+z^2]^2} + \left(\frac{(z-1)}{[1+(z-1)^2]} + \frac{z}{[1+z^2]}\right)^2\right]^{1/2}$$

A plot of this over the specified range is shown in Prob3.8.pdf.

3.9. A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\,\text{mm} < r < 10\,\text{mm}$. Let $\rho_v = 0$ for $0 < r < 8\,\text{mm}$.

a) Find the total charge inside the spherical surface $r = 10$ mm: This will be

$$\begin{aligned}Q &= \int_0^{2\pi}\int_0^{\pi}\int_{.008}^{.010}(80\times10^{-6})r^2\sin\theta\,dr\,d\theta\,d\phi = 4\pi\times(80\times10^{-6})\frac{r^3}{3}\Big|_{.008}^{.010} \\ &= 1.64\times10^{-10}\,\text{C} = \underline{164\,\text{pC}}\end{aligned}$$

b) Find D_r at $r = 10$ mm: Using a spherical gaussian surface at $r = 10$, Gauss' law is written as $4\pi r^2 D_r = Q = 164\times10^{-12}$, or

$$D_r(10\,\text{mm}) = \frac{164\times10^{-12}}{4\pi(.01)^2} = 1.30\times10^{-7}\,\text{C/m}^2 = \underline{130\,\text{nC/m}^2}$$

c) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm: This will be the same computation as in part b, except the gaussian surface now lies at 20 mm. Thus

$$D_r(20\,\text{mm}) = \frac{164\times10^{-12}}{4\pi(.02)^2} = 3.25\times10^{-8}\,\text{C/m}^2 = \underline{32.5\,\text{nC/m}^2}$$

3.10. Let $\rho_s = 8\,\mu\text{C/m}^2$ in the region where $x = 0$ and $-4 < z < 4$ m, and let $\rho_s = 0$ elsewhere. Find $\mathbf{D}$ at $P(x, 0, z)$, where $x > 0$: The sheet charge can be thought of as an assembly of infinitely-long parallel strips that lie parallel to the y axis in the yz plane, and where each is of thickness dz. The field from each strip is that of an infinite line charge, and so we can construct the field at P from a single strip as:

$$d\mathbf{D}_P = \frac{\rho_s\,dz'}{2\pi}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2}$$

3.10 (continued) where $\mathbf{r} = x\mathbf{a}_x + z\mathbf{a}_z$ and $\mathbf{r}' = z'\mathbf{a}_z$ We distinguish between the fixed coordinate of P, z, and the variable coordinate, z', that determines the location of each charge strip. To find the net field at P, we sum the contributions of each strip by integrating over z':

$$\mathbf{D}_P = \int_{-4}^{4} \frac{8\times 10^{-6}\, dz'\,(x\mathbf{a}_x + (z-z')\mathbf{a}_z)}{2\pi[x^2+(z-z')^2]}$$

We can re-arrange this to determine the integral forms:

$$\mathbf{D}_P = \frac{8\times 10^{-6}}{2\pi}\left[(x\mathbf{a}_x + z\mathbf{a}_z)\int_{-4}^{4}\frac{dz'}{(x^2+z^2)-2zz'+(z')^2} - \mathbf{a}_z\int_{-4}^{4}\frac{z'\,dz'}{(x^2+z^2)-2zz'+(z')^2}\right]$$

Using integral tables, we find

$$\mathbf{D}_P = \frac{4\times 10^{-6}}{\pi}\left[(x\mathbf{a}_x + z\mathbf{a}_z)\frac{1}{x}\tan^{-1}\left(\frac{2z'-2z}{2x}\right)\right.$$
$$\left. - \left[\frac{1}{2}\ln(x^2+z^2-2zz'+(z')^2) + \frac{2z}{2}\frac{1}{x}\tan^{-1}\left(\frac{2z'-2z}{2x}\right)\right]\mathbf{a}_z\right]_{-4}^{4}$$

which evaluates as

$$\mathbf{D}_P = \frac{4\times 10^{-6}}{\pi}\left\{\left[\tan^{-1}\left(\frac{z+4}{x}\right) - \tan^{-1}\left(\frac{z-4}{x}\right)\right]\mathbf{a}_x + \frac{1}{2}\ln\left[\frac{x^2+(z+4)^2}{x^2+(z-4)^2}\right]\mathbf{a}_z\right\}\ \text{C/m}^2$$

The student is invited to verify that for very small x or for a very large sheet (allowing z' to approach infinity), the above expression reduces to the expected form, $\mathbf{D}_P = \rho_s/2$. Note also that the expression is valid for all x (positive or negative values).

3.11. In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for 1 mm $< \rho <$ 1.5 mm, and $\rho_v = 0$ for $\rho > 1.5$ mm. Find $\mathbf{D}$ everywhere: Since the charge varies only with radius, and is in the form of a cylinder, symmetry tells us that the flux density will be radially-directed and will be constant over a cylindrical surface of a fixed radius. Gauss' law applied to such a surface of unit length in z gives:

a) for $\rho < 1$ mm, $D_\rho = 0$, since no charge is enclosed by a cylindrical surface whose radius lies within this range.

b) for 1 mm $< \rho <$ 1.5 mm, we have

$$2\pi\rho D_\rho = 2\pi\int_{.001}^{\rho} 2\times 10^{-9}\sin(2000\pi\rho')\rho'\,d\rho'$$
$$= 4\pi\times 10^{-9}\left[\frac{1}{(2000\pi)^2}\sin(2000\pi\rho) - \frac{\rho}{2000\pi}\cos(2000\pi\rho)\right]_{.001}^{\rho}$$

or finally,

$$D_\rho = \frac{10^{-15}}{2\pi^2\rho}\Big[\sin(2000\pi\rho) + 2\pi\left[1 - 10^3\rho\cos(2000\pi\rho)\right]\Big]\ \text{C/m}^2 \quad (1\,\text{mm} < \rho < 1.5\,\text{mm})$$

3.11. (continued)

c) for $\rho > 1.5\,\text{mm}$, the gaussian cylinder now lies at radius ρ *outside* the charge distribution, so the integral that evaluates the enclosed charge now includes the entire charge distribution. To accomplish this, we change the upper limit of the integral of part b from ρ to 1.5 mm, finally obtaining:

$$D_\rho = \frac{2.5 \times 10^{-15}}{\pi\rho}\ \text{C/m}^2 \quad (\rho > 1.5\,\text{mm})$$

3.12. A nonuniform volume charge density, $\rho_v = 120r\ \text{C/m}^3$, lies within the spherical surface $r = 1\,\text{m}$, and $\rho_v = 0$ everywhere else.

a) Find D_r everywhere. For $r < 1$ m, we apply Gauss' law to a spherical surface of radius r within this range to find

$$4\pi r^2 D_r = 4\pi \int_0^r 120r'(r')^2\,dr' = 120\pi r^4$$

Thus $D_r = \underline{(30r^2)}$ for $r < 1$ m. For $r > 1$ m, the gaussian surface lies outside the charge distribution. The set up is the same, except the upper limit of the above integral is 1 instead of r. This results in $D_r = \underline{(30/r^2)}$ for $r > 1$ m.

b) What surface charge density, ρ_{s2}, should be on the surface $r = 2$ such that $D_{r,r=2-} = 2D_{r,r=2+}$? At $r = 2^-$, we have $D_{r,r=2-} = 30/2^2 = 15/2$, from part a. The flux density in the region $r > 2$ arising from a surface charge at $r = 2$ is found from Gauss' law through

$$4\pi r^2 D_{rs} = 4\pi(2)^2\rho_{s2} \ \Rightarrow\ D_{rs} = \frac{4\rho_{s2}}{r^2}$$

The total flux density in the region $r > 2$ arising from the two distributions is

$$D_{rT} = \frac{30}{r^2} + \frac{4\rho_{s2}}{r^2}$$

Our requirement that $D_{r,r=2-} = 2D_{r,r=2+}$ becomes

$$\frac{30}{2^2} = 2\left(\frac{30}{2^2} + \rho_{s2}\right) \ \Rightarrow\ \rho_{s2} = \underline{-\frac{15}{4}\ \text{C/m}^2}$$

c) Make a sketch of D_r vs. r for $0 < r < 5$ m with both distributions present. With both charges, $D_r(r < 1) = 30r^2$, $D_r(1 < r < 2) = 30/r^2$, and $D_r(r > 2) = 15/r^2$. These are plotted on the next page.

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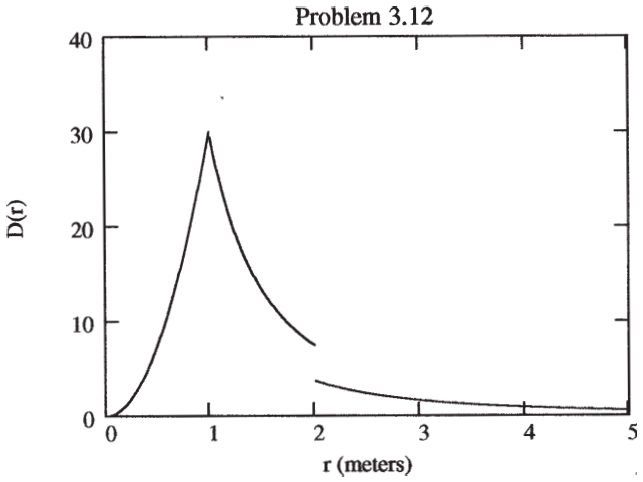


3.13. Spherical surfaces at $r = 2,\ 4,$ and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{s0}, respectively.

a) Find **D** at $r = 1,\ 3$ and 5 m: Noting that the charges are spherically-symmetric, we ascertain that **D** will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{80 \times 10^{-9}}{r^2}\ \text{C/m}^2$$

So $D_r(r = 3) = \underline{8.9 \times 10^{-9}\ \text{C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) + 4\pi(4)^2(-4 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{16 \times 10^{-9}}{r^2}$$

So $D_r(r = 5) = \underline{6.4 \times 10^{-10}\ \text{C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16 \times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9) \times 10^{-9}\ \text{C/m}^2}$.

3.14. If $\rho_v = 5$ nC/m^3 for $0 < \rho < 1$ mm and no other charges are present:

a) find D_ρ for $\rho < 1$ mm: Applying Gauss' law to a cylindrical surface of unit length in z, and of radius $\rho < 1$ mm, we find

$$2\pi\rho D_\rho = \pi\rho^2(5 \times 10^{-9}) \quad \Rightarrow \quad D_\rho = \underline{2.5\,\rho \times 10^{-9}\ \text{C/m}^2}$$

3.14b. find D_ρ for $\rho > 1$ mm: The Gaussian cylinder now lies outside the charge, so

$$2\pi\rho D_\rho = \pi(.001)^2(5\times10^{-9}) \;\Rightarrow\; D_\rho = \underline{\frac{2.5\times10^{-15}}{\rho}\,\text{C/m}^2}$$

c) What line charge ρ_L at $\rho = 0$ would give the same result for part b? The line charge field will be

$$D_r = \frac{\rho_L}{2\pi\rho} = \frac{2.5\times10^{-15}}{\rho} \;\;(\text{part } b)$$

Thus $\rho_L = \underline{5\pi\times10^{-15}\,\text{C/m}}$. In all answers, ρ is expressed in meters.

3.15. Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\ \mu\text{C/m}^3$ for $1 < \rho < 2$ mm.

a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm: We find

$$Q = \int_0^L \int_0^{2\pi} \int_{.001}^{\rho_1} 4\rho\ \rho\, d\rho\, d\phi\, dz = \underline{\frac{8\pi L}{3}[\rho_1^3 - 10^{-9}]\ \mu\text{C}}$$

where ρ_1 is in meters.

b) Use Gauss' law to determine D_ρ at $\rho = \rho_1$: Gauss' law states that $2\pi\rho_1 L D_\rho = Q$, where Q is the result of part a. Thus

$$D_\rho(\rho_1) = \underline{\frac{4(\rho_1^3 - 10^{-9})}{3\rho_1}\ \mu\text{C/m}^2}$$

where ρ_1 is in meters.

c) Evaluate D_ρ at $\rho = 0.8\,\text{mm},\ 1.6\,\text{mm},$ and $2.4\,\text{mm}$: At $\rho = 0.8$ mm, no charge is enclosed by a cylindrical gaussian surface of that radius, so $D_\rho(0.8\text{mm}) = \underline{0}$. At $\rho = 1.6$ mm, we evaluate the part b result at $\rho_1 = 1.6$ to obtain:

$$D_\rho(1.6\text{mm}) = \frac{4[(.0016)^3 - (.0010)^3]}{3(.0016)} = \underline{3.6\times10^{-6}\ \mu\text{C/m}^2}$$

At $\rho = 2.4$, we evaluate the charge integral of part a from .001 to .002, and Gauss' law is written as

$$2\pi\rho L D_\rho = \frac{8\pi L}{3}[(.002)^2 - (.001)^2]\ \mu\text{C}$$

from which $D_\rho(2.4\text{mm}) = \underline{3.9\times10^{-6}\ \mu\text{C/m}^2}$.

3.16. Given the electric flux density, $\mathbf{D} = 2xy\,\mathbf{a}_x + x^2\,\mathbf{a}_y + 6z^3\,\mathbf{a}_z\ \text{C/m}^2$:

a) use Gauss' law to evaluate the total charge enclosed in the volume $0 < x, y, z < a$: We call the surfaces at $x = a$ and $x = 0$ the front and back surfaces respectively, those at $y = a$ and $y = 0$ the right and left surfaces, and those at $z = a$ and $z = 0$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D}\cdot\mathbf{n}$ over all six surfaces and sum the results:

$$\Phi = Q = \oint \mathbf{D}\cdot\mathbf{n}\,da = \underbrace{\int_0^a\int_0^a 2ay\,dy\,dz}_{\text{front}} + \underbrace{\int_0^a\int_0^a -2(0)y\,dy\,dz}_{\text{back}}$$
$$+ \underbrace{\int_0^a\int_0^a -x^2\,dx\,dz}_{\text{left}} + \underbrace{\int_0^a\int_0^a x^2\,dx\,dz}_{\text{right}} + \underbrace{\int_0^a\int_0^a -6(0)^3\,dx\,dy}_{\text{bottom}} + \underbrace{\int_0^a\int_0^a 6a^3\,dx\,dy}_{\text{top}}$$

3.16a. (continued) Noting that the back and bottom integrals are zero, and that the left and right integrals cancel, we evaluate the remaining two (front and top) to obtain $Q = \underline{6a^5 + a^4}$.

b) use Eq. (8) to find an approximate value for the above charge. Evaluate the derivatives at $P(a/2, a/2, a/2)$: In this application, Eq. (8) states that $Q \doteq (\nabla \cdot \mathbf{D}\big|_P)\Delta v$. We find $\nabla \cdot \mathbf{D} = 2x + 18z^2$, which when evaluated at P becomes $\nabla \cdot \mathbf{D}\big|_P = a + 4.5a^2$. Thus $Q \doteq (a + 4.5a^2)a^3 = \underline{4.5a^5 + a^4}$

c) Show that the results of parts a and b agree in the limit as $a \to 0$. In this limit, both expressions reduce to $Q = a^4$, and so they agree.

3.17. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y$ C/m^2:

a) apply Gauss' law to find the total flux leaving the closed surface of the cube. We call the surfaces at $x = 1.2$ and $x = 1$ the front and back surfaces respectively, those at $y = 1.2$ and $y = 1$ the right and left surfaces, and those at $z = 1.2$ and $z = 1$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. We note that there is no z component of $\mathbf{D}$, so there will be no outward flux contributions from the top and bottom surfaces. The fluxes through the remaining four are

$$\Phi = Q = \oint \mathbf{D} \cdot \mathbf{n}\, da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\, dy\, dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\, dy\, dz}_{\text{back}}$$
$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\, dx\, dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\, dx\, dz}_{\text{right}} = \underline{0.1028\,\text{C}}$$

b) evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube: This is

$$\nabla \cdot \mathbf{D} = \Big[4xy + 6x^2y\Big]_{(1.1,1.1)} = 4(1.1)^2 + 6(1.1)^3 = \underline{12.83}$$

c) Estimate the total charge enclosed within the cube by using Eq. (8): This is

$$Q \doteq \nabla \cdot \mathbf{D}\big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026}\ \text{Close!}$$

3.18. Let a vector field by given by $\mathbf{G} = 5x^4y^4z^4\,\mathbf{a}_y$. Evaluate both sides of Eq. (8) for this $\mathbf{G}$ field and the volume defined by $x = 3$ and 3.1, $y = 1$ and 1.1, and $z = 2$ and 2.1. Evaluate the partial derivatives at the center of the volume. First find

$$\nabla \cdot \mathbf{G} = \frac{\partial G_y}{\partial y} = 20x^4y^3z^4$$

The center of the cube is located at (3.05,1.05,2.05), and the volume is $\Delta v = (0.1)^3 = 0.001$. Eq. (8) then becomes

$$\Phi \doteq 20(3.05)^4(1.05)^3(2.05)^4(0.001) = \underline{35.4}$$

3.19. A spherical surface of radius 3 mm is centered at $P(4, 1, 5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use $\Phi \doteq \nabla \cdot \mathbf{D} \Delta v$, where in this case $\nabla \cdot \mathbf{D} = (\partial/\partial x)x = 1$ C/m^3. Thus

$$\Phi \doteq \frac{4}{3}\pi(.003)^3(1) = 1.13 \times 10^{-7}\,\text{C} = \underline{113\,\text{nC}}$$

3.20. A cube of volume a^3 has its faces parallel to the cartesian coordinate surfaces. It is centered at $P(3, -2, 4)$. Given the field $\mathbf{D} = 2x^3\mathbf{a}_x$ C/m^2:

a) calculate div $\mathbf{D}$ at P: In the present case, this will be

$$\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} = \frac{dD_x}{dx} = \underline{54\,\text{C/m}^3}$$

b) evaluate the fraction in the rightmost side of Eq. (13) for $a = 1$ m, 0.1 m, and 1 mm: With the field having only an x component, flux will pentrate only the two surfaces at $x = 3 \pm a/2$, each of which has surface area a^2. The cube volume is $\Delta v = a^3$. The equation reads:

$$\frac{\oint \mathbf{D} \cdot d\mathbf{S}}{\Delta v} = \frac{1}{a^3}\left[2\left(3+\frac{a}{2}\right)^3 a^2 - 2\left(3-\frac{a}{2}\right)^3 a^2\right] = \frac{2}{a}\left[(3+\frac{a}{2})^3 - (3-\frac{a}{2})^3\right]$$

evaluating the above formula at $a = 1$ m, .1 m, and 1 mm, yields respectively

$$\underline{54.50,\ 54.01,\ \text{and } 54.00\ \text{C/m}^3},$$

thus demonstrating the approach to the exact value as Δv gets smaller.

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2z\,\mathbf{a}_y + (2z^3 - 5x^2y)\,\mathbf{a}_z\right]$ at $P(-2, 3, 5)$: We find

$$\nabla \cdot \mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3, -45^\circ, 5)$: In cylindrical coordinates, we have

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45^\circ,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3, 45^\circ, -45^\circ)$: In spherical coordinates, we have

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} \\ &= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta\sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45^\circ,-45^\circ)} = \underline{-2}\end{aligned}$$

3.22. Let $\mathbf{D} = 8\rho \sin\phi\, \mathbf{a}_\rho + 4\rho\cos\phi\, \mathbf{a}_\phi$ C/m^2.

a) Find div $\mathbf{D}$: Using the divergence formula for cylindrical coordinates (see problem 3.21), we find $\nabla \cdot \mathbf{D} = \underline{12\sin\phi}$.

b) Find the volume charge density at $P(2.6, 38^\circ, -6.1)$: Since $\rho_v = \nabla \cdot \mathbf{D}$, we evaluate the result of part a at this point to find $\rho_{vP} = 12\sin 38^\circ = \underline{7.39\ \text{C/m}^3}$.

c) How much charge is located inside the region defined by $0 < \rho < 1.8$, $20^\circ < \phi < 70^\circ$, $2.4 < z < 3.1$? We use

$$Q = \int_{vol} \rho_v dv = \int_{2.4}^{3.1}\int_{20^\circ}^{70^\circ}\int_0^{1.8} 12\sin\phi\rho\, d\rho\, d\phi\, dz = -(3.1-2.4)12\cos\phi\Big|_{20^\circ}^{70^\circ}\frac{\rho^2}{2}\Big|_0^{1.8}$$
$$= \underline{8.13\ \text{C}}$$

3.23. a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. For a point charge at the origin we know that $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$. Using the formula for divergence in spherical coordinates (see problem 3.21 solution), we find in this case that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0$$

The above is true provided $r > 0$. When $r = 0$, we have a singularity in $\mathbf{D}$, so its divergence is not defined.

b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere: To achieve the same net charge, we require that $(4/3)\pi a^3 \rho_{v0} = Q$, so $\underline{\rho_{v0} = 3Q/(4\pi a^3)\ \text{C/m}^3}$. Gauss' law tells us that inside the charged sphere

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_{v0} = \frac{Qr^3}{a^3}$$

Thus

$$D_r = \frac{Qr}{4\pi a^3}\ \text{C/m}^2 \text{ and } \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{Qr^3}{4\pi a^3}\right) = \frac{3Q}{4\pi a^3}$$

as expected. Outside the charged sphere, $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$ as before, and the divergence is zero.

3.24. Inside the cylindrical shell, $3 < \rho < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(\rho - 3)^3\, \mathbf{a}_\rho\ \text{C/m}^2$$

a) What is the volume charge density at $\rho = 4$ m? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{d}{d\rho}[5\rho(\rho-3)^3] = \frac{5(\rho-3)^2}{\rho}(4\rho - 3)\ \text{C/m}^3$$

Evaluating this at $\rho = 4$ m, we find $\rho_v(4) = \underline{16.25\ \text{C/m}^3}$

b) What is the electric flux density at $\rho = 4$ m? We evaluate the given $\mathbf{D}$ at this point to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_\rho\ \text{C/m}^2}$

3.24c. How much electric flux leaves the closed surface $3 < \rho < 4$, $0 < \phi < 2\pi$, $-2.5 < z < 2.5$? We note that **D** has only a radial component, and so flux would leave only through the cylinder sides. Also, **D** does not vary with ϕ or z, so the flux is found by a simple product of the side area and the flux density. We further note that $\mathbf{D} = 0$ at $\rho = 3$, so only the outer side (at $\rho = 4$) will contribute. We use the result of part *b*, and write the flux as

$$\Phi = [2.5 - (-2.5)]2\pi(4)(5) = \underline{200\pi \text{ C}}$$

d) How much charge is contained within the volume used in part *c*? By Gauss' law, this will be the same as the net outward flux through that volume, or again, $\underline{200\pi \text{ C}}$.

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3 \, \mathbf{a}_r \text{ C/m}^2$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6) \text{ C/m}^3$$

which we evaluate at $r = 4$ to find $\rho_v(r=4) = \underline{17.50 \text{ C/m}^3}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r \text{ C/m}^2}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part *b*, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi \text{ C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi \text{ C}}$.

3.26. Given the field

$$\mathbf{D} = \frac{5\sin\theta\cos\phi}{r} \, \mathbf{a}_r \text{ C/m}^2,$$

find:

a) the volume charge density: Use

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \underline{\frac{5\sin\theta\cos\phi}{r^2} \text{ C/m}^3}$$

b) the total charge contained within the region $r < 2$ m: To find this, we integrate over the volume:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \frac{5\sin\theta\cos\phi}{r^2} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Before plunging into this one notice that the ϕ integration is of $\cos\phi$ from zero to 2π. This yields a zero result, and so the total enclosed charge is $\underline{Q = 0}$.

c) the value of **D** at the surface $r = 2$: Substituting $r = 2$ into the given field produces

$$\mathbf{D}(r=2) = \underline{\frac{5}{2}\sin\theta\cos\phi \, \mathbf{a}_r \text{ C/m}^2}$$

3.26d. the total electric flux leaving the surface $r = 2$ Since the total enclosed charge is zero (from part *b*), the net outward flux is also zero, from Gauss' law.

3.27. Let $\mathbf{D} = 5.00r^2\mathbf{a}_r$ mC/m^2 for $r \leq 0.08$ m and $\mathbf{D} = 0.205\,\mathbf{a}_r/r^2$ μC/m^2 for $r \geq 0.08$ m (note error in problem statement).

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5.00r^4) = 20r \text{ mC/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is zero at 0.1 m.

c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m? The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{.08} 20r(\text{mC/m}^3)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 2.57 \times 10^{-3} \text{ mC} = 2.57\,\mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi(.08)^2}\right] = \underline{-32\,\mu\text{C/m}^2}$$

3.28. The electric flux density is given as $\mathbf{D} = 20\rho^3\,\mathbf{a}_\rho$ C/m^2 for $\rho < 100\,\mu$m, and $k\,\mathbf{a}_\rho/\rho$ for $\rho > 100\,\mu$m.

a) Find k so that $\mathbf{D}$ is continuous at $\rho = 100\,\mu$m: We require

$$20 \times 10^{-12} = \frac{k}{10^{-4}} \quad \Rightarrow \quad k = \underline{2 \times 10^{-15} \text{ C/m}}$$

b) Find and sketch ρ_v as a function of ρ: In cylindrical coordinates, with only a radial component of $\mathbf{D}$, we use

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(20\rho^4) = \underline{80\rho^2 \text{ C/m}^3 \;\; (\rho < 100\,\mu\text{m})}$$

For $\rho > 100\,\mu$m, we obtain

$$\rho_v = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{k}{\rho}) = \underline{0}$$

The sketch of ρ_v vs. ρ would be a parabola, starting at the origin, reaching a maximum value of 8×10^{-7} C/m^3 at $\rho = 100$ μm. The plot is zero at larger radii.

3.29. In the region of free space that includes the volume $2 < x, y, z < 3$,

$$\mathbf{D} = \frac{2}{z^2}(yz\,\mathbf{a}_x + xz\,\mathbf{a}_y - 2xy\,\mathbf{a}_z) \text{ C/m}^2$$

a) Evaluate the volume integral side of the divergence theorem for the volume defined above: In cartesian, we find $\nabla \cdot \mathbf{D} = 8xy/z^3$. The volume integral side is now

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_2^3\int_2^3\int_2^3 \frac{8xy}{z^3}\, dx dy dz = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

3.29b. Evaluate the surface integral side for the corresponding closed surface: We call the surfaces at $x = 3$ and $x = 2$ the front and back surfaces respectively, those at $y = 3$ and $y = 2$ the right and left surfaces, and those at $z = 3$ and $z = 2$ the top and bottom surfaces. To evaluate the surface integral side, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. Note that since the x component of $\mathbf{D}$ does not vary with x, the outward fluxes from the front and back surfaces will cancel each other. The same is true for the left and right surfaces, since D_y does not vary with y. This leaves only the top and bottom surfaces, where the fluxes are:

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{3^2}\, dxdy}_{\text{top}} - \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{2^2}\, dxdy}_{\text{bottom}} = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

3.30. If $\mathbf{D} = 15\rho^2 \sin 2\phi\, \mathbf{a}_\rho + 10\rho^2 \cos 2\phi\, \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region $1 < \rho < 2$ m, $1 < \phi < 2$ rad, $1 < z < 2$ m: Taking the surface integral side first, the six sides over which the flux must be evaluated are only four, since there is no z component of $\mathbf{D}$. We are left with the sides at $\phi = 1$ and $\phi = 2$ rad (left and right sides, respectively), and those at $\rho = 1$ and $\rho = 2$ (back and front sides). We evaluate

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_1^2 \int_1^2 15(2)^2 \sin(2\phi)\,(2) d\phi dz}_{\text{front}} - \underbrace{\int_1^2 \int_1^2 15(1)^2 \sin(2\phi)\,(1) d\phi dz}_{\text{back}}$$
$$- \underbrace{\int_1^2 \int_1^2 10\rho^2 \cos(2)\, d\rho dz}_{\text{left}} + \underbrace{\int_1^2 \int_1^2 10\rho^2 \cos(4)\, d\rho dz}_{\text{right}} = \underline{6.93 \text{ C}}$$

For the volume integral side, we first evaluate the divergence of $\mathbf{D}$, which is

$$\nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(15\rho^3 \sin 2\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(10\rho^2 \cos 2\phi) = 25\rho \sin 2\phi$$

Next

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_1^2 \int_1^2 \int_1^2 25\rho \sin(2\phi)\, \rho d\rho\, d\phi\, dz = \frac{25}{3}\rho^3\Big|_1^2 \left[\frac{-\cos(2\phi)}{2}\right]_1^2 = \underline{6.93 \text{ C}}$$

3.31. Given the flux density

$$\mathbf{D} = \frac{16}{r}\cos(2\theta)\, \mathbf{a}_\theta \text{ C/m}^2,$$

use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad: We use the divergence theorem and first evaluate the surface integral side. We are evaluating the net outward flux through a curvilinear "cube", whose boundaries are defined by the specified ranges. The flux contributions will be only through the surfaces of constant θ, however, since $\mathbf{D}$ has only a θ component. On a constant-theta surface, the differential area is $da = r \sin\theta dr d\phi$, where θ is fixed at the surface location. Our flux integral becomes

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{-\int_1^2 \int_1^2 \frac{16}{r}\cos(2)\, r \sin(1)\, dr d\phi}_{\theta=1} + \underbrace{\int_1^2 \int_1^2 \frac{16}{r}\cos(4)\, r \sin(2)\, dr d\phi}_{\theta=2}$$
$$= -16\left[\cos(2)\sin(1) - \cos(4)\sin(2)\right] = \underline{-3.91 \text{ C}}$$

3.31. (continued) We next evaluate the volume integral side of the divergence theorem, where in this case,

$$\nabla \cdot \mathbf{D} = \frac{1}{r \sin\theta} \frac{d}{d\theta}(\sin\theta \, D_\theta) = \frac{1}{r \sin\theta} \frac{d}{d\theta} \left[\frac{16}{r} \cos 2\theta \sin\theta \right] = \frac{16}{r^2} \left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2 \sin 2\theta \right]$$

We now evaluate:

$$\int_{vol} \nabla \cdot \mathbf{D} \, dv = \int_1^2 \int_1^2 \int_1^2 \frac{16}{r^2} \left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2 \sin 2\theta \right] r^2 \sin\theta \, dr d\theta d\phi$$

The integral simplifies to

$$\int_1^2 \int_1^2 \int_1^2 16[\cos 2\theta \cos\theta - 2 \sin 2\theta \sin\theta] \, dr d\theta d\phi = 8 \int_1^2 [3 \cos 3\theta - \cos\theta] \, d\theta = \underline{-3.91 \text{ C}}$$

3.32. If $\mathbf{D} = 2r \, \mathbf{a}_r$ C/m^2, find the total electric flux leaving the surface of the cube, $0 < x, y, z < 0.4$: This is where the divergence theorem really saves you time! First find

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2} \frac{d}{dr}(r^2 \times 2r) = 6$$

Then the net outward flux will be

$$\int_{vol} \nabla \cdot \mathbf{D} \, dv = 6(0.4)^3 = \underline{0.38 \text{ C}}$$

CHAPTER 4

4.1. The value of $\mathbf{E}$ at $P(\rho = 2,\ \phi = 40^\circ,\ z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E}\cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6\times10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20\times10^{-6}\,\text{C})(100\,\text{V/m})(6\times10^{-6}\,\text{m}) = -12\times10^{-9}\,\text{J} = \underline{-12\,\text{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6\times10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20\times10^{-6})(-200)(6\times10^{-6}) = 2.4\times10^{-8}\,\text{J} = \underline{24\,\text{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6\times10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20\times10^{-6})(300)(6\times10^{-6}) = -3.6\times10^{-8}\,\text{J} = \underline{-36\,\text{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6\times10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$\begin{aligned} dW &= -(20\times10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times10^{-6}) \\ &= \underline{-44.9\,\text{nJ}} \end{aligned}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$\begin{aligned} dW &= -(20\times10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times10^{-6}) \\ &= -(20\times10^{-6})\left[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y)\right. \\ &\quad \left. + 222.9\right](6\times10^{-6}) \end{aligned}$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40^\circ) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40^\circ) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40^\circ) = -0.643$. Substituting these results in

$$dW = -(20\times10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times10^{-6}) = \underline{-41.8\,\text{nJ}}$$

4.2. Let $\mathbf{E} = 400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z$ in the neighborhood of point $P(6, 2, -3)$. Find the incremental work done in moving a 4-C charge a distance of 1 mm in the direction specified by:

a) $\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$: We write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z) \cdot \frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}} (10^{-3})$$
$$= -\frac{(4 \times 10^{-3})}{\sqrt{3}}(400 - 300 + 500) = \underline{-1.39\,\text{J}}$$

b) $-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$: The computation is similar to that of part a, but we change the direction:

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -4(400\mathbf{a}_x - 300\mathbf{a}_y + 500\mathbf{a}_z) \cdot \frac{(-2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z)}{\sqrt{14}} (10^{-3})$$
$$= -\frac{(4 \times 10^{-3})}{\sqrt{14}}(-800 - 900 - 500) = \underline{2.35\,\text{J}}$$

4.3. If $\mathbf{E} = 120\,\mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a 50 μm charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -(50 \times 10^{-6}) \left[120\mathbf{a}_\rho \cdot \frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}\right] (2 \times 10^{-3})$$
$$= -(50 \times 10^{-6})(120) \left[(\mathbf{a}_\rho \cdot \mathbf{a}_x) - (\mathbf{a}_\rho \cdot \mathbf{a}_y)\right] \frac{1}{\sqrt{3}}(2 \times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4^\circ$. Thus $(\mathbf{a}_\rho \cdot \mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho \cdot \mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\,\mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\,\mu\text{J}}$ as in part a. This is also found by going through the same procedure as in part a, but with the direction (roles of P and Q) reversed.

4.4. Find the amount of energy required to move a 6-C charge from the origin to $P(3, 1, -1)$ in the field $\mathbf{E} = 2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z$ V/m along the straight-line path $x = -3z$, $y = x + 2z$: We set up the computation as follows, and find the the result *does not depend on the path*.

$$W = -q \int \mathbf{E} \cdot d\mathbf{L} = -6 \int (2x\mathbf{a}_x - 3y^2\mathbf{a}_y + 4\mathbf{a}_z) \cdot (dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z)$$
$$= -6 \int_0^3 2x dx \; + \; 6 \int_0^1 3y^2 dy \; - \; 6 \int_0^{-1} 4dz = \underline{-24\,\text{J}}$$

4.5. Compute the value of $\int_A^P \mathbf{G}\cdot d\mathbf{L}$ for $\mathbf{G}=2y\mathbf{a}_x$ with $A(1,-1,2)$ and $P(2,1,2)$ using the path:

a) straight-line segments $A(1,-1,2)$ to $B(1,1,2)$ to $P(2,1,2)$: In general we would have

$$\int_A^P \mathbf{G}\cdot d\mathbf{L}=\int_A^P 2y\,dx$$

The change in x occurs when moving between B and P, during which $y=1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L}=\int_B^P 2y\,dx=\int_1^2 2(1)dx=\underline{2}$$

b) straight-line segments $A(1,-1,2)$ to $C(2,-1,2)$ to $P(2,1,2)$: In this case the change in x occurs when moving from A to C, during which $y=-1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L}=\int_A^C 2y\,dx=\int_1^2 2(-1)dx=\underline{-2}$$

4.6. Let $\mathbf{G}=4x\mathbf{a}_x+2z\mathbf{a}_y+2y\mathbf{a}_z$. Given an initial point $P(2,1,1)$ and a final point $Q(4,3,1)$, find $\int \mathbf{G}\cdot d\mathbf{L}$ using the path: a) straight line: $y=x-1$, $z=1$; b) parabola: $6y=x^2+2$, $z=1$:

With $\mathbf{G}$ as given, the line integral will be

$$\int \mathbf{G}\cdot d\mathbf{L}=\int_2^4 4x\,dx+\int_1^3 2z\,dy+\int_1^1 2y\,dz$$

Clearly, we are going nowhere in z, so the last integral is zero. With $z=1$, the first two evaluate as

$$\int \mathbf{G}\cdot d\mathbf{L}=2x^2\Big|_2^4+2y\Big|_1^3=\underline{28}$$

The paths specified in parts *a* and *b* did not play a role, meaning that the integral between the specified points is path-independent.

4.7. Repeat Problem 4.6 for $\mathbf{G}=3xy^3\mathbf{a}_x+2z\mathbf{a}_y$. Now things are different in that the path does matter:

a) straight line: $y=x-1$, $z=1$: We obtain:

$$\int \mathbf{G}\cdot d\mathbf{L}=\int_2^4 3xy^2\,dx+\int_1^3 2z\,dy=\int_2^4 3x(x-1)^2\,dx+\int_1^3 2(1)\,dy=\underline{90}$$

b) parabola: $6y=x^2+2$, $z=1$: We obtain:

$$\int \mathbf{G}\cdot d\mathbf{L}=\int_2^4 3xy^2\,dx+\int_1^3 2z\,dy=\int_2^4 \frac{1}{12}x(x^2+2)^2\,dx+\int_1^3 2(1)\,dy=\underline{82}$$

4.8. A point charge Q_1 is located at the origin in free space. Find the work done in carrying a charge Q_2 from: (a) $B(r_B, \theta_B, \phi_B)$ to $C(r_A, \theta_B, \phi_B)$ with θ and ϕ held constant; (b) $C(r_A, \theta_B, \phi_B)$ to $D(r_A, \theta_A, \phi_B)$ with r and ϕ held constant; (c) $D(r_A, \theta_A, \phi_B)$ to $A(r_A, \theta_A, \phi_A)$ with r and θ held constant: The general expression for the work done in this instance is

$$W = -Q_2 \int \mathbf{E} \cdot d\mathbf{L} = -Q_2 \int \frac{Q_1}{4\pi\epsilon_0 r^2} \mathbf{a}_r \cdot (dr\mathbf{a}_r + rd\theta\mathbf{a}_\theta + r\sin\theta d\phi\mathbf{a}_\phi) = -\frac{Q_1 Q_2}{4\pi\epsilon_0} \int \frac{dr}{r^2}$$

We see that only changes in r will produce non-zero results. Thus for part a we have

$$W = -\frac{Q_1 Q_2}{4\pi\epsilon_0} \int_{r_B}^{r_A} \frac{dr}{r^2} = \underline{\frac{Q_1 Q_2}{4\pi\epsilon_0} \left[\frac{1}{r_A} - \frac{1}{r_B}\right] \text{ J}}$$

The answers to parts b and c (involving paths over which r is held constant) are both $\underline{0}$.

4.9. A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space.

a) Find the absolute potential at $P(r = 1\,\text{cm}, \theta = 25^\circ, \phi = 50^\circ)$: Since the charge density is uniform and is spherically-symmetric, the angular coordinates do not matter. The potential function for $r > 0.6$ cm will be that of a point charge of $Q = 4\pi a^2 \rho_s$, or

$$V(r) = \frac{4\pi(0.6 \times 10^{-2})^2(20 \times 10^{-9})}{4\pi\epsilon_0 r} = \frac{0.081}{r} \text{ V with } r \text{ in meters}$$

At $r = 1$ cm, this becomes $V(r = 1\,\text{cm}) = \underline{8.14 \text{ V}}$

b) Find V_{AB} given points $A(r = 2\,\text{cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3\,\text{cm}, \theta = 45^\circ, \phi = 90^\circ)$: Again, the angles do not matter because of the spherical symmetry. We use the part a result to obtain

$$V_{AB} = V_A - V_B = 0.081 \left[\frac{1}{0.02} - \frac{1}{0.03}\right] = \underline{1.36 \text{ V}}$$

4.10. Given a surface charge density of 8 nC/m^2 on the plane $x = 2$, a line charge density of 30 nC/m on the line $x = 1$, $y = 2$, and a 1-μC point charge at $P(-1, -1, 2)$, find V_{AB} for points $A(3, 4, 0)$ and $B(4, 0, 1)$: We need to find a potential function for the combined charges. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho} d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0} \ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(x) = -\int \frac{\rho_s}{2\epsilon_0} dx + C_2 = -\frac{\rho_s}{2\epsilon_0} x + C_2$$

4.10. (continued) The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}x + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and x are *scalar distances* from the charges, and will be treated as such here. For point A we have $r_A = \sqrt{(3-(-1))^2 + (4-(-1))^2 + (-2)^2} = \sqrt{45}$, $\rho_A = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{8}$, and its distance from the sheet charge is $x_A = 3 - 2 = 1$. The potential at A is then

$$V_A = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{45}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{8} - \frac{8\times10^{-9}}{2\epsilon_0}(1) + C$$

At point B, $r_B = \sqrt{(4-(-1))^2 + (0-(-1))^2 + (1-2)^2} = \sqrt{27}$,
$\rho_B = \sqrt{(4-1)^2 + (0-2)^2} = \sqrt{13}$, and the distance from the sheet charge is $x_B = 4 - 2 = 2$.
The potential at A is then

$$V_B = \frac{10^{-6}}{4\pi\epsilon_0\sqrt{27}} - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\sqrt{13} - \frac{8\times10^{-9}}{2\epsilon_0}(2) + C$$

Then

$$V_A - V_B = \frac{10^{-6}}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{45}} - \frac{1}{\sqrt{27}}\right] - \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left(\sqrt{\frac{8}{13}}\right) - \frac{8\times10^{-9}}{2\epsilon_0}(1-2) = \underline{193\text{ V}}$$

4.11. Let a uniform surface charge density of $5\,\text{nC/m}^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,\text{nC/m}$ be located at $x = 0$, $z = 4$, and a point charge of $2\,\mu\text{C}$ be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$: We need to find a potential function for the combined charges which is zero at M. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho}d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(z) = -\int \frac{\rho_s}{2\epsilon_0}dz + C_2 = -\frac{\rho_s}{2\epsilon_0}z + C_2$$

The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}z + C$$

4.11. (continued) The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and z are *scalar distances* from the charges, and will be treated as such here. To evaluate the constant, C, we first look at point M, where $V_T = 0$. At M, $r = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\rho = 1$, and $z = 5$. We thus have

$$0 = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{29}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(1) - \frac{5 \times 10^{-9}}{2\epsilon_0}5 + C \quad \Rightarrow \quad C = -1.93 \times 10^3 \text{ V}$$

At point N, $r = \sqrt{1 + 4 + 9} = \sqrt{14}$, $\rho = \sqrt{2}$, and $z = 3$. The potential at N is thus

$$V_N = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{14}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(\sqrt{2}) - \frac{5 \times 10^{-9}}{2\epsilon_0}(3) - 1.93 \times 10^3 = 1.98 \times 10^3 \text{ V} = \underline{1.98 \text{ kV}}$$

4.12. Three point charges, 0.4 μC each, are located at $(0, 0, -1)$, $(0, 0, 0)$, and $(0, 0, 1)$, in free space.

a) Find an expression for the absolute potential as a function of z along the line $x = 0$, $y = 1$: From a point located at position z along the given line, the distances to the three charges are $R_1 = \sqrt{(z-1)^2 + 1}$, $R_2 = \sqrt{z^2 + 1}$, and $R_3 = \sqrt{(z+1)^2 + 1}$. The total potential will be

$$V(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right]$$

Using $q = 4 \times 10^{-7}$ C, this becomes

$$V(z) = \underline{(3.6 \times 10^3)\left[\frac{1}{\sqrt{(z-1)^2 + 1}} + \frac{1}{\sqrt{z^2 + 1}} + \frac{1}{\sqrt{(z+1)^2 + 1}}\right] \text{ V}}$$

b) Sketch $V(z)$. The sketch will show that V maximizes to a value of 8.68×10^3 at $z = 0$, and then monotonically decreases with increasing $|z|$ symmetrically on either side of $z = 0$.

4.13. Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them? This will be the magnitude of the charge times the potential difference between the finishing and starting positions, or

$$W = \frac{(4 \times 10^{-12})^2}{2\pi\epsilon_0}\left[\frac{1}{2.5} - \frac{1}{5}\right] \times 10^4 = 5.76 \times 10^{-10} \text{ J} = \underline{576 \text{ pJ}}$$

4.14. two 6-nC point charges are located at $(1, 0, 0)$ and $(-1, 0, 0)$ in free space.

a) Find V at $P(0, 0, z)$: Since the charges are positioned symmetrically about the z axis, the potential at z will be double that from one charge. This becomes:

$$V(z) = (2)\frac{q}{4\pi\epsilon_0\sqrt{z^2 + 1}} = \underline{\frac{q}{2\pi\epsilon_0\sqrt{z^2 + 1}}}$$

b) Find V_{max}: It is clear from the part *a* result that V will maximize at $z = 0$, or $v_{max} = q/(2\pi\epsilon_0) = \underline{108 \text{ V}}$.

4.14. (continued)

c) Calculate $|dV/dz|$ on the z axis: Differentiating the part a result, we find

$$\left|\frac{dV}{dz}\right| = \underline{\frac{qz}{\pi\epsilon_0(z^2+1)^{3/2}}\ \text{V/m}}$$

d) Find $|dV/dz|_{max}$: To find this we need to differentiate the part c result and find its zero:

$$\frac{d}{dz}\left|\frac{dV}{dz}\right| = \frac{q(1-2z^2)}{\pi\epsilon_0(z^2+1)^{5/2}} = 0 \quad \Rightarrow \quad z = \pm\frac{1}{\sqrt{2}}$$

Substituting $z = 1/\sqrt{2}$ into the part c result, we find

$$\left|\frac{dV}{dz}\right|_{max} = \frac{q}{\sqrt{2}\pi\epsilon_0(3/2)^{3/2}} = \underline{83.1\ \text{V/m}}$$

4.15. Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$: The net potential function for the two charges would in general be:

$$V = -\frac{\rho_l}{2\pi\epsilon_0}\ln(R_1) - \frac{\rho_l}{2\pi\epsilon_0}\ln(R_2) + C$$

At the origin, $R_1 = R_2 = \sqrt{5}$, and $V = 100$ V. Thus, with $\rho_l = 8 \times 10^{-9}$,

$$100 = -2\frac{(8\times10^{-9})}{2\pi\epsilon_0}\ln(\sqrt{5}) + C \quad \Rightarrow \quad C = 331.6\ \text{V}$$

At $P(4, 1, 3)$, $R_1 = |(4, 1, 3) - (1, 1, 2)| = \sqrt{10}$ and $R_2 = |(4, 1, 3) - (-1, 2, 3)| = \sqrt{26}$. Therefore

$$V_P = -\frac{(8\times10^{-9})}{2\pi\epsilon_0}\left[\ln(\sqrt{10}) + \ln(\sqrt{26})\right] + 331.6 = \underline{-68.4\ \text{V}}$$

4.16. Uniform surface charge densities of 6, 4, and 2 nC/m^2 are present at $r = 2$, 4, and 6 cm, respectively, in free space.

a) Assume $V = 0$ at infinity, and find $V(r)$. We keep in mind the definition of absolute potential as the work done in moving a unit positive charge from infinity to location r. At radii outside all three spheres, the potential will be the same as that of a point charge at the origin, whose charge is the sum of the three sphere charges:

$$V(r)\,(r > 6\,\text{cm}) = \frac{q_1 + q_2 + q_3}{4\pi\epsilon_0 r} = \frac{[4\pi(.02)^2(6) + 4\pi(.04)^2(4) + 4\pi(.06)^2(2)] \times 10^{-9}}{4\pi\epsilon_0 r}$$

$$= \frac{(96 + 256 + 288)\pi \times 10^{-13}}{4\pi(8.85 \times 10^{-12})r} = \underline{\frac{1.81}{r}\ \text{V}} \quad \text{where r is in meters}$$

As the unit charge is moved inside the outer sphere to positions $4 < r < 6$ cm, the outer sphere contribution to the energy is fixed at its value at $r = 6$. Therefore,

$$V(r)\,(4 < r < 6\,\text{cm}) = \frac{q_1 + q_2}{4\pi\epsilon_0 r} + \frac{q_3}{4\pi\epsilon_0(.06)} = \underline{\frac{0.994}{r} + 13.6\ \text{V}}$$

In moving inside the sphere at $r = 4$ cm, the contribution from that sphere becomes fixed at its potential function at $r = 4$:

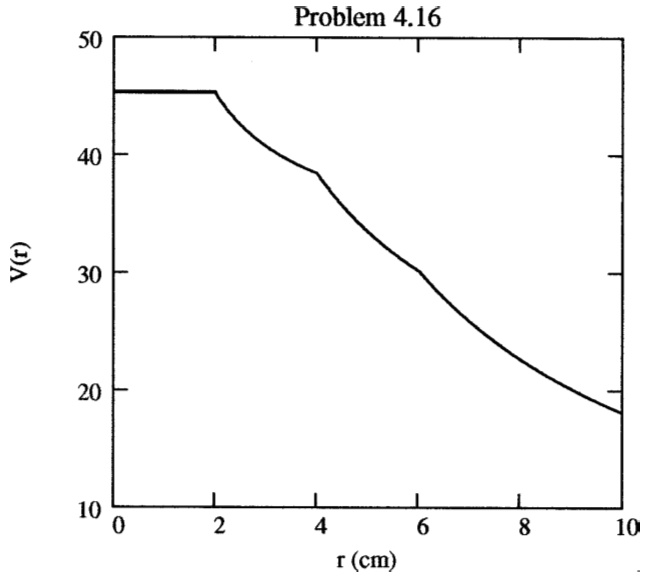
$$V(r)\,(2 < r < 4\,\text{cm}) = \frac{q_1}{4\pi\epsilon_0 r} + \frac{q_2}{4\pi\epsilon_0(.04)} + \frac{q_3}{4\pi\epsilon_0(.06)} = \underline{\frac{0.271}{r} + 31.7\ \text{V}}$$

Finally, using the same reasoning, the potential inside the inner sphere becomes

$$V(r)\,(r < 2\,\text{cm}) = \frac{0.271}{.02} + 31.7 = \underline{45.3\,\text{V}}$$

b) Calculate V at $r = 1, 3, 5,$ and 7 cm: Using the results of part a, we substitute these distances (in meters) into the appropriate formulas to obtain: $V(1) = \underline{45.3\,\text{V}}$, $V(3) = \underline{40.7\,\text{V}}$, $V(5) = \underline{33.5\,\text{V}}$, and $V(7) = \underline{25.9\,\text{V}}$.

c) Sketch V versus r for $0 < r < 10$ cm.



4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E} \cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6 \times 10^{-9})}{\epsilon_0} \ln\left(\frac{5}{4}\right) = \underline{-3.026\,\text{V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho} d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho} d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6 \times 10^{-9}) + (.06)(2 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\text{V}}$$

4.18. A nonuniform linear charge density, $\rho_L = 8/(z^2 + 1)$ nC/m lies along the z axis. Find the potential at $P(\rho = 1, 0, 0)$ in free space if $V = 0$ at infinity: This last condition enables us to write the potential at P as a superposition of point charge potentials. The result is the integral:

$$V_P = \int_{-\infty}^{\infty} \frac{\rho_L dz}{4\pi\epsilon_0 R}$$

where $R = \sqrt{z^2 + 1}$ is the distance from a point z on the z axis to P. Substituting the given charge distribution and R into the integral gives us

$$V_P = \int_{-\infty}^{\infty} \frac{8 \times 10^{-9} dz}{4\pi\epsilon_0 (z^2 + 1)^{3/2}} = \frac{2 \times 10^{-9}}{\pi\epsilon_0} \frac{z}{\sqrt{z^2 + 1}}\Big|_{-\infty}^{\infty} = \underline{144\,\text{V}}$$

4.19. The annular surface, $1\,\text{cm} < \rho < 3\,\text{cm}$, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho\,\text{nC/m}^2$. Find V at $P(0, 0, 2\,\text{cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\int \frac{\rho_s\, da}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi} \int_{.01}^{.03} \frac{(5 \times 10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\epsilon_0 \sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5 \times 10^{-9})}{2\epsilon_0}\right] \left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2} \ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\,\text{V}}$$

4.20. Fig. 4.11 shows three separate charge distributions in the $z = 0$ plane in free space.

a) find the total charge for each distribution: Line charge along the y axis:

$$Q_1 = \int_3^5 \pi \times 10^{-9} dy = 2\pi \times 10^{-9}\,\text{C} = \underline{6.28\,\text{nC}}$$

Line charge in an arc at radius $\rho = 3$:

$$Q_2 = \int_{10^\circ}^{70^\circ} (10^{-9})\,3\,d\phi = 4.5 \times 10^{-9}\,(70 - 10)\frac{2\pi}{360} = 4.71 \times 10^{-9}\,\text{C} = \underline{4.71\,\text{nC}}$$

Sheet charge:

$$Q_3 = \int_{10^\circ}^{70^\circ} \int_{1.6}^{3.5} (10^{-9})\,\rho\,d\rho\,d\phi = 5.07 \times 10^{-9}\,\text{C} = \underline{5.07\,\text{nC}}$$

b) Find the potential at $P(0, 0, 6)$ caused by each of the three charge distributions acting alone: Line charge along y axis:

$$V_{P1} = \int_3^5 \frac{\rho_L dL}{4\pi\epsilon_0 R} = \int_3^5 \frac{\pi \times 10^{-9} dy}{4\pi\epsilon_0\sqrt{y^2 + 6^2}} = \frac{10^3}{4 \times 8.854} \ln(y + \sqrt{y^2 + 6^2})\Big|_3^5 = \underline{7.83\,\text{V}}$$

Line charge in an arc a radius $\rho = 3$:

$$V_{P2} = \int_{10^\circ}^{70^\circ} \frac{(1.5 \times 10^{-9})\,3\,d\phi}{4\pi\epsilon_0\sqrt{3^2 + 6^2}} = \frac{Q_2}{4\pi\epsilon_0\sqrt{45}} = \underline{6.31\,\text{V}}$$

Sheet charge:

$$V_{P3} = \int_{10^\circ}^{70^\circ} \int_{1.6}^{3.5} \frac{(10^{-9})\,\rho\,d\rho\,d\phi}{4\pi\epsilon_0\sqrt{\rho^2 + 6^2}} = \frac{60 \times 10^{-9}}{4\pi(8.854 \times 10^{-12}} \left(\frac{2\pi}{360}\right) \int_{1.6}^{3.5} \frac{\rho\,d\rho}{\sqrt{\rho^2 + 36}}$$

$$= 9.42\sqrt{\rho^2 + 36}\Big|_{1.6}^{3.5} = \underline{6.93\,\text{V}}$$

c) Find V_P: This will be the sum of the three results of part *b*, or

$$V_P = V_{P1} + V_{P2} + V_{P3} = 7.83 + 6.31 + 6.93 = \underline{21.1\,\text{V}}$$

4.21. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$:

a) V: Substitute P directly to obtain: $V = \underline{-15.0\,\text{V}}$

b) $|V|$. This will be just $\underline{15.0\,\text{V}}$.

c) **E**: We have

$$\mathbf{E}\Big|_P = -\nabla V\Big|_P = -\left[\left(2y^2z^3 + \frac{6x}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_x + \left(4xyz^3 + \frac{12y}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_y \right.$$

$$\left. + \left(6xy^2z^2 + \frac{18z}{x^2 + 2y^2 + 3z^2}\right)\mathbf{a}_z\right]_P = \underline{7.1\mathbf{a}_x + 22.8\mathbf{a}_y - 71.1\mathbf{a}_z\,\text{V/m}}$$

4.21d. $|\mathbf{E}|_P$: taking the magnitude of the part c result, we find $|\mathbf{E}|_P = \underline{75.0\,\text{V/m}}$.

e) $\mathbf{a}_N$: By definition, this will be

$$\mathbf{a}_N\Big|_P = -\frac{\mathbf{E}}{|\mathbf{E}|} = \underline{-0.095\,\mathbf{a}_x - 0.304\,\mathbf{a}_y + 0.948\,\mathbf{a}_z}$$

f) $\mathbf{D}$: This is $\mathbf{D}\Big|_P = \epsilon_0\mathbf{E}\Big|_P = \underline{62.8\,\mathbf{a}_x + 202\,\mathbf{a}_y - 629\,\mathbf{a}_z\ \text{pC/m}^2}$.

4.22. It is known that the potential is given as $V = 80r^{0.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = -(0.6)80r^{-0.4}\,\mathbf{a}_r = \underline{-48r^{-0.4}\,\mathbf{a}_r\ \text{V/m}}$$

b) the volume charge density at $r = 0.5$ m: Begin by finding

$$\mathbf{D} = \epsilon_0\mathbf{E} = -48r^{-0.4}\epsilon_0\,\mathbf{a}_r\ \text{C/m}^2$$

We next find

$$\rho_v = \nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2D_r\right) = \frac{1}{r^2}\frac{d}{dr}\left(-48\epsilon_0 r^{1.6}\right) = -\frac{76.8\epsilon_0}{r^{1.4}}\ \text{C/m}^3$$

Then at $r = 0.5$ m,

$$\rho_v(0.5) = \frac{-76.8(8.854\times10^{-12})}{(0.5)^{1.4}} = -1.79\times10^{-9}\ \text{C/m}^3 = \underline{-1.79\ \text{nC/m}^3}$$

c) the total charge lying within the surface $r = 0.6$: The easiest way is to use Gauss' law, and integrate the flux density over the spherical surface $r = 0.6$. Since the field is constant at constant radius, we obtain the product:

$$Q = 4\pi(0.6)^2(-48\epsilon_0(0.6)^{-0.4}) = -2.36\times10^{-9}\ \text{C} = \underline{-2.36\ \text{nC}}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) $\mathbf{E}$: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4}\,\text{V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0\mathbf{E}$, we find the charge density through

$$\rho_v\Big|_{.5} = [\nabla\cdot\mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}\left(\rho D_\rho\right)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673\,\text{pC/m}^3}$$

4.23c. the total charge lying within the closed surface $\rho = .6, 0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part a, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521\,\text{pC/m}^2$. Thus $Q = -2\pi(.6)(1)521\times10^{-12}\,\text{C} = \underline{-1.96\,\text{nC}}$.

4.24. Given the potential field $V = 80r^2\cos\theta$ and a point $P(2.5, \theta = 30^\circ, \phi = 60^\circ)$ in free space, find at P:

a) V: Substitute the coordinates into the function and find $V_P = 80(2.5)^2\cos(30) = \underline{433\ \text{V}}$.

b) **E**:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta = -160r\cos\theta\,\mathbf{a}_r + 80r\sin\theta\,\mathbf{a}_\theta\ \text{V/m}$$

Evaluating this at P yields $\mathbf{E}_p = \underline{-346\mathbf{a}_r + 100\mathbf{a}_\theta\ \text{V/m}}$.

c) **D**: In free space, $\mathbf{D}_P = \epsilon_0\mathbf{E}_P = (-346\mathbf{a}_r + 100\mathbf{a}_\theta)\epsilon_0 = \underline{-3.07\,\mathbf{a}_r + 0.885\,\mathbf{a}_\theta\ \text{nC/m}^2}$.

d) ρ_v:

$$\rho_v = \nabla\cdot\mathbf{D} = \epsilon_0\nabla\cdot\mathbf{E} = \epsilon_0\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2E_r\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(E_\theta\sin\theta\right)\right]$$

Substituting the components of **E**, we find

$$\rho_v = \left[-\frac{160\cos\theta}{r^2}3r^2 + \frac{1}{r\sin\theta}80r(2\sin\theta\cos\theta)\right]\epsilon_0 = -320\epsilon_0\cos\theta = \underline{-2.45\ \text{nC/m}^3}$$

with $\theta = 30^\circ$.

e) dV/dN: This will be just $|\mathbf{E}|$ evaluated at P, which is

$$\frac{dV}{dN}\Big|_P = |-346\mathbf{a}_r + 100\mathbf{a}_\theta| = \sqrt{(346)^2 + (100)^2} = \underline{360\ \text{V/m}}$$

f) $\mathbf{a}_N$: This will be

$$\mathbf{a}_N = -\frac{\mathbf{E}_P}{|\mathbf{E}_P|} = -\left[\frac{-346\mathbf{a}_r + 100\mathbf{a}_\theta}{\sqrt{(346)^2 + (100)^2}}\right] = \underline{0.961\,\mathbf{a}_r - 0.278\,\mathbf{a}_\theta}$$

4.25. Within the cylinder $\rho = 2, 0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho\sin\phi$ V.

a) Find V, **E**, **D**, and ρ_v at $P(1, 60^\circ, 0.5)$ in free space: First, substituting the given point, we find $V_P = \underline{279.9\,\text{V}}$. Then,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi = -\left[50 + 150\sin\phi\right]\mathbf{a}_\rho - \left[150\cos\phi\right]\mathbf{a}_\phi$$

Evaluate the above at P to find $\mathbf{E}_P = \underline{-179.9\mathbf{a}_\rho - 75.0\mathbf{a}_\phi\ \text{V/m}}$

Now $\mathbf{D} = \epsilon_0\mathbf{E}$, so $\mathbf{D}_P = \underline{-1.59\mathbf{a}_\rho - .664\mathbf{a}_\phi\ \text{nC/m}^2}$. Then

$$\rho_v = \nabla\cdot\mathbf{D} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}\left(\rho D_\rho\right) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} = \left[-\frac{1}{\rho}(50 + 150\sin\phi) + \frac{1}{\rho}150\sin\phi\right]\epsilon_0 = -\frac{50}{\rho}\epsilon_0\,\text{C}$$

At P, this is $\rho_{vP} = \underline{-443\ \text{pC/m}^3}$.

4.25b. How much charge lies within the cylinder? We will integrate ρ_v over the volume to obtain:

$$Q = \int_0^1 \int_0^{2\pi} \int_0^2 -\frac{50\epsilon_0}{\rho} \rho \, d\rho \, d\phi \, dz = -2\pi(50)\epsilon_0(2) = \underline{-5.56\,\text{nC}}$$

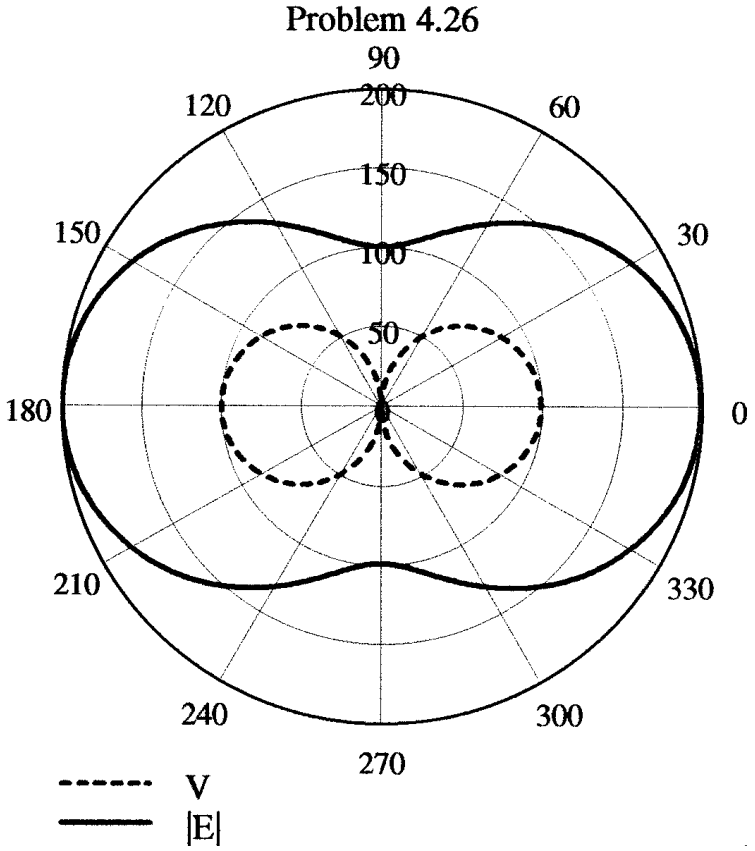
4.26. A dipole having $Qd/(4\pi\epsilon_0) = 100\,\text{V}\cdot\text{m}^2$ is located at the origin in free space and aligned so that its moment is in the $\mathbf{a}_z$ direction. a) Sketch $|V(r=1,\theta,\phi=0)|$ versus θ on polar graph paper (homemade if you wish). b) Sketch $|\mathbf{E}(r=1,\theta,\phi=0)|$ versus θ on polar graph paper:

$$V = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{100\cos\theta}{r^2} \quad \Rightarrow \quad |V(r=1,\theta,\phi=0)| = \underline{|100\cos\theta|}$$

$$\mathbf{E} = \frac{Qd}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) = \frac{100}{r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

$$|\mathbf{E}(r=1,\theta,\phi=0)| = 100\left(4\cos^2\theta + \sin^2\theta\right)^{1/2} = \underline{100\left(1+3\cos^2\theta\right)^{1/2}}$$

These results are plotted below:



4.27. Two point charges, 1 nC at (0, 0, 0.1) and -1 nC at (0, 0, -0.1), are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0|\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0|\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\text{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0(.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0(.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}\left[2.42\mathbf{a}_x + 1.41\mathbf{a}_z\right]\ \text{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\text{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9^\circ$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9^\circ)}{4\pi\epsilon_0(.5)^2} = \underline{5.76\,\text{V}}$$

4.28. A dipole located at the origin in free space has a moment $\mathbf{p} 2 \times 10^{-9}\,\mathbf{a}_z$ C · m. At what points on the line $y = z,\ x = 0$ is:

a) $|E_\theta| = 1$ mV/m? We note that the line $y = z$ lies at $\theta = 45^\circ$. Begin with

$$\mathbf{E} = \frac{2 \times 10^{-9}}{4\pi\epsilon_0 r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta) = \frac{10^{-9}}{2\sqrt{2}\pi\epsilon_0 r^3}(2\mathbf{a}_r + \mathbf{a}_\theta) \text{ at } \theta = 45^\circ$$

from which

$$E_\theta = \frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad \Rightarrow \quad r^3 = 1.27 \times 10^{-4} \text{ or } r = 23.3\,\text{m}$$

The y and z values are thus $y = z = \pm 23.3/\sqrt{2} = \underline{\pm 16.5\,\text{m}}$

b) $|E_r| = 1$ mV/m? From the above field expression, the radial component magnitude is twice that of the theta component. Using the same development, we then find

$$E_r = 2\frac{10^{-9}}{2\pi\epsilon_0 r^3} = 10^{-3}\ \text{V/m (required)} \quad \Rightarrow \quad r^3 = 2(1.27 \times 10^{-4}) \text{ or } r = 29.4\,\text{m}$$

The y and z values are thus $y = z = \pm 29.4/\sqrt{2} = \underline{\pm 20.8\,\text{m}}$

4.29. A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC · m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$: We use the general expression for the potential in the far field:

$$V = \frac{\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = P - Q = (1, 1, 8)$. So

$$V_P = \frac{(3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z) \cdot (\mathbf{a}_x + \mathbf{a}_y + 8\mathbf{a}_z) \times 10^{-9}}{4\pi\epsilon_0 [1^2 + 1^2 + 8^2]^{1.5}} = \underline{1.31 \text{ V}}$$

4.30. A dipole, having a moment $\mathbf{p} = 2\mathbf{a}_z$ nC · m is located at the origin in free space. Give the magnitude of $\mathbf{E}$ and its direction $\mathbf{a}_E$ in cartesian components at $r = 100$ m, $\phi = 90^\circ$, and $\theta =$: a) 0°; b) 30°; c) 90°. Begin with

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} [2\cos\theta\, \mathbf{a}_r + \sin\theta\, \mathbf{a}_\theta]$$

from which

$$|\mathbf{E}| = \frac{p}{4\pi\epsilon_0 r^3} \left[4\cos^2\theta + \sin^2\theta\right]^{1/2} = \frac{p}{4\pi\epsilon_0 r^3} \left[1 + 3\cos^2\theta\right]^{1/2}$$

Now

$$E_x = \mathbf{E} \cdot \mathbf{a}_x = \frac{p}{4\pi\epsilon_0 r^3} [2\cos\theta\, \mathbf{a}_r \cdot \mathbf{a}_x + \sin\theta\, \mathbf{a}_\theta \cdot \mathbf{a}_x] = \frac{p}{4\pi\epsilon_0 r^3} [3\cos\theta \sin\theta \cos\phi]$$

then

$$E_y = \mathbf{E} \cdot \mathbf{a}_y = \frac{p}{4\pi\epsilon_0 r^3} \left[2\cos\theta\, \mathbf{a}_r \cdot \mathbf{a}_y + \sin\theta\, \mathbf{a}_\theta \cdot \mathbf{a}_y\right] = \frac{p}{4\pi\epsilon_0 r^3} [3\cos\theta \sin\theta \sin\phi]$$

and

$$E_z = \mathbf{E} \cdot \mathbf{a}_z = \frac{p}{4\pi\epsilon_0 r^3} \left[2\cos\theta\, \mathbf{a}_r \cdot \mathbf{a}_z + \sin\theta\, \mathbf{a}_\theta \cdot \mathbf{a}_z\right] = \frac{p}{4\pi\epsilon_0 r^3} \left[2\cos^2\theta - \sin^2\theta\right]$$

Since ϕ is given as 90°, $E_x = 0$, and the field magnitude becomes

$$|\mathbf{E}(\phi = 90^\circ)| = \sqrt{E_y^2 + E_z^2} = \frac{p}{4\pi\epsilon_0 r^3} \left[9\cos^2\theta \sin^2\theta + (2\cos^2\theta - \sin^2\theta)^2\right]^{1/2}$$

Then the unit vector becomes (again at $\phi = 90^\circ$):

$$\mathbf{a}_E = \frac{3\cos\theta \sin\theta\, \mathbf{a}_y + (2\cos^2\theta - \sin^2\theta)\, \mathbf{a}_z}{\left[9\cos^2\theta \sin^2\theta + (2\cos^2\theta - \sin^2\theta)^2\right]^{1/2}}$$

Now with $r = 100$ m and $p = 2 \times 10^{-9}$,

$$\frac{p}{4\pi\epsilon_0 r^3} = \frac{2 \times 10^{-9}}{4\pi(8.854 \times 10^{-12})10^6} = 1.80 \times 10^{-5}$$

Using the above formulas, we find at $\theta = 0^\circ$, $|\mathbf{E}| = (1.80 \times 10^{-5})(2) = \underline{36.0\, \mu\text{V/m}}$ and $\mathbf{a}_E = \underline{\mathbf{a}_z}$. At $\theta = 30^\circ$, we find $|\mathbf{E}| = (1.80 \times 10^{-5})[1.69 + 1.56]^{1/2} = \underline{32.5\, \mu\text{V/m}}$ and $\mathbf{a}_E = (1.30\mathbf{a}_y + 1.25\mathbf{a}_z)/1.80 = \underline{0.72\, \mathbf{a}_x + 0.69\, \mathbf{a}_z}$. At $\theta = 90^\circ$, $|\mathbf{E}| = (1.80 \times 10^{-5})(1) = \underline{18.0\ \mu\text{V/m}}$ and $\mathbf{a}_E = \underline{-\mathbf{a}_z}$.

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0 \mathbf{E} \cdot \mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20\left[\frac{1}{x^2yz}\mathbf{a}_x + \frac{1}{xy^2z}\mathbf{a}_y + \frac{1}{xyz^2}\mathbf{a}_z\right] \text{ V/m}$$

The energy is now

$$W_E = 200\epsilon_0 \int_1^2 \int_1^2 \int_1^2 \left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4}\right] dx\, dy\, dz$$

The integral evaluates as follows:

$$\begin{aligned}
W_E &= 200\epsilon_0 \int_1^2 \int_1^2 \left[-\left(\frac{1}{3}\right)\frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4}\right]_1^2 dy\, dz \\
&= 200\epsilon_0 \int_1^2 \int_1^2 \left[\left(\frac{7}{24}\right)\frac{1}{y^2z^2} + \left(\frac{1}{2}\right)\frac{1}{y^4z^2} + \left(\frac{1}{2}\right)\frac{1}{y^2z^4}\right] dy\, dz \\
&= 200\epsilon_0 \int_1^2 \left[-\left(\frac{7}{24}\right)\frac{1}{yz^2} - \left(\frac{1}{6}\right)\frac{1}{y^3z^2} - \left(\frac{1}{2}\right)\frac{1}{yz^4}\right]_1^2 dz \\
&= 200\epsilon_0 \int_1^2 \left[\left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{1}{4}\right)\frac{1}{z^4}\right] dz \\
&= 200\epsilon_0(3)\left[\frac{7}{96}\right] = \underline{387 \text{ pJ}}
\end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0(3)\left[\frac{1}{(1.5)^4(1.5)^2(1.5)^2}\right] = 2.07 \times 10^{-10} \text{ J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207 \text{ pJ}}$.

4.32. In the region of free space where $2 < r < 3$, $0.4\pi < \theta < 0.6\pi$, $0 < \phi < \pi/2$, let $\mathbf{E} = k/r^2\, \mathbf{a}_r$.

a) Find a positive value for k so that the total energy stored is exactly 1 J: The energy is found through

$$\begin{aligned}
W_E &= \int_v \frac{1}{2}\epsilon_0 E^2\, dv = \int_0^{\pi/2} \int_{0.4\pi}^{0.6\pi} \int_2^3 \frac{1}{2}\epsilon_0 \frac{k^2}{r^2}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= \frac{\pi}{2}(-\cos\theta)\Big|_{.4\pi}^{.6\pi} \left(\frac{1}{2}\right)\epsilon_0 k^2 \left(-\frac{1}{r}\right)\Big|_2^3 = \frac{0.616\pi}{24}\epsilon_0 k^2 = 1 \text{ J}
\end{aligned}$$

Solve for k to find $k = \underline{1.18 \times 10^6 \text{ V} \cdot \text{m}}$.

4.32b. Show that the surface $\theta = 0.6\pi$ is an equipotential surface: This will be the surface of a cone, centered at the origin, along which **E**, in the $\mathbf{a}_r$ direction, will exist. Therefore, the given surface *cannot* be an equipotential (the problem was ill-conceived). Only a surface of constant r could be an equipotential in this field.

c) Find V_{AB}, given points $A(2, \theta = \pi/2, \phi = \pi/3)$ and $B(3, \pi/2, \pi/4)$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_2^3 \frac{k}{r^2}\mathbf{a}_r \cdot \mathbf{a}_r \, dr = k\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{k}{6}$$

Using the result of part *a*, we find $V_{AB} = (1.18 \times 10^6)/6 = \underline{197\text{ kV}}$.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of 5 μC in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5 \times 10^{-6}}{4\pi r^2}\mathbf{a}_r \text{ C/m}^2$$

b) Calculate the total energy stored in the electrostatic field: Use

$$\begin{aligned} W_E &= \int_{vol} \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \, dv = \int_0^{2\pi}\int_0^{\pi}\int_{.04}^{\infty} \frac{1}{2}\frac{(5 \times 10^{-6})^2}{16\pi^2\epsilon_0 r^4} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= (4\pi)\left(\frac{1}{2}\right)\frac{(5 \times 10^{-6})^2}{16\pi^2\epsilon_0}\int_{.04}^{\infty}\frac{dr}{r^2} = \frac{25 \times 10^{-12}}{8\pi\epsilon_0}\frac{1}{.04} = \underline{2.81\text{ J}} \end{aligned}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5 \times 10^{-6})^2}{2(2.81)} = 4.45 \times 10^{-12}\text{ F} = \underline{4.45\text{ pF}}$$

4.34. Given the potential field in free space, $V = 80\phi$ V (note that $\mathbf{a}_phi$ should not be present), find:

a) the energy stored in the region $2 < \rho < 4$ cm, $0 < \phi < 0.2\pi$, $0 < z < 1$ m: First we find

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{80}{\rho}\mathbf{a}_\phi \text{ V/m}$$

Then

$$W_E = \int_v w_E dv = \int_0^1\int_0^{0.2\pi}\int_{.02}^{.04}\frac{1}{2}\epsilon_0\frac{(80)^2}{\rho^2}\rho \, d\rho \, d\phi \, dz = 640\pi\epsilon_0 \ln\left(\frac{.04}{.02}\right) = \underline{12.3\text{ nJ}}$$

b) the potential difference, V_{AB}, for $A(3\text{ cm}, \phi = 0, z = 0)$ and $B(3\text{cm}, 0.2\pi, 1\text{m})$: Use

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} = -\int_{.2\pi}^0 -\frac{80}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi \, \rho \, d\phi = -80(0.2\pi) = \underline{-16\pi\text{ V}}$$

4.34c. the maximum value of the energy density in the specified region: The energy density is

$$w_E = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 \frac{6400}{\rho^2}$$

This will maximize at the lowest value of ρ in the specified range, which is $\rho = 2$ cm. So

$$w_{E,max} = \frac{1}{2}\epsilon_0 \frac{6400}{.02^2} = 7.1 \times 10^{-5}\ \text{J/m}^3 = \underline{71\ \mu\text{J/m}^3}$$

4.35. Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side.

a) Find the total potential energy stored: This will be given by

$$W_E = \frac{1}{2}\sum_{n=1}^{4} q_n V_n$$

where V_n in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{21} + V_{31} + V_{41} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{.04} + \frac{1}{.04} + \frac{1}{.04\sqrt{2}}\right]$$

Taking the summation produces a factor of 4, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}(4)q_1 V_1 = \frac{(.8 \times 10^{-9})^2}{2\pi\epsilon_0(.04)}\left[2 + \frac{1}{\sqrt{2}}\right] = 7.79 \times 10^{-7}\ \text{J} = \underline{0.779\,\mu\text{J}}$$

b) A fifth 0.8 μC charge is installed at the center of the square. Again find the total stored energy: This will be the energy found in part *a* plus the amount of work done in moving the fifth charge into position from infinity. The latter is just the potential at the square center arising from the original four charges, times the new charge value, or

$$\Delta W_E = \frac{4(.8 \times 10^{-9})^2}{4\pi\epsilon_0(.04\sqrt{2}/2)} = .813\,\mu\text{J}$$

The total energy is now

$$W_{E\,net} = W_E(\text{part a}) + \Delta W_E = .779 + .813 = \underline{1.59\,\mu\text{J}}$$

CHAPTER 5

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]\,\text{kA/m}^2$:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1 \left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right] dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1-e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part *b*, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.2. Let the current density be $\mathbf{J} = 2\phi\cos^2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$ A/m^2 within the region $2.1 < \rho < 2.5$, $0 < \phi < 0.1$ rad, $6 < z < 6.1$. Find the total current I crossing the surface:

a) $\rho = 2.2$, $0 < \phi < 0.1$, $6 < z < 6.1$ in the $\mathbf{a}_\rho$ direction: This is a surface of constant ρ, so only the radial component of $\mathbf{J}$ will contribute: At $\rho = 2.2$ we write:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_0^{0.1} 2(2)\cos^2\phi\;\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,2d\phi dz = 2(2.2)^2(0.1)\int_0^{0.1}\frac{1}{2}(1+\cos 2\phi)\,d\phi$$
$$= 0.2(2.2)^2\left[\frac{1}{2}(0.1) + \frac{1}{4}\sin 2\phi\Big|_0^{0.1}\right] = \underline{97\,\text{mA}}$$

b) $\phi = 0.05$, $2.2 < \rho < 2.5$, $6 < z < 6.1$ in the $\mathbf{a}_\phi$ direction: In this case only the ϕ component of $\mathbf{J}$ will contribute:

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_6^{6.1}\int_{2.2}^{2.5} -\rho\sin 2\phi\big|_{\phi=0.05}\,\mathbf{a}_\phi\cdot\mathbf{a}_\phi\,d\rho\,dz = -(0.1)^2\frac{\rho^2}{2}\Big|_{2.2}^{2.5} = \underline{-7\,\text{mA}}$$

5.2c. Evaluate $\nabla \cdot \mathbf{J}$ at $P(\rho = 2.4, \phi = 0.08, z = 6.05)$:

$$\nabla \cdot \mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho J_\rho) + \frac{1}{\rho}\frac{\partial J_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2 \cos^2\phi) - \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho \sin 2\phi) = 4\cos^2\phi - 2\cos 2\phi \Big|_{0.08}$$
$$= \underline{2.0\ \text{A/m}^3}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2 + 4}\,\mathbf{a}_r\ \text{A/m}^2$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J} \cdot \mathbf{n}\Big|_S\, da = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} \frac{400\sin\theta}{(.8)^2 + 4}(.8)^2 \sin\theta\, d\theta\, d\phi = \frac{400(.8)^2 2\pi}{4.64}\int_{.1\pi}^{.3\pi} \sin^2\, d\theta$$
$$= 346.5\int_{.1\pi}^{.3\pi} \frac{1}{2}[1 - \cos(2\theta)]\, d\theta = \underline{77.4\ \text{A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta\, d\theta\, d\phi = 1.46\ \text{m}^2$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\,\mathbf{a}_r = \underline{53.0\,\mathbf{a}_r\ \text{A/m}^2}$.

5.4. The cathode of a planar vacuum tube is at $z = 0$. Let $\mathbf{E} = -4 \times 10^6\,\mathbf{a}_z$ V/m for $z > 0$. An electron ($e = 1.602 \times 10^{-19}$ C, $m = 9.11 \times 10^{-31}$ kg) is emitted from the cathode with zero initial velocity at $t = 0$.

a) Find $v(t)$: Using Newton's second law, we write:

$$\mathbf{F} = m\mathbf{a} = q\mathbf{E} \quad \Rightarrow \quad \mathbf{a} = \frac{(-1.602 \times 10^{-19})(-4 \times 10^6)\mathbf{a}_z}{(9.11 \times 10^{-31})} = 7.0 \times 10^{17}\mathbf{a}_z\ \text{m/s}^2$$

Then $v(t) = at = \underline{7.0 \times 10^{17} t\ \text{m/s}}$.

b) Find $z(t)$, the electron location as a function of time: Use

$$z(t) = \int_0^t v(t')dt' = \frac{1}{2}(7.0 \times 10^{17})t^2 = \underline{3.5 \times 10^{17} t^2\ \text{m}}$$

c) Determine $v(z)$: Solve the result of part b for t, obtaining

$$t = \frac{\sqrt{z}}{\sqrt{3.5 \times 10^{17}}} = 1.7 \times 10^9\sqrt{z}$$

Substitute into the result of part a to find $v(z) = 7.0 \times 10^{17}(1.7 \times 10^{-9})\sqrt{z} = \underline{1.2 \times 10^9\sqrt{z}\ \text{m/s}}$.

5.4d. Make the assumption that the electrons are emitted continuously as a beam with a 0.25 mm radius and a total current of 60 μA. Find $\mathbf{J}(z)$ and $\rho(z)$:

$$\mathbf{J}(z) = \frac{-60 \times 10^{-6}}{\pi(0.25)^2(10^{-6})}\mathbf{a}_z = \underline{-3.1 \times 10^2\, \mathbf{a}_z \text{ A/m}^2}$$

(negative since we have electrons flowing in the positive z direction) Next we use $\mathbf{J}(z) = \rho_v(z)\mathbf{v}(z)$, or

$$\rho_v(z) = \frac{J}{v} = \frac{-3.1 \times 10^2}{1.2 \times 10^9\sqrt{z}} = -\frac{2.6 \times 10^{-7}}{\sqrt{z}} \text{ C/m}^3 = \underline{\frac{-26}{\sqrt{z}}\ \mu\text{C/m}^3}$$

5.5. Let

$$\mathbf{J} = \frac{25}{\rho}\,\mathbf{a}_\rho - \frac{20}{\rho^2 + 0.01}\,\mathbf{a}_z \text{ A/m}^2$$

a) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$: Use

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_{z=.2}\, da = \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\rho\, d\rho\, d\phi$$
$$= -\left(\frac{1}{2}\right) 20\ln(.01 + \rho^2)\Big|_0^{.4}(2\pi) = -20\pi\ln(17) = \underline{-178.0\text{ A}}$$

b) Calculate $\partial\rho_v/\partial t$: This is found using the equation of continuity:

$$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2 + .01}\right) = \underline{0}$$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$: This will be

$$I = \int_0^{.2}\int_0^{2\pi} \frac{25}{.01}\mathbf{a}_\rho\cdot(-\mathbf{a}_\rho)(.01)\, d\phi\, dz + \int_0^{.2}\int_0^{2\pi} \frac{25}{.4}\mathbf{a}_\rho\cdot(\mathbf{a}_\rho)(.4)\, d\phi\, dz$$
$$+ \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(-\mathbf{a}_z)\,\rho\, d\rho\, d\phi + \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(\mathbf{a}_z)\,\rho\, d\rho\, d\phi = \underline{0}$$

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for **J** and the surface specified in part *b*. In part *c*, the net outward flux was found to be zero, and in part *b*, the divergence of **J** was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.

5.6. Let $\epsilon = \epsilon_0$ and $V = 90z^{4/3}$ in the region $z = 0$.

a) Obtain expressions for **E**, **D**, and ρ_v as functions of z: First,

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{4}{3}(90)z^{1/3}\mathbf{a}_z = \underline{-120z^{1/3}\mathbf{a}_z \text{ V/m}}$$

5.6a. (continued)

Next, $\mathbf{D} = \epsilon_0 \mathbf{E} = \underline{1.06z^{1/3}\mathbf{a}_z \text{ nC/m}^2}$. Then

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{dD_z}{dz} = -\frac{1}{3}(120)\epsilon_0 z^{-2/3} = \underline{-354z^{-2/3} \text{ pC/m}^3}$$

b) If the velocity of the charge density is given as $v_z = 5 \times 10^6 z^{2/3}$ m/s, find J_z at $z = 0$ and $z = 0.1$ m (note that v_z is written as v_x through a missprint): Use $J_z = \rho_v v_z = (-354 \times 10^{-12})z^{-2/3}(5 \times 10^6)z^{2/3} = \underline{-1.8 \text{ mA/m}^2}$ at any z.

5.7. Assuming that there is no transformation of mass to energy or vice-versa, it is possible to write a continuity equation for mass.

a) If we use the continuity equation for charge as our model, what quantities correspond to $\mathbf{J}$ and ρ_v? These would be, respectively, <u>mass flux density</u> in (kg/m^2 − s) and <u>mass density</u> in (kg/m^3).

b) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, -9.85, 1.75, -2.00, -4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center. We may write the continuity equation for mass as follows, also invoking the divergence theorem:

$$\int_v \frac{\partial \rho_m}{\partial t} dv = -\int_v \nabla \cdot \mathbf{J}_m \, dv = -\oint_s \mathbf{J}_m \cdot \mathbf{dS}$$

where

$$\oint_s \mathbf{J}_m \cdot \mathbf{dS} = 10.25 - 9.85 + 1.75 - 2.00 - 4.05 + 4.45 = 0.550 \text{ mg/s}$$

Treating our 1 cm^3 volume as differential, we find

$$\frac{\partial \rho_m}{\partial t} \doteq -\frac{0.550 \times 10^{-3} \text{ g/s}}{10^{-6} \text{ m}^3} = \underline{-550 \text{ g/m}^3 - \text{s}}$$

5.8. The continuity equation for mass equates the divergence of the mass rate of flow (mass per second per square meter) to the negative of the density (mass per cubic meter). After setting up a cartesian coordinate system inside a star, Captain Kirk and his intrepid crew make measurements over the faces of a cube centered at the origin with edges 40 km long and parallel to the coordinate axes. They find the mass rate of flow of material outward across the six faces to be -1112, 1183, 201, -196, 1989, and -1920 kg/m$^2 \cdot$ s.

a) Estimate the divergence of the mass rate of flow at the origin: We make the estimate using the definition of divergence, but without taking the limit as the volume shrinks to zero:

$$\text{Div } \mathbf{J}_m \doteq \frac{\oint \mathbf{J}_m \cdot d\mathbf{S}}{\Delta v} = \frac{(-1112 + 1183 + 201 - 196 + 1989 - 1920)(40)^2}{(40)^3} = \underline{3.63 \text{ kg/km}^3 \cdot \text{s}}$$

b) Estimate the rate of change of the density at the origin: The continuity equation for mass reads: $\text{Div } \mathbf{J}_m = -\partial \rho_m / \partial t$. Therefore, the rate of change of density at the origin will be just the negative of the part a result, or $\partial \rho_m / \partial t \doteq \underline{-3.63 \text{ kg/km}^3 \cdot \text{s}}$.

5.9a. Using data tabulated in Appendix C, calculate the required diameter for a 2-m long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it:

The required resistance will be

$$R = \frac{V^2}{P} = \frac{l}{\sigma(\pi a^2)}$$

Thus the diameter will be

$$d = 2a = 2\sqrt{\frac{lP}{\sigma\pi V^2}} = 2\sqrt{\frac{2(450)}{(10^6)\pi(120)^2}} = 2.8\times10^{-4}\,\text{m} = \underline{0.28\,\text{mm}}$$

b) Calculate the rms current density in the wire: The rms current will be $I = 450/120 = 3.75\,\text{A}$. Thus

$$J = \frac{3.75}{\pi\left(2.8\times10^{-4}/2\right)^2} = \underline{6.0\times10^7\,\text{A/m}^2}$$

5.10. A steel wire has a radius of 2 mm and a conductivity of 2×10^6 S/m. The steel wire has an aluminum ($\sigma = 3.8\times10^7$ S/m) coating of 2 mm thickness. Let the total current carried by this hybrid conductor be 80 A dc. Find:

a) J_{st}. We begin with the fact that electric field must be the same in the aluminum and steel regions. This comes from the requirement that **E** tangent to the boundary between two media must be continuous, and from the fact that when integrating E over the wire length, the applied voltage value must be obtained, regardless of the medium within which this integral is evaluated. We can therefore write

$$E_{Al} = E_{st} = \frac{J_{Al}}{\sigma_{Al}} = \frac{J_{st}}{\sigma_{st}} \quad\Rightarrow\quad J_{Al} = \frac{\sigma_{Al}}{\sigma_{st}}J_{st}$$

The net current is now expressed as the sum of the currents in each region, written as the sum of the products of the current densities in each region times the appropriate cross-sectional area:

$$I = \pi(2\times10^{-3})^2 J_{st} + \pi[(4\times10^{-3})^2 - (2\times10^{-3})^2]J_{Al} = 80\,\text{A}$$

Using the above relation between J_{st} and J_{Al}, we find

$$80 = \pi\left[(2\times10^{-3})^2\left[1-\left(\frac{3.8\times10^7}{6\times10^6}\right)\right] + (4\times10^{-3})^2\left(\frac{3.8\times10^7}{6\times10^6}\right)\right]J_{st}$$

Solve for J_{st} to find $J_{st} = \underline{3.2\times10^5\,\text{A/m}^2}$.

b)

$$J_{Al} = \frac{3.8\times10^7}{6\times10^6}(3.2\times10^5) = \underline{2.0\times10^6\,\text{A/m}^2}$$

c,d) $E_{st} = E_{Al} = J_{st}/\sigma_{st} = J_{Al}/\sigma_{Al} = \underline{5.3\times10^{-2}\,\text{V/m}}$.

e) the voltage between the ends of the conductor if it is 1 mi long: Using the fact that 1 mi $= 1.61\times10^3$ m, we have $V = El = (5.3\times10^{-2})(1.61\times10^3) = \underline{85.4\,\text{V}}$.

5.11. Two perfectly-conducting cylindrical surfaces are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc. *Assume the cylinders are both of length l.*

a) Find the voltage and resistance between the cylinders, and **E** in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho \text{ A/m}^2$$

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot\mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}\text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\ \Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E}\cdot\mathbf{J}\,dv = \int_0^l\int_0^{2\pi}\int_{.03}^{.05} \frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\,\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l}\ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\text{ W}$$

which is in agreement with the power density integration.

5.12. The spherical surfaces $r = 3$ and $r = 5$ cm are perfectly conducting, and the total current passing radially outward through the medium between the surfaces is 3 A dc.

a) Find the voltage and resistance between the spheres, and **E** in the region between them, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < r < 5$ cm. We first find **J** as a function of radius by dividing the current by the area of a sphere of radius r:

$$\mathbf{J} = \frac{I}{4\pi r^2}\,\mathbf{a}_r = \frac{3}{4\pi r^2}\,\mathbf{a}_r \text{ A/m}^2$$

Then

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{3}{4\pi r^2(0.05)}\,\mathbf{a}_r = \underline{\frac{4.77}{r^2}\,\mathbf{a}_r \text{ V/m}}$$

5.12a. (continued)

$$V = -\int_{r_2}^{r_1} \mathbf{E}\cdot d\mathbf{L} = -\int_{.05}^{.03} \frac{4.77}{r^2}\,dr = 4.77\left[\frac{1}{.03} - \frac{1}{.05}\right] = \underline{63.7\text{ V}}$$

Finally, $R = V/I = 63.7/3 = \underline{21.2\ \Omega}$.

b) Repeat if $\sigma = 0.0005/r$ for $3 < r < 5$ cm: First, $\mathbf{J} = 3\mathbf{a}_r/(4\pi r^2)$ as before. The electric field is now

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{3r\mathbf{a}_r}{4\pi(.0005)r^2} = \underline{\frac{477}{r}\,\mathbf{a}_r\text{ V/m}}$$

Now

$$V = -\int_{r_2}^{r_1} \mathbf{E}\cdot d\mathbf{L} = -\int_{.05}^{.03} \frac{477}{r}\,dr = -477\ln\left(\frac{.03}{.05}\right) = \underline{244\text{ V}}$$

Finally, $R = V/I = 244/3 = \underline{81.3\ \Omega}$.

c) Show that integrating the power dissipated per unit volume in part *b* over the volume gives the total dissipated power: The dissipated power density is

$$p_d = \mathbf{E}\cdot\mathbf{J} = \left(\frac{3}{4\pi(.0005)r}\right)\left(\frac{3}{4\pi r^2}\right) = \frac{114}{r^3}\text{ W/m}^2$$

We integrate this over the volume between spheres:

$$P_d = \int_0^{2\pi}\int_0^{\pi}\int_{.03}^{.05} \frac{114}{r^3}\,r^2\sin\theta\,dr\,d\theta\,d\phi = 4\pi(114)\ln\left(\frac{5}{3}\right) = \underline{732\text{ W}}$$

The dissipated power should be just $I^2R = (3)^2(81.3) = 732$ W. So it works.

5.13. A hollow cylindrical tube with a rectangular cross-section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5\times10^7$ S/m. A current of 200 A dc is flowing down the tube.

a) What voltage drop is present across a 1m length of the tube? Converting all measurements to meters, the tube resistance over a 1 m length will be:

$$R_1 = \frac{1}{(1.5\times10^7)\left[(2.54)(2.54/2)\times10^{-4} - 2.54(1-.1)(2.54/2)(1-.2)\times10^{-4}\right]}$$
$$= 7.38\times10^{-4}\ \Omega$$

The voltage drop is now $V = IR_1 = 200(7.38\times10^{-4} = \underline{0.147\text{ V}}$.

b) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5\times10^5$ S/m: The resistance of the filling will be:

$$R_2 = \frac{1}{(1.5\times10^5)(1/2)(2.54)^2\times10^{-4}(.9)(.8)} = 2.87\times10^{-2}\ \Omega$$

The total resistance is now the parallel combination of R_1 and R_2:
$R_T = R_1R_2/(R_1+R_2) = 7.19\times10^{-4}\ \Omega$, and the voltage drop is now $V = 200R_T = \underline{.144\text{ V}}$.

5.14. Find the magnitude of the electric field intensity in a conductor if:

a) the current density is 5 MA/m^2, the electron mobility is 3×10^{-3} m^2/V $\cdot$ s, and the volume charge density is -2.4×10^{10} C/m^3: In magnitude, we have

$$E=\frac{J}{\mu_e\rho_v}=\frac{5\times10^6}{(2.4\times10^{10})(3\times10^{-3})}=\underline{6.9\times10^{-2}\text{ V/m}}$$

b) $J=3$ MA/m^2 and the resistivity is $3\times10^{-8}\ \Omega\cdot$m: $E=J\rho=(3\times10^6)(3\times10^{-8})=\underline{9\times10^{-2}\text{ V/m}}$.

5.15. Let $V=10(\rho+1)z^2\cos\phi$ V in free space.

a) Let the equipotential surface $V=20$ V define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\underline{(\rho+1)z^2\cos\phi=2}$$

b) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi=0.2\pi$ and $z=1.5$: At the given values of ϕ and z, we solve the equation of the surface found in part a for ρ, obtaining $\rho=\underline{.10}$. Then

$$\begin{aligned}\mathbf{E}=-\nabla V&=-\frac{\partial V}{\partial\rho}\mathbf{a}_\rho-\frac{1}{\rho}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi-\frac{\partial V}{\partial z}\mathbf{a}_z\\&=-10z^2\cos\phi\,\mathbf{a}_\rho+10\,\frac{\rho+1}{\rho}z^2\sin\phi\,\mathbf{a}_\phi-20(\rho+1)z\cos\phi\,\mathbf{a}_z\end{aligned}$$

Then

$$\mathbf{E}(.10,.2\pi,1.5)=\underline{-18.2\,\mathbf{a}_\rho+145\,\mathbf{a}_\phi-26.7\,\mathbf{a}_z\text{ V/m}}$$

c) Find $|\rho_s|$ at that point: Since $\mathbf{E}$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s=\epsilon_0\mathbf{E}\cdot\mathbf{n}\Big|_{\text{surface}}=\epsilon_0\frac{\mathbf{E}\cdot\mathbf{E}}{|\mathbf{E}|}=\epsilon_0\sqrt{\mathbf{E}\cdot\mathbf{E}}=\epsilon_0\sqrt{(18.2)^2+(145)^2+(26.7)^2}=\underline{1.32\text{ nC/m}^2}$$

5.16. A potential field in free space is given as $V=(80\cos\theta\sin\phi)/r^3$ V. Point $P(r=2,\theta=\pi/3,\phi=\pi/2)$ lies on a conducting surface.

a) Write the equation of the conducting surface: The surface will be an equipotential, where the value of the potential is V_P:

$$V_P=\frac{80\cos(\pi/3)\sin(\pi/2)}{(2)^3}=5$$

So the equation of the surface is

$$\frac{80\cos\theta\sin\phi}{r^3}=5\ \text{ or }\ \underline{16\cos\theta\sin\phi=r^3}$$

5.16c. (I will work parts *b* and *c* in reverse order)

Find **E** at P:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\,\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\,\mathbf{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi$$
$$= \frac{80(3)\cos\theta\sin\phi}{r^4}\,\mathbf{a}_r + \frac{80\sin\theta\sin\phi}{r^4}\,\mathbf{a}_\theta - \frac{80\cos\theta\cos\phi}{r^4\sin\theta}\,\mathbf{a}_\phi$$

Now

$$\mathbf{E}_P = \frac{80(1/2)(1)(3)}{16}\,\mathbf{a}_r + \frac{80(\sqrt{3}/2)(1)}{16}\,\mathbf{a}_\theta - 0\,\mathbf{a}_\phi = \underline{7.5\,\mathbf{a}_r + 4.3\,\mathbf{a}_\theta \text{ V/m}}$$

b) Find a unit vector directed outward to the surface, assuming the origin is inside the surface: Such a unit normal can be construced from the result of part *c*:

$$\mathbf{a}_N = \frac{7.5\,\mathbf{a}_r + 4.3\,\mathbf{a}_\theta}{4.33} = \underline{0.87\,\mathbf{a}_r + 0.50\,\mathbf{a}_\theta}$$

5.17. Given the potential field

$$V = \frac{100xz}{x^2+4}\text{ V}$$

in free space:

a) Find **D** at the surface $z = 0$: Use

$$\mathbf{E} = -\nabla V = -100z\frac{\partial}{\partial x}\left(\frac{x}{x^2+4}\right)\mathbf{a}_x - 0\,\mathbf{a}_y - \frac{100x}{x^2+4}\,\mathbf{a}_z \text{ V/m}$$

At $z = 0$, we use this to find

$$\mathbf{D}(z=0) = \epsilon_0\mathbf{E}(z=0) = \underline{-\frac{100\epsilon_0 x}{x^2+4}\,\mathbf{a}_z \text{ C/m}^2}$$

b) Show that the $z = 0$ surface is an equipotential surface: There are two reasons for this: 1) **E** at $z = 0$ is everywhere z-directed, and so moving a charge around on the surface involves doing no work; 2) When evaluating the given potential function at $z = 0$, the result is 0 for all x and y.

c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$: We have

$$\rho_s = \mathbf{D}\cdot\mathbf{a}_z\Big|_{z=0} = -\frac{100\epsilon_0 x}{x^2+4}\text{ C/m}^2$$

So

$$Q = \int_{-3}^{0}\int_{0}^{2} -\frac{100\epsilon_0 x}{x^2+4}\,dx\,dy = -(3)(100)\epsilon_0\left(\frac{1}{2}\right)\ln(x^2+4)\Big|_0^2 = -150\epsilon_0\ln 2 = \underline{-0.92\text{ nC}}$$

5.18. Let us assume a field $\mathbf{E} = 3y^2z^3\,\mathbf{a}_x + 6xyz^3\,\mathbf{a}_y + 9xy^2z^2\,\mathbf{a}_z$ V/m in free space, and also assume that point $P(2, 1, 0)$ lies on a conducting surface.

a) Find ρ_v just adjacent to the surface at P:

$$\rho_v = \nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} = 6xz^3 + 18xy^2z \text{ C/m}^3$$

Then at P, $\rho_v = \underline{0}$, since $z = 0$.

b) Find ρ_s at P:

$$\rho_s = \mathbf{D} \cdot \mathbf{n}\Big|_P = \epsilon_0 \mathbf{E}\dot{\mathbf{n}}\Big|_P$$

Note however, that this computation involves evaluating $\mathbf{E}$ at the surface, yielding a value of 0. Therefore the surface charge density at P is $\underline{0}$.

c) Show that $V = -3xy^2z^3$ V: The simplest way to show this is just to take $-\nabla V$, which yields the given field: A more general method involves deriving the potential from the given field: We write

$$E_x = -\frac{\partial V}{\partial x} = 3y^2z^3 \;\Rightarrow\; V = -3xy^2z^3 + f(y, z)$$

$$E_y = -\frac{\partial V}{\partial y} = 6xyz^3 \;\Rightarrow\; V = -3xy^2z^3 + f(x, z)$$

$$E_z = -\frac{\partial V}{\partial z} = 9xy^2z^2 \;\Rightarrow\; V = -3xy^2z^3 + f(x, y)$$

where the integration "constants" are functions of all variables other than the integration variable. The general procedure is to adjust the functions, f, such that the result for V is the same in all three integrations. In this case we see that $f(x, y) = f(x, z) = f(y, z) = 0$ accomplishes this, and the potential function is $V = -3xy^2z^3$ as given.

d) Determine V_{PQ}, given $Q(1, 1, 1)$: Using the potential function of part c, we have

$$V_{PQ} = V_P - V_Q = 0 - (-3) = \underline{3 \text{ V}}$$

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At } 0 \text{ V}: \quad 2x^2y - z = 0$$

$$\text{At } 60 \text{ V}: \quad 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

5.19b. (continued) Now

$$\mathbf{E} = -\nabla V = -40xyz\,\mathbf{a}_x - 20x^2z\,\mathbf{a}_y - [20xy - 20z]\,\mathbf{a}_z$$

Then, at the given point, we have

$$\mathbf{D}(2, 7/8, 1) = \epsilon_0\mathbf{E}(2, 7/8, 1) = -\epsilon_0[70\,\mathbf{a}_x + 80\,\mathbf{a}_y + 50\,\mathbf{a}_z]\,\text{C/m}^2$$

We know that since this is the higher potential surface, **D** must be directed away from it, and so the charge density would be positive. Thus

$$\rho_s = \sqrt{\mathbf{D}\cdot\mathbf{D}} = 10\epsilon_0\sqrt{7^2 + 8^2 + 5^2} = \underline{1.04\,\text{nC/m}^2}$$

c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface: This will be in the direction of **E** and **D** as found in part *b*, or

$$\mathbf{a}_n = -\left[\frac{7\mathbf{a}_x + 8\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{7^2 + 8^2 + 5^2}}\right] = \underline{-[0.60\mathbf{a}_x + 0.68\mathbf{a}_y + 0.43\mathbf{a}_z]}$$

5.20. A conducting plane is located at $z = 0$ in free space, and a 20 nC point charge is present at $Q(2, 4, 6)$.

a) If $V = 0$ at $z = 0$, find V at $P(5, 3, 1)$: The plane can be replaced by an image charge of -20 nC at $Q'(2, 4, -6)$. Vectors **R** and $\mathbf{R}'$ directed from Q and Q' to P are $\mathbf{R} = (5, 3, 1) - (2, 4, 6) = (3, -1, -5)$ and $\mathbf{R}' = (5, 3, 1) - (2, 4, -6) = (3, -1, 7)$. Their magnitudes are $R = \sqrt{35}$ and $R' = \sqrt{59}$. The potential at P is given by

$$V_P = \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_0 R'} = \frac{20\times10^{-9}}{4\pi\epsilon_0\sqrt{35}} - \frac{20\times10^{-9}}{4\pi\epsilon_0\sqrt{59}} = \underline{7.0\text{ V}}$$

b) Find **E** at P:

$$\begin{aligned}\mathbf{E}_P &= \frac{q\mathbf{R}}{4\pi\epsilon_0 R^3} - \frac{q\mathbf{R}'}{4\pi\epsilon_0 (R')^3} = \frac{(20\times10^{-9})(3, -1, -5)}{4\pi\epsilon_0(35)^{3/2}} - \frac{(20\times10^{-9})(3, -1, 7)}{4\pi\epsilon_0(59)^{3/2}} \\ &= \frac{20\times10^{-9}}{4\pi\epsilon_0}\left[(3\mathbf{a}_x - \mathbf{a}_y)\left(\frac{1}{(35)^{3/2}} - \frac{1}{(59)^{3/2}}\right) - \left(\frac{7}{(59)^{3/2}} + \frac{5}{(35)^{3/2}}\right)\mathbf{a}_z\right] \\ &= \underline{1.4\mathbf{a}_x - 0.47\mathbf{a}_y - 7.1\mathbf{a}_z\text{ V/m}}\end{aligned}$$

c) Find ρ_s at $A(5, 3, 0)$: First, find the electric field there:

$$\mathbf{E}_A = \frac{20\times10^{-9}}{4\pi\epsilon_0}\left[\frac{(5, 3, 0) - (2, 4, 6)}{(46)^{3/2}} - \frac{(5, 3, 0) - (2, 4, -6)}{(46)^{3/2}}\right] = -6.9\mathbf{a}_z\text{ V/m}$$

Then $\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_{surface} = -6.9\epsilon_0\mathbf{a}_z\cdot\mathbf{a}_z = \underline{-61\text{ pC/m}^2}$.

5.21. Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$.

a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1, 2, 0)$: The line charges will image across the plane, producing image line charges of -30 nC/m each at $x = 0$, $y = -1$, and $x = 0$, $y = -2$. We find the potential at P by evaluating the work done in moving a unit positive charge from the $y = 0$ plane (we choose the origin) to P: For each line charge, this will be:

$$V_P - V_{0,0,0} = -\frac{\rho_l}{2\pi\epsilon_0} \ln\left[\frac{\text{final distance from charge}}{\text{initial distance from charge}}\right]$$

where $V_{0,0,0} = 0$. Considering the four charges, we thus have

$$\begin{aligned} V_P &= -\frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{2}}{1}\right) - \ln\left(\frac{\sqrt{10}}{1}\right) - \ln\left(\frac{\sqrt{17}}{2}\right)\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\ln(2) + \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{10}\right) + \ln\left(\frac{\sqrt{17}}{2}\right)\right] = \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{10}\sqrt{17}}{\sqrt{2}}\right] \\ &= \underline{1.20\,\text{kV}} \end{aligned}$$

b) Find $\mathbf{E}$ at P: Use

$$\begin{aligned} \mathbf{E}_P &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,2,0)-(0,1,0)}{|(1,1,0)|^2} + \frac{(1,2,0)-(0,2,0)}{|(1,0,0)|^2}\right. \\ &\quad \left. - \frac{(1,2,0)-(0,-1,0)}{|(1,3,0)|^2} - \frac{(1,2,0)-(0,-2,0)}{|(1,4,0)|^2}\right] \\ &= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,1,0)}{2} + \frac{(1,0,0)}{1} - \frac{(1,3,0)}{10} - \frac{(1,4,0)}{17}\right] = \underline{723\,\mathbf{a}_x - 18.9\,\mathbf{a}_y\ \text{V/m}} \end{aligned}$$

5.22. Let the plane $x = 0$ be a perfect conductor in free space. Locate a point charge of 4nC at $P_1(7, 1, -2)$ and a point charge of $-3nC$ at $P_2(4, 2, 1)$.

a) Find $\mathbf{E}$ at $A(5, 0, 0)$: Image charges will be located at $P_1'(-7, 1, -2)$ (-4nC) and at $P_2'(-4, 2, 1)$ (3nC). Vectors from all four charges to point A are:

$$\mathbf{R}_1 = (5,0,0) - (7,1,-2) = (-2,-1,2)$$

$$\mathbf{R}_1' = (5,0,0) - (-7,1,-2) = (12,-1,2)$$

$$\mathbf{R}_2 = (5,0,0) - (4,2,1) = (1,-2,-1)$$

and

$$\mathbf{R}_2' = (5,0,0) - (-4,2,1) = (9,-2,-1)$$

Replacing the plane by the image charges enables the field at A to be calculated through:

$$\begin{aligned} \mathbf{E}_A &= \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{(4)(-2,-1,2)}{9^{3/2}} - \frac{(3)(1,-2,-1)}{6^{3/2}} - \frac{(4)(12,-1,2)}{(149)^{3/2}} + \frac{(3)(9,-2,-1)}{(86)^{3/2}}\right] \\ &= \underline{-4.43\mathbf{a}_x + 2.23\mathbf{a}_y + 4.42\mathbf{a}_z\ \text{V/m}} \end{aligned}$$

5.22b. Find $|\rho_s|$ at $B(0, 0, 0)$ (note error in problem statement): First, $\mathbf{E}$ at the origin is done as per the setup in part a, except the vectors are directed from the charges to the origin:

$$\mathbf{E}_B = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{(4)(-7,-1,2)}{(54)^{3/2}} - \frac{(3)(-4,-2,-1)}{(21)^{3/2}} - \frac{(4)(7,-1,2)}{(54)^{3/2}} + \frac{(3)(4,-2,-1)}{(21)^{3/2}}\right]$$

Now $\rho_s = \mathbf{D}\cdot\mathbf{n}|_{surface} = \mathbf{D}\cdot\mathbf{a}_x$ in our case (note the other components cancel anyway as they must, but we still need to express ρ_s as a scalar):

$$\rho_{sB} = \epsilon_0\mathbf{E}_B\cdot\mathbf{a}_x = \frac{10^{-9}}{4\pi}\left[\frac{(4)(-7)}{(54)^{3/2}} - \frac{(3)(-4)}{(21)^{3/2}} - \frac{(4)(7)}{(54)^{3/2}} + \frac{(3)(4)}{(21)^{3/2}}\right] = \underline{8.62\,\text{pC/m}^2}$$

5.23. A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\text{C}\cdot\text{m}$ is located at $A(1, 0, 0)$ in free space, and the $x = 0$ plane is perfectly-conducting.

a) Find V at $P(2, 0, 1)$. We use the far-field potential for a z-directed dipole:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0}\frac{z}{[x^2+y^2+z^2]^{1.5}}$$

The dipole at $x = 1$ will image in the plane to produce a second dipole of the opposite orientation at $x = -1$. The potential at any point is now:

$$V = \frac{p}{4\pi\epsilon_0}\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right]$$

Substituting $P(2, 0, 1)$, we find

$$V = \frac{.1\times10^6}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right] = \underline{289.5\ \text{V}}$$

b) Find the equation of the 200-V equipotential surface in cartesian coordinates: We just set the potential exression of part a equal to 200 V to obtain:

$$\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right] = 0.222$$

5.24. The mobilities for intrinsic silicon at a certain temperature are $\mu_e = 0.14\ \text{m}^2/\text{V}\cdot\text{s}$ and $\mu_h = 0.035\ \text{m}^2/\text{V}\cdot\text{s}$. The concentration of both holes and electrons is $2.2\times10^{16}\,\text{m}^{-3}$. Determine both the conductivity and the resistivity of this silicon sample: Use

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = (1.6\times10^{-19}\text{C})(2.2\times10^{16}\,\text{m}^{-3})(0.14\,\text{m}^2/\text{V}\cdot\text{s} + 0.035\,\text{m}^2/\text{V}\cdot\text{s})$$
$$= \underline{6.2\times10^{-4}\ \text{S/m}}$$

Conductivity is $\rho = 1/\sigma = \underline{1.6\times10^3\ \Omega\cdot\text{m}}$.

5.25. Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ C/m^3. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ m^2/V $\cdot$ s and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ m^2/V $\cdot$ s, where the temperature, T, *is in degrees Kelvin.* The conductivity will thus be

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = 6200T^{1.5}e^{-7000/T}\left[2.1 \times 10^5 T^{-2.5} + 2.3 \times 10^5 T^{-2.7}\right]$$
$$= \frac{1.30 \times 10^9}{T}e^{-7000/T}\left[1 + 1.095T^{-.2}\right] \text{ S/m}$$

Find σ at:

a) 0° C: With $T = 273$°K, the expression evaluates as $\sigma(0) = \underline{4.7 \times 10^{-5} \text{ S/m}}$.

b) 40° C: With $T = 273 + 40 = 313$, we obtain $\sigma(40) = \underline{1.1 \times 10^{-3} \text{ S/m}}$.

c) 80° C: With $T = 273 + 80 = 353$, we obtain $\sigma(80) = \underline{1.2 \times 10^{-2} \text{ S/m}}$.

5.26. A little donor impurity, such as arsenic, is added to pure silicon so that the electron concentration is 2×10^{17} conduction electrons per cubic meter while the number of holes per cubic meter is only 1.1×10^{15}. If $\mu_e = 0.15$ m^2/V $\cdot$ s for this sample, and $\mu_h = 0.045$ m^2/V $\cdot$ s, determine the conductivity and resistivity:

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = (1.6 \times 10^{-19})\left[(2 \times 10^{17})(0.15) + (1.1 \times 10^{15})(0.045)\right] = \underline{4.8 \times 10^{-3} \text{ S/m}}$$

Then $\rho = 1/\sigma = \underline{2.1 \times 10^2 \ \Omega \cdot \text{m}}$.

5.27. Atomic hydrogen contains 5.5×10^25 atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m.

a) Find P: With all identical dipoles, we have

$$P = Nqd = (5.5 \times 10^{25})(1.602 \times 10^{-19})(7.1 \times 10^{-19}) = 6.26 \times 10^{-12} \text{ C/m}^2 = \underline{6.26 \text{ pC/m}^2}$$

b) Find ϵ_R: We use $P = \epsilon_0\chi_e E$, and so

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

Then $\epsilon_R = 1 + \chi_e = \underline{1.000176}$.

5.28. In a certain region where the relative permittivity is 2.4, $\mathbf{D} = 2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z$ nC/m^2. Find:

$$a) \qquad \mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{(2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z) \times 10^{-9}}{(2.4)(8.85 \times 10^{-12})} = \underline{94\mathbf{a}_x - 188\mathbf{a}_y + 235\mathbf{a}_z \text{ V/m}}$$

$$b) \qquad \mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E} = \epsilon_0\mathbf{E}(\epsilon_R - 1) = \frac{(2\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z) \times 10^{-9}}{2.4}(2.4 - 1)$$
$$= \underline{1.2\mathbf{a}_x - 2.3\mathbf{a}_y + 2.9\mathbf{a}_z \text{ nC/m}^2}$$

$$c) \qquad |\nabla V| = |\mathbf{E}| = [(94.1)^2 + (188)^2 + (235)^2]^{1/2} = \underline{315 \text{ V/m}}$$

5.29. A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_R = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho$ nC/m^2 in the dielectric, find:

a) **D** and **E** as functions of ρ: Use

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0(\epsilon_R - 1)} = \frac{(2/\rho) \times 10^{-9}\mathbf{a}_\rho}{(8.85 \times 10^{-12})(1.56)} = \underline{\frac{144.9}{\rho}\mathbf{a}_\rho \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \frac{2 \times 10^{-9}\mathbf{a}_\rho}{\rho}\left[\frac{1}{1.56} + 1\right] = \frac{3.28 \times 10^{-9}\mathbf{a}_\rho}{\rho} \text{ C/m}^2 = \underline{\frac{3.28\mathbf{a}_\rho}{\rho} \text{ nC/m}^2}$$

b) Find V_{ab} and χ_e: Use

$$V_{ab} = -\int_3^{0.8} \frac{144.9}{\rho}\, d\rho = 144.9 \ln\left(\frac{3}{0.8}\right) = \underline{192 \text{ V}}$$

$\chi_e = \epsilon_r - 1 = \underline{1.56}$, as found in part a.

c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$: Use

$$\mathbf{p} = \frac{\mathbf{P}}{N} = \frac{(2 \times 10^{-9}/\rho)}{4 \times 10^{19}}\mathbf{a}_\rho = \underline{\frac{5.0 \times 10^{-29}}{\rho}\mathbf{a}_\rho \text{ C} \cdot \text{m}}$$

5.30. Given the potential field $V = 200 - 50x + 20y$ V in a dielectric material for which $\epsilon_R = 2.1$, find:

a) $\mathbf{E} = -\nabla V = \underline{50\mathbf{a}_x - 20\mathbf{a}_y \text{ V/m}}$.

b) $\mathbf{D} = \epsilon\mathbf{E} = (2.1)(8.85 \times 10^{-12})(50\mathbf{a}_x - 20\mathbf{a}_y) = \underline{930\mathbf{a}_x - 372\mathbf{a}_y \text{ pC/m}^2}$.

c) $\mathbf{P} = \epsilon_0\mathbf{E}(\epsilon_R - 1) = (8.85 \times 10^{-12})(50\mathbf{a}_x - 20\mathbf{a}_y)(1.1) = \underline{487\mathbf{a}_x - 195\mathbf{a}_y \text{ pC/m}^2}$.

d) $\rho_v = \nabla \cdot \mathbf{D} = \underline{0}$.

e) $\rho_b = -\nabla \cdot \mathbf{P} = \underline{0}$

f) $\rho_T = \nabla \cdot \epsilon_0\mathbf{E} = \underline{0}$

5.31. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_R = \epsilon_{R1} = 3$, while $\epsilon_{R2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find:

a) E_{N1}: This will be $\mathbf{E}_1 \cdot \mathbf{a}_x = \underline{80 \text{ V/m}}$.

b) $\mathbf{E}_{T1}$. This consists of components of $\mathbf{E}_1$ *not* normal to the surface, or $\mathbf{E}_{T1} = \underline{-60\mathbf{a}_y - 30\mathbf{a}_z \text{ V/m}}$.

c) $E_{T1} = \sqrt{(60)^2 + (30)^2} = \underline{67.1 \text{ V/m}}$.

d) $E_1 = \sqrt{(80)^2 + (60)^2 + (30)^2} = \underline{104.4 \text{ V/m}}$.

e) The angle θ_1 between $\mathbf{E}_1$ and a normal to the surface: Use

$$\cos\theta_1 = \frac{\mathbf{E}_1 \cdot \mathbf{a}_x}{E_1} = \frac{80}{104.4} \Rightarrow \theta_1 = \underline{40.0^\circ}$$

5.31 (continued)

f) $D_{N2} = D_{N1} = \epsilon_{R1}\epsilon_0 E_{N1} = 3(8.85 \times 10^{-12})(80) = \underline{2.12\,\text{nC/m}^2}$.

g) $D_{T2} = \epsilon_{R2}\epsilon_0 E_{T1} = 5(8.85 \times 10^{-12})(67.1) = \underline{2.97\,\text{nC/m}^2}$.

h) $\mathbf{D}_2 = \epsilon_{R1}\epsilon_0 E_{N1}\mathbf{a}_x + \epsilon_{R2}\epsilon_0 \mathbf{E}_{T1} = \underline{2.12\mathbf{a}_x - 2.66\mathbf{a}_y - 1.33\mathbf{a}_z\ \text{nC/m}^2}$.

i) $\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0\mathbf{E}_2 = \mathbf{D}_2\,[1 - (1/\epsilon_{R2})] = (4/5)\mathbf{D}_2 = \underline{1.70\mathbf{a}_x - 2.13\mathbf{a}_y - 1.06\mathbf{a}_z\ \text{nC/m}^2}$.

j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface: Use

$$\cos\theta_2 = \frac{\mathbf{E}_2 \cdot \mathbf{a}_x}{E_2} = \frac{\mathbf{D}_2 \cdot \mathbf{a}_x}{D_2} = \frac{2.12}{\sqrt{(2.12)^2 = (2.66)^2 + (1.33)^2}} = .581$$

Thus $\theta_2 = \cos^{-1}(.581) = \underline{54.5^\circ}$.

5.32. In Fig. 5.18, let $\mathbf{D} = 3\mathbf{a}_x - 4\mathbf{a}_y + 5\mathbf{a}_z\ \text{nC/m}^2$ and find:

a) $\mathbf{D}_2$: First, the electric field in region 1 is

$$\mathbf{E}_1 = \left[\frac{3}{2\epsilon_0}\mathbf{a}_x - \frac{4}{2\epsilon_0}\mathbf{a}_y + \frac{5}{2\epsilon_0}\mathbf{a}_z\right] \times 10^{-9}\ \text{V/m}$$

Since, at the dielectric interface, tangential electric field and normal electric flux density are continuous, we may write

$$\mathbf{D}_2 = \epsilon_{R2}\epsilon_0\mathbf{E}_{T1} + \mathbf{D}_{N1} = \left(\frac{5}{2}\right)3\mathbf{a}_x - \left(\frac{5}{2}\right)4\mathbf{a}_y + 5\mathbf{a}_z = \underline{7.5\mathbf{a}_x - 10\mathbf{a}_y + 5\mathbf{a}_z\ \text{nC/m}^2}$$

b) $\mathbf{D}_{N2} = \underline{5\mathbf{a}_z}$, as explained above.

c) $\mathbf{D}_{T2} = \epsilon_{R2}\epsilon_0\mathbf{E}_{T2} = \epsilon_{R2}\epsilon_0\mathbf{E}_{T1} = \underline{7.5\mathbf{a}_x - 10\mathbf{a}_y\ \text{nC/m}^2}$.

d) the energy density in each region:

$$w_{e1} = \frac{1}{2}\epsilon_{R1}\epsilon_0\mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{1}{2}(2)\epsilon_0\left[\left(\frac{3}{2\epsilon_0}\right)^2 + \left(\frac{4}{2\epsilon_0}\right)^2 + \left(\frac{5}{2\epsilon_0}\right)^2\right] \times 10^{-18} = \underline{1.41\ \mu\text{J/m}^3}$$

$$w_{e2} = \frac{1}{2}\epsilon_{R2}\epsilon_0\mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{1}{2}(5)\epsilon_0\left[\left(\frac{3}{2\epsilon_0}\right)^2 + \left(\frac{4}{2\epsilon_0}\right)^2 + \left(\frac{5}{5\epsilon_0}\right)^2\right] \times 10^{-18} = \underline{2.04\ \mu\text{J/m}^3}$$

e) the angle that $\mathbf{D}_2$ makes with $\mathbf{a}_z$: Use $\mathbf{D}_2 \cdot \mathbf{a}_z = |\mathbf{D}_2|\cos\theta = D_z = 5$. where $|\mathbf{D}_2| = \left[(7.5)^2 + (10)^2 + (5)^2\right]^{1/2} = 13.5$. So $\theta = \cos^{-1}(5/13.5) = \underline{68^\circ}$.

f) $D_2/D_1 = \left[(7.5)^2 + (10)^2 + (5)^2\right]^{1/2} / \left[(3)^2 + (4)^2 + (5)^2\right]^{1/2} = \underline{1.91}$.

g) P_2/P_1: First $\mathbf{P}_1 = \epsilon_0\mathbf{E}_1(\epsilon_{R1} - 1) = 1.5\mathbf{a}_x - 2\mathbf{a}_y + 2.5\mathbf{a}_z\ \text{nC/m}^2$.
Then $\mathbf{P}_2 = \epsilon_0\mathbf{E}_2(\epsilon_{R2} - 1) = 6\mathbf{a}_x - 8\mathbf{a}_y + 4\mathbf{a}_z\ \text{nC/m}^2$. So

$$\frac{P_2}{P_1} = \frac{[(6)^2 + (8)^2 + (4)^2]^{1/2}}{[(1.5)^2 + (2)^2 + (2.5)^2]^{1/2}} = \underline{3.04}$$

5.33. Two perfect dielectrics have relative permittivities $\epsilon_{R1} = 2$ and $\epsilon_{R2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$: We need to find the components of $\mathbf{E}_1$ that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = \mathbf{E}_1 \cdot \mathbf{n}$. Taking $f = x - y + 2z$, the unit vector that is normal to the surface is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right]$$

This normal will point in the direction of increasing f, which will be away from the origin, or into region 2 (you can visualize a portion of the surface as a triangle whose vertices are on the three coordinate axes at $x = 5$, $y = -5$, and $z = 2.5$). So $E_{N1} = (1/\sqrt{6})[100 - 200 - 100] = -81.7$ V/m. Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$\mathbf{E}_{N1} = -81.65\left(\frac{1}{\sqrt{6}}\right)[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z] = -33.33\mathbf{a}_x + 33.33\mathbf{a}_y - 66.67\mathbf{a}_z \text{ V/m}$$

Now, the tangential component will be

$$\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z$$

Our boundary conditions state that $\mathbf{E}_{T2} = \mathbf{E}_{T1}$ and $\mathbf{E}_{N2} = (\epsilon_{R1}/\epsilon_{R2})\mathbf{E}_{N1} = (1/4)\mathbf{E}_{N1}$. Thus

$$\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = \mathbf{E}_{T1} + \frac{1}{4}\mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z - 8.3\mathbf{a}_x + 8.3\mathbf{a}_y - 16.67\mathbf{a}_z$$
$$= \underline{125\mathbf{a}_x + 175\mathbf{a}_y \text{ V/m}}$$

5.34. Let the spherical surfaces $r = 4$ cm and $r = 9$ cm be separated by two perfect dielectric shells, $\epsilon_{R1} = 2$ for $4 < r < 6$ cm and $\epsilon_{R2} = 5$ for $6 < r < 9$ cm. If $\mathbf{E}_1 = (2000/r^2)\mathbf{a}_r$ V/m, find:

a) $\mathbf{E}_2$: Since $\mathbf{E}$ is normal to the interface between ϵ_{R1} and ϵ_{R2}, $\mathbf{D}$ will be continuous across the boundary, and so

$$\mathbf{D}_1 = \frac{2\epsilon_0(2000)}{r^2}\mathbf{a}_r = \mathbf{D}_2$$

Then

$$\mathbf{E}_2 = \frac{\mathbf{D}_2}{5\epsilon_0} = \left(\frac{2}{5}\right)\frac{2000}{r^2}\mathbf{a}_r = \underline{\frac{800}{r^2}\mathbf{a}_r \text{ V/m}}$$

b) the total electrostatic energy stored in each region: In region 1, the energy density is

$$w_{e1} = \frac{1}{2}\epsilon_{R1}\epsilon_0|\mathbf{E}_1|^2 = \frac{1}{2}(2)\epsilon_0\frac{(2000)^2}{r^4} \text{ J/m}^3$$

In region 2:

$$w_{e2} = \frac{1}{2}\epsilon_{R2}\epsilon_0|\mathbf{E}_2|^2 = \frac{1}{2}(5)\epsilon_0\frac{(800)^2}{r^4} \text{ J/m}^3$$

5.34. (continued)

The energies in each region are then

$$\text{Region 1:} \qquad W_{e1} = (2000)^2\epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_{.04}^{.06} \frac{1}{r^2}\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 4\pi\epsilon_0(2000)^2\left[\frac{1}{.04} - \frac{1}{.06}\right] = \underline{3.7 \text{ mJ}}$$

$$\text{Region 2:} \qquad W_{e2} = (800)^2\left(\frac{5}{2}\right)\epsilon_0 \int_0^{2\pi}\int_0^{\pi}\int_{.06}^{.09} \frac{1}{r^2}\, r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= 4\pi\epsilon_0(800)^2\left(\frac{5}{2}\right)\left[\frac{1}{.06} - \frac{1}{.09}\right] = \underline{0.99 \text{ mJ}}$$

5.35. Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{R1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{R2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{E}_2$: The interfaces between the two media will lie on planes of constant ϕ, to which $\mathbf{E}_1$ is parallel. Thus the field is the same on either side of the boundaries, and so $\underline{\mathbf{E}_2 = \mathbf{E}_1}$.

b) the total electrostatic energy stored in a 1m length of each region: In general we have $w_E = (1/2)\epsilon_R\epsilon_0 E^2$. So in region 1:

$$W_{E1} = \int_0^1\int_0^{\pi/2}\int_4^9 \frac{1}{2}(2)\epsilon_0\frac{(2000)^2}{\rho_2}\,\rho\, d\rho\, d\phi\, dz = \frac{\pi}{2}\epsilon_0(2000)^2\ln\left(\frac{9}{4}\right) = \underline{45.1\,\mu\text{J}}$$

In region 2, we have

$$W_{E2} = \int_0^1\int_{\pi/2}^{2\pi}\int_4^9 \frac{1}{2}(5)\epsilon_0\frac{(2000)^2}{\rho_2}\,\rho\, d\rho\, d\phi\, dz = \frac{15\pi}{4}\epsilon_0(2000)^2\ln\left(\frac{9}{4}\right) = \underline{338\,\mu\text{J}}$$

5.36. Let $S = 120\,\text{cm}^2$, $d = 4$ mm, and $\epsilon_R = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:
$C = \epsilon_R\epsilon_0 S/d = [12\epsilon_0(120\times10^{-4})]/[4\times10^{-3}] = 3.19\times10^{-10} = \underline{319\,\text{pF}}$.

b) After connecting a 40 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: $E = V/d = 40/(4\times10^{-3}) = \underline{10^4\,\text{V/m}}$. $D = \epsilon_R\epsilon_0 E = 12\epsilon_0\times10^4 = \underline{1.06\,\mu\text{C/m}^2}$. Then $Q = \mathbf{D}\cdot\mathbf{n}|_{surface}\times S = 1.06\times10^{-6}\times(120\times10^{-4}) = 1.27\times10^{-8}\text{C} = \underline{12.7\,\text{nC}}$. Finally $W_e = (1/2)CV_0^2 = (1/2)(319\times10^{-12})(40)^2 = \underline{255\,\text{nJ}}$.

c) The source is now removed and the dielectric is carefully withdrawn from between the plates. Again calculate E, D, Q, and the energy: With the source disconnected, the charge is constant, and thus so is D: Therefore, $Q = \underline{12.7\,\text{nC}}$, $D = \underline{1.06\,\mu\text{C/m}^2}$, and $E = D/\epsilon_0 = 10^4/8.85\times10^{-12} = \underline{1.2\times10^5\,\text{V/m}}$. The energy is then

$$W_e = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\times S = \frac{1}{2}(1.06\times10^{-6})(1.2\times10^5)(120\times10^{-4})(4\times10^{-3}) = \underline{3.05\,\mu\text{J}}$$

d) What is the voltage between the plates? $V = E\times d = (1.2\times10^5)(4\times10^{-3}) = \underline{480\,\text{V}}$.

5.37. Capacitors tend to be more expensive as their capacitance and maximum voltage, V_{max}, increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: $\epsilon_R = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_R = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_R = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_R = 2.26$, $E_{BD} = 4.7$ MV/m? Note that $V_{max} = EBDd$, where d is the plate separation. Also, $C = \epsilon_R\epsilon_0 A/d$, and so $V_{max}C = \epsilon_R\epsilon_0 AE_{BD}$, where A is the plate area. The maximum CV_{max} product is found through the maximum $\epsilon_R E_{BD}$ product. Trying this with the given materials yields the winner, which is barium titanate.

5.38. A dielectric circular cylinder used between the plates of a capacitor has a thickness of 0.2 mm and a radius of 1.4 cm. The dielectric properties are $\epsilon_R = 400$ and $\sigma = 10^{-5}$ S/m.

a) Calculate C:

$$C = \frac{\epsilon_R\epsilon_0 S}{d} = \frac{(400)(8.854\times10^{-12})\pi(1.4\times10^{-2})^2}{2\times10^{-4}} = 1.09\times10^{-8} = \underline{10.9\text{ nF}}$$

b) Find the quality factor Q_{QF} ($Q_QF = \omega RC$) of the capacitor at $f = 10$ kHz: Use the relation $RC = \epsilon/\sigma$ to write

$$Q_{QF} = \omega RC = \frac{2\pi f\epsilon}{\sigma} = \frac{(2\pi\times10^4)(400)(8.854\times10^{-12})}{10^{-5}} = \underline{22.3}$$

c) If the maximum field strength permitted in the dielectric is 2 MV/m, what is the maximum permissible voltage across the capacitor? $V_{max} = E_{BD}d = (2\times10^6)(2\times10^{-4}) = \underline{400\text{ V}}$.

d) What energy is stored when this voltage is applied?

$$W_{e,max} = \frac{1}{2}CV_{max}^2 = \frac{1}{2}(10.9\times10^{-9})(400)^2 = 8.7\times10^{-4} = \underline{0.87\text{ mJ}}$$

5.39. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_R = 2 + 2\times10^6x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. **D** will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that **D** *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that **D**, which is normal to the layers, will be continuous across each boundary, and so **D** is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0\epsilon_R} = \frac{\rho_s}{\epsilon_0(2+2\times10^6x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\,dx}{\epsilon_0(2+2\times10^6x^2)} = \frac{\rho_s}{\epsilon_0}\frac{1}{\sqrt{4\times10^6}}\tan^{-1}\left(\frac{x\sqrt{4\times10^6}}{2}\right)\Bigg|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0}\frac{1}{2\times10^3}\left(\frac{\pi}{4}\right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s(.02)\epsilon_0(2\times10^3)(4)}{\rho_s\pi} = 4.51\times10^{-10}\,\text{F} = \underline{451\,\text{pF}}$$

5.40a. The width of the region containing ϵ_{R1} in Fig. 5.19 is 1.2 m. Find ϵ_{R1} if $\epsilon_{R2} = 2.5$ and the total capacitance is 60 nF: The plate areas associated with each capacitor are $A_1 = 1.2(2) = 2.4$ m^2 and $A_2 = 0.8(2) = 1.6$ m^2. Having parallel capacitors, the capacitances will add, so

$$C = C_1 + C_2 \;\Rightarrow\; 60 \times 10^{-9} = \frac{\epsilon_{R1}\epsilon_0(2.4)}{2 \times 10^{-3}} + \frac{2.5\epsilon_0(1.6)}{2 \times 10^{-3}}$$

Solve this to obtain $\epsilon_{R1} = \underline{4.0}$.

b) Find the width of each region (containing ϵ_{R1} and ϵ_{R2}) if $C_{total} = 80$ nF, $\epsilon_{R2} = 3\epsilon_{R1}$, and $C_1 = 2C_2$: Let w_1 be the width of region 1. The above conditions enable us to write:

$$\left[\frac{\epsilon_{R1}\epsilon_0 w_1(2)}{2 \times 10^{-3}}\right] = 2\left[\frac{3\epsilon_{R1}\epsilon_0(2 - w_1)(2)}{2 \times 10^{-3}}\right] \;\Rightarrow\; w_1 = 6(2 - w_1)$$

So that $w_1 = 12/7 = \underline{1.7\,\text{m}}$ and $w_2 = \underline{0.3\,\text{m}}$.

5.41. Let $\epsilon_{R1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{R2} = 4$ for $1 < y < 3$ mm, and ϵ_{R3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: a) ϵ_{R3} is that of air; b) $\epsilon_{R3} = \epsilon_{R1}$; c) $\epsilon_{R3} = \epsilon_{R2}$; d) ϵ_{R3} is silver: The combination will be three capacitors in series, for which

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{d_1}{\epsilon_{R1}\epsilon_0(1)} + \frac{d_2}{\epsilon_{R2}\epsilon_0(1)} + \frac{d_3}{\epsilon_{R3}\epsilon_0(1)} = \frac{10^{-3}}{\epsilon_0}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\epsilon_{R3}}\right]$$

So that

$$C = \frac{(5 \times 10^{-3})\epsilon_0\epsilon_{R3}}{10 + 4.5\epsilon_{R3}}$$

Evaluating this for the four cases, we find a) $C = \underline{3.05\,\text{nF}}$ for $\epsilon_{R3} = 1$, b) $C = \underline{5.21\,\text{nF}}$ for $\epsilon_{R3} = 2.5$, c) $C = \underline{6.32\,\text{nF}}$ for $\epsilon_{R3} = 4$, and d) $C = \underline{9.83\,\text{nF}}$ if silver (taken as a perfect conductor) forms region 3; this has the effect of removing the term involving ϵ_{R3} from the original formula (first equation line), or equivalently, allowing ϵ_{R3} to approach infinity.

5.42. Cylindrical conducting surfaces are located at $\rho = 0.8$ cm and 3.6 cm. The region $0.8 < \rho < a$ contains a dielectric for which $\epsilon_R = 4$, while $\epsilon_R = 2$ for $a < \rho < 3.6$.

a) Find a so that the voltage across each dielectric layer is the same: Assuming charge density ρ_s on the inner cylinder, we have $\mathbf{D} = \rho_s(0.8)/\rho\,\mathbf{a}_\rho$, which gives $\mathbf{E}(0.8 < \rho < a) = (0.8\rho_s)/(4\epsilon_0\rho)\mathbf{a}_\rho$ and $\mathbf{E}(a < \rho < 3.6) = (0.8\rho_s)/(2\epsilon_0\rho)\mathbf{a}_\rho$. The voltage between conductors is now

$$V_0 = -\int_{3.6}^{a} \frac{0.8\rho_s}{2\epsilon_0\rho}\,d\rho - \int_{a}^{0.8} \frac{0.8\rho_s}{4\epsilon_0\rho}\,d\rho = \frac{0.8\rho_s}{2\epsilon_0}\left[\ln\left(\frac{3.6}{a}\right) + \frac{1}{2}\ln\left(\frac{a}{0.8}\right)\right]$$

We require

$$\ln\left(\frac{3.6}{a}\right) = \frac{1}{2}\ln\left(\frac{a}{0.8}\right) \;\Rightarrow\; \frac{3.6}{a} = \sqrt{\frac{a}{0.8}} \;\Rightarrow\; a = \underline{2.2\,\text{cm}}$$

b) Find the total capacitance per meter: Using the part a result, have

$$V_0 = \frac{0.8\rho_s}{2\epsilon_0}\left[\ln\left(\frac{3.6}{2.2}\right) + \frac{1}{2}\ln\left(\frac{2.2}{0.8}\right)\right] = \frac{0.4\rho_s}{\epsilon_0}$$

5.42b. (continued) The charge on a unit length of the inner conductor is $Q = 2\pi(0.8)(1)\rho_s$. The capacitance is now

$$C = \frac{Q}{V_0} = \frac{2\pi(0.8)(1)\rho_s}{0.4\rho_s/\epsilon_0} = 4\pi\epsilon_0 = \underline{111 \text{ pF/m}}$$

Note that throughout this problem, I left all dimensions in cm, knowing that all cm units would cancel, leaving the units of capacitance to be those used for ϵ_0.

5.43. Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_R = 4$. Find the capacitance if

a) $c = 2$ cm, $d = 3$ cm: This is two capacitors in series, and so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\frac{1}{4}\ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\right] \Rightarrow C = \underline{143 \text{ pF}}$$

b) $d = 4$ cm, and the volume of the dielectric is the same as in part a: Having equal volumes requires that $3^2 - 2^2 = 4^2 - c^2$, from which $c = 3.32$ cm. Now

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\ln\left(\frac{3.32}{2}\right) + \frac{1}{4}\ln\left(\frac{4}{3.32}\right)\right] \Rightarrow C = \underline{101 \text{ pF}}$$

5.44. Conducting cylinders lie at $\rho = 3$ and $\rho = 12$ mm; both extend from $z = 0$ to $z = 1$ m. Perfect dielectrics occupy the interior region: $\epsilon_R = 1$ for $3 < \rho < 6$ mm, $\epsilon_R = 4$ for $6 < \rho < 9$ mm, and $\epsilon_R = 8$ for $9 < \rho < 12$ mm.

a) Calculate C: First we know that $\mathbf{D} = (3\rho_s/\rho)\mathbf{a}_\rho$ C/m^2, with ρ expressed in mm. Then, with ρ in mm,

$$\mathbf{E}_1 = \frac{3\rho_s}{\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (3 < \rho < 6)$$

$$\mathbf{E}_2 = \frac{3\rho_s}{4\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (6 < \rho < 9)$$

and

$$\mathbf{E}_3 = \frac{3\rho_s}{8\epsilon_0\rho}\mathbf{a}_\rho \text{ V/m} \quad (9 < \rho < 12)$$

The voltage between conductors will be:

$$\begin{aligned} V_0 &= \left[-\int_{12}^{9}\frac{3\rho_s}{8\epsilon_0\rho}\,d\rho - \int_{9}^{6}\frac{3\rho_s}{4\epsilon_0\rho}\,d\rho - \int_{6}^{3}\frac{3\rho_s}{\epsilon_0\rho}\,d\rho\right] \times 10^{-3}(\text{m/mm}) \\ &= \frac{.003\rho_s}{\epsilon_0}\left[\frac{1}{8}\ln\left(\frac{12}{9}\right) + \frac{1}{4}\ln\left(\frac{9}{6}\right) + \ln\left(\frac{6}{3}\right)\right] = \frac{.003\rho_s}{\epsilon_0}(0.830) \text{ V} \end{aligned}$$

Now, the charge on the 1 m length of the inner conductor is $Q = 2\pi(.003)(1)\rho_s$. The capacitance is then

$$C = \frac{Q}{V_0} = \frac{2\pi(.003)(1)\rho_s}{(.003)\rho_s(.830)/\epsilon_0} = \frac{2\pi\epsilon_0}{.830} = \underline{67 \text{ pF}}$$

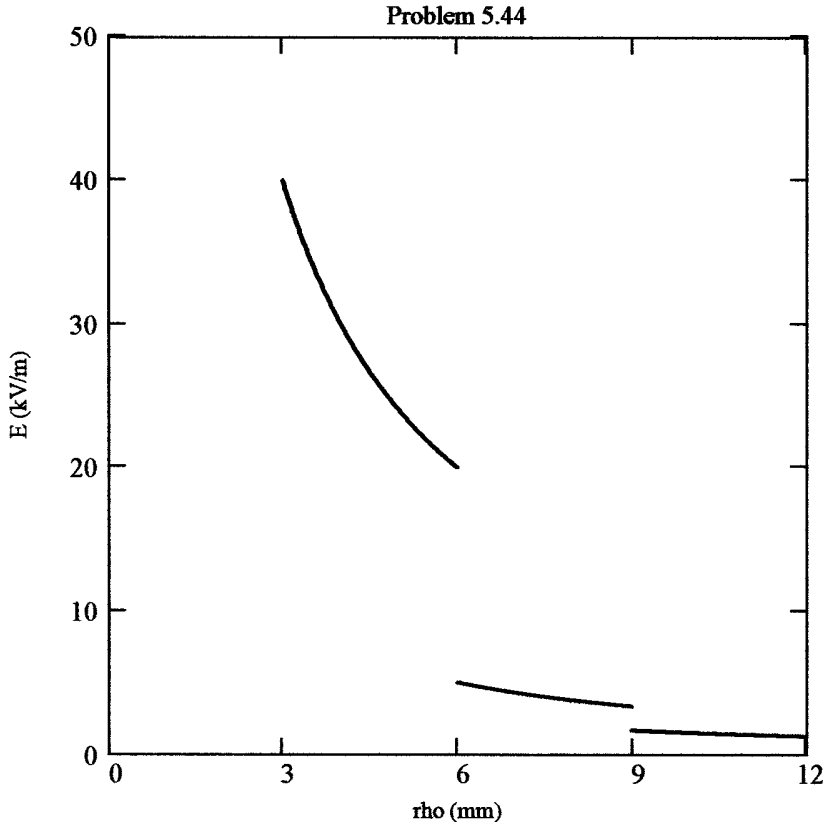
5.44b. If the voltage between the cylinders is 100 V, plot $|E_\rho|$ vs. ρ:
Have $Q = CV_0 = (67 \times 10^{-12})(100) = 6.7\text{nC}$. Then

$$\rho_s = \frac{6.7 \times 10^{-9}}{2\pi(.003)(1)} = 355 \text{ nC/m}^2$$

Then, using the electric field expressions from part a, we find

$$E_1 = \left(\frac{3}{\rho}\right) \frac{355 \times 10^{-9}}{8.854 \times 10^{-12}} = \frac{12 \times 10^4}{\rho} \text{ V/m} = \frac{120}{\rho} \text{ kV/m } (3 < \rho < 6)$$

where ρ is expressed in mm. Similarly, we find $E_2 = E_1/4 = 30/\rho$ kV/m $(6 < \rho < 9)$ and $E_3 = E_1/8 = 15$ kV/m $(9 < \rho < 12)$. These fields are plotted below.



5.45. Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_R = 8$.

a) Find C: For a spherical capacitor, we know that:

$$C = \frac{4\pi\epsilon_R\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi(8)\epsilon_0}{\left(\frac{1}{3} - \frac{1}{6}\right)(100)} = 1.92\pi\epsilon_0 = \underline{53.3\,\text{pF}}$$

b) A portion of the dielectric is now removed so that $\epsilon_R = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_R = 8$, $\pi/2 < \phi < 2\pi$. Again, find C: We recognize here that removing that portion leaves us with two capacitors in parallel (whose C's will add). We use the fact that with the dielectric *completely* removed, the capacitance would be $C(\epsilon_R = 1) = 53.3/8 = 6.67\,\text{pF}$. With one-fourth the dielectric removed, the total capacitance will be

$$C = \frac{1}{4}(6.67) + \frac{3}{4}(53.4) = \underline{41.7\,\text{pF}}$$

5.46. (see Problem 5.44).

5.47. With reference to Fig. 5.17, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C: We have

$$K_1 = \left[\frac{h + \sqrt{h^2 + b^2}}{b}\right]^2 = \left[\frac{15 + \sqrt{(15)^2 + (6)^2}}{6}\right]^2 = \underline{23.0}$$

We then have

$$\rho_L = \frac{4\pi\epsilon_0 V_0}{\ln K_1} = \frac{4\pi\epsilon_0(250)}{\ln(23)} = \underline{8.87\,\text{nC/m}}$$

Next, $a = \sqrt{h^2 - b^2} = \sqrt{(15)^2 - (6)^2} = \underline{13.8\,\text{m}}$. Finally,

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi\epsilon_0}{\cosh^{-1}(15/6)} = \underline{35.5\,\text{pF}}$$

5.48. A potential function in free space is given by

$$V = -20 + 10 \ln \left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right]$$

a) Describe the 0-V equipotential surface: Setting the given expression equal to zero, we find

$$\left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right] = e^2 = 7.39$$

So $6.39x^2 + 6.39y^2 - 83.9y + 160 = 0$. Completing the square in the y trinomial leads to $x^2 + (y - 6.56)^2 = 18.1 = (4.25)^2$, which we recognize as a right circular cylinder whose axis is located at $x = 0$, $y = 6.56$, and whose radius is 4.25.

b) Describe the 10-V equipotential surface: In this case, the given expression is set equal to ten, leading to

$$\left[\frac{(5+y)^2 + x^2}{(5-y)^2 + x^2} \right] = e^3 = 20.1$$

So $19.1x^2 + 19.1y^2 - 211y + 477 = 0$. Following the same procedure as in part a, this becomes $x^2 + (y - 5.52)^2 = 5.51 = (2.35)^2$, which we recognize again as a right circular cylinder with axis at $x = 0$, $y = 5.52$, and of radius 2.35.

5.49. A 2 cm diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the:

a) cylinder at a point nearest the plane: The cylinder will image across the plane, producing an equivalent two-cylinder problem, with the second one at location 5 cm below the plane. We will take the plane as the zy plane, with the cylinder positions at $x = \pm 5$. Now $b = 1$ cm, $h = 5$ cm, and $V_0 = 100$ V. Thus $a = \sqrt{h^2 - b^2} = 4.90$ cm. Then $K_1 = [(h+a)/b]^2 = 98.0$, and $\rho_L = (4\pi\epsilon_0 V_0)/\ln K_1 = 2.43\,\text{nC/m}$. Now

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -\frac{\rho_L}{2\pi} \left[\frac{(x+a)\mathbf{a}_x + y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\mathbf{a}_x + y\mathbf{a}_y}{(x-a)^2 + y^2} \right]$$

and

$$\rho_{s,\,max} = \mathbf{D} \cdot (-\mathbf{a}_x)\Big|_{x=h-b,\,y=0} = \frac{\rho_L}{2\pi} \left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2} \right] = \underline{473\,\text{nC/m}^2}$$

b) plane at a point nearest the cylinder: At $x = y = 0$,

$$\mathbf{D}(0,0) = -\frac{\rho_L}{2\pi} \left[\frac{a\mathbf{a}_x}{a^2} - \frac{-a\mathbf{a}_x}{a^2} \right] = -\frac{\rho_L}{2\pi} \frac{2}{a} \mathbf{a}_x$$

from which

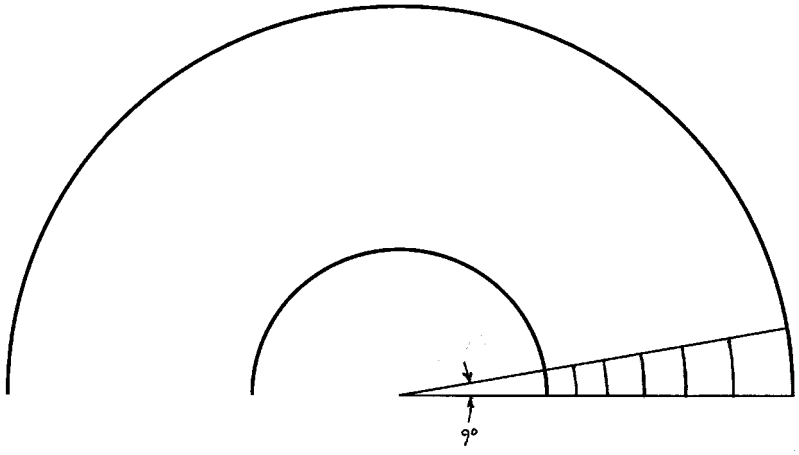
$$\rho_s = \mathbf{D}(0,0) \cdot \mathbf{a}_x = -\frac{\rho_L}{\pi a} = \underline{-15.8\,\text{nC/m}^2}$$

CHAPTER 6.

6.1 Construct a curvilinear square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing.

a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$: The sketch is shown below. Note that only a 9° sector was drawn, since this would then be duplicated 40 times around the circumference to complete the drawing. The capacitance is thus

$$C \doteq \epsilon_0 \frac{N_Q}{N_V} = \epsilon_0 \frac{40}{6} = \underline{59 \text{ pF/m}}$$



b) Calculate an exact value for the capacitance per unit length: This will be

$$C = \frac{2\pi\epsilon_0}{\ln(8/3)} = \underline{57 \text{ pF/m}}$$

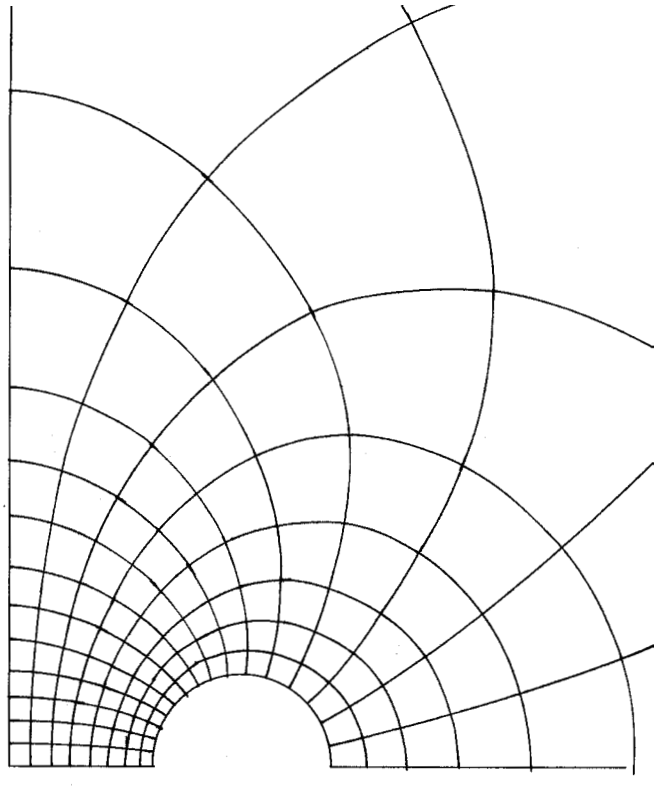
6.2 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

Symmetry allows us to plot the field lines and equipotentials over just the first quadrant, as is done in the sketch below (shown to one-half scale). The capacitance is found from the formula $C = (N_Q/N_V)\epsilon_0$, where N_Q is twice the number of squares around the perimeter of the half-circle and N_V is twice the number of squares between the half-circle and the left vertical plane. The result is

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{32}{16}\epsilon_0 = 2\epsilon_0 = \underline{17.7 \text{ pF/m}}$$

We check this result with that using the exact formula:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(13/5)} = 1.95\epsilon_0 = \underline{17.3 \text{ pF/m}}$$



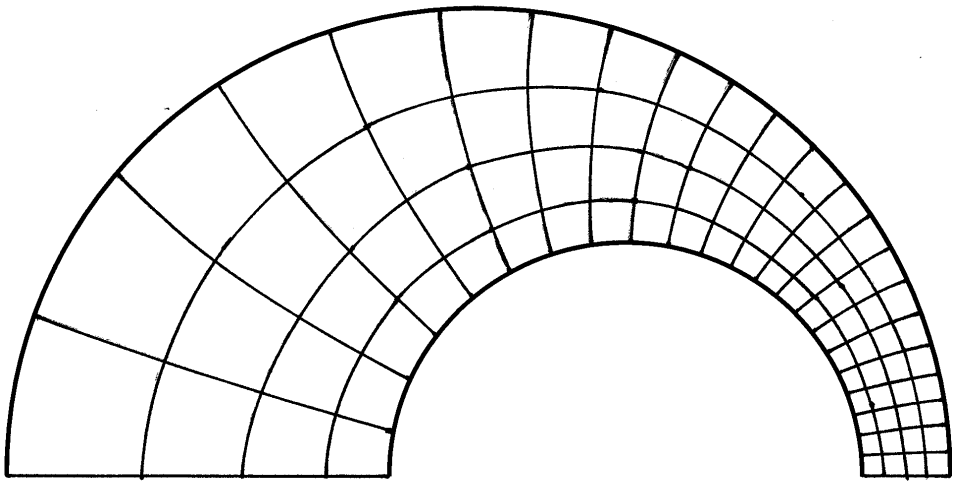
6.3. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2 + b^2 - D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

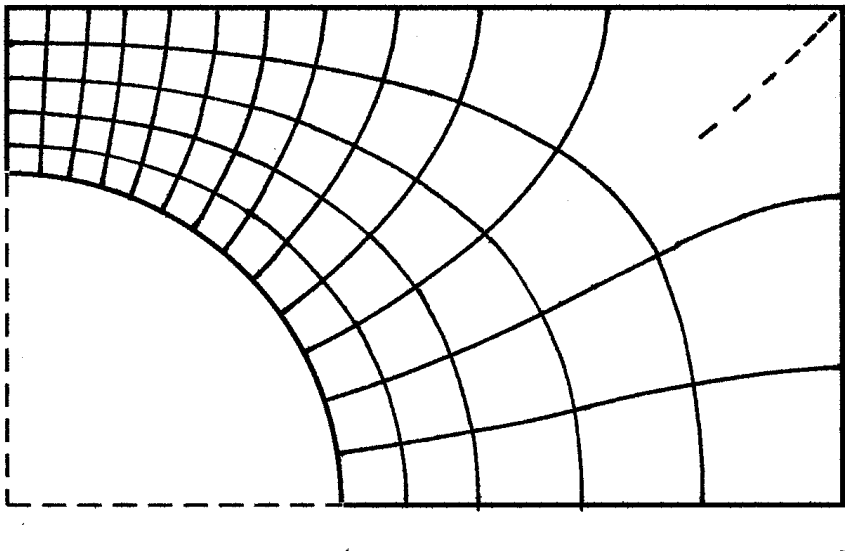
The drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.4. A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross-section.

a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior: The result below could still be improved a little, but is nevertheless sufficient for a reasonable capacitance estimate. Note that the five-sided region in the upper right corner has been partially subdivided (dashed line) in anticipation of how it would look when the next-level subdivision is done (doubling the number of field lines and equipotentials).

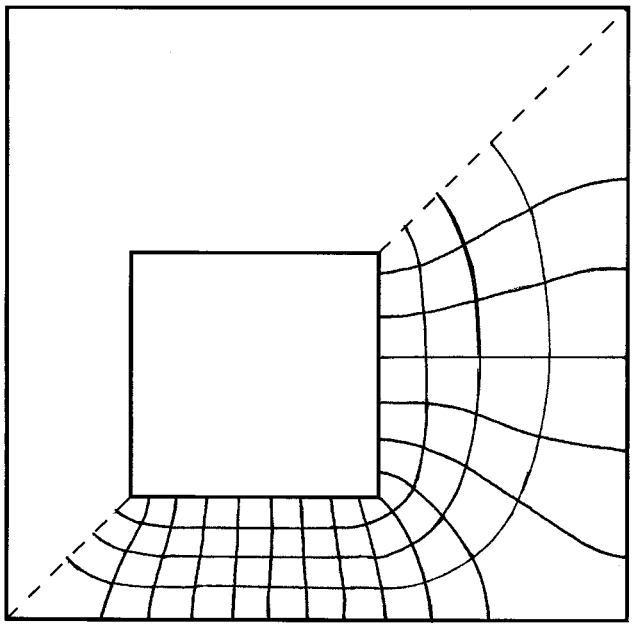


b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length: In this case N_Q is the number of squares around the full perimeter of the circular conductor, or four times the number of squares shown in the drawing. N_V is the number of squares between the circle and the rectangle, or 5. The capacitance is estimated to be

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{4 \times 13}{5}\epsilon_0 = 10.4\epsilon_0 \doteq \underline{90 \text{ pF/m}}$$

6.5. The inner conductor of the transmission line shown in Fig. 6.12 has a square cross-section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of the transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part *b* change if $a = 0.6$ cm?

a) The plot is shown below. Some improvement is possible, depending on how much time one wishes to spend.



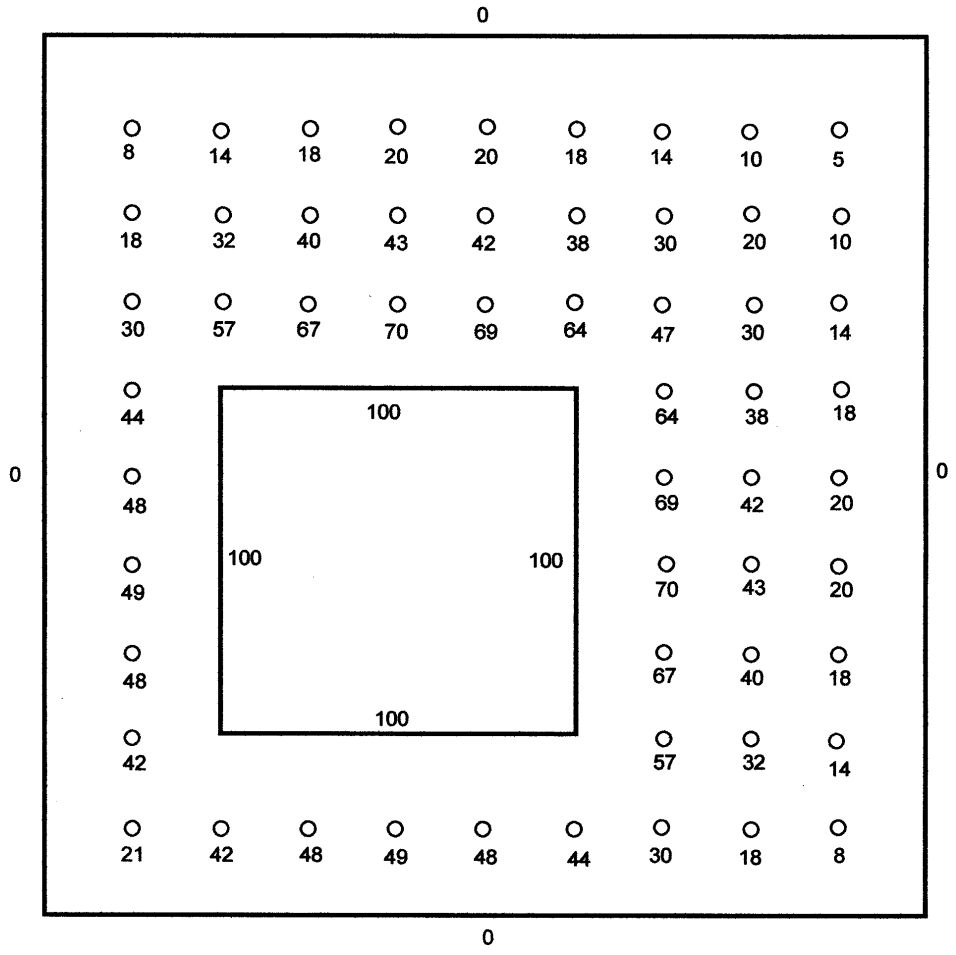
b) From the plot, the capacitance is found to be

$$C \doteq \frac{16 \times 2}{4}(1.6)\epsilon_0 = 12.8\epsilon_0 \doteq \underline{110 \text{ pF/m}}$$

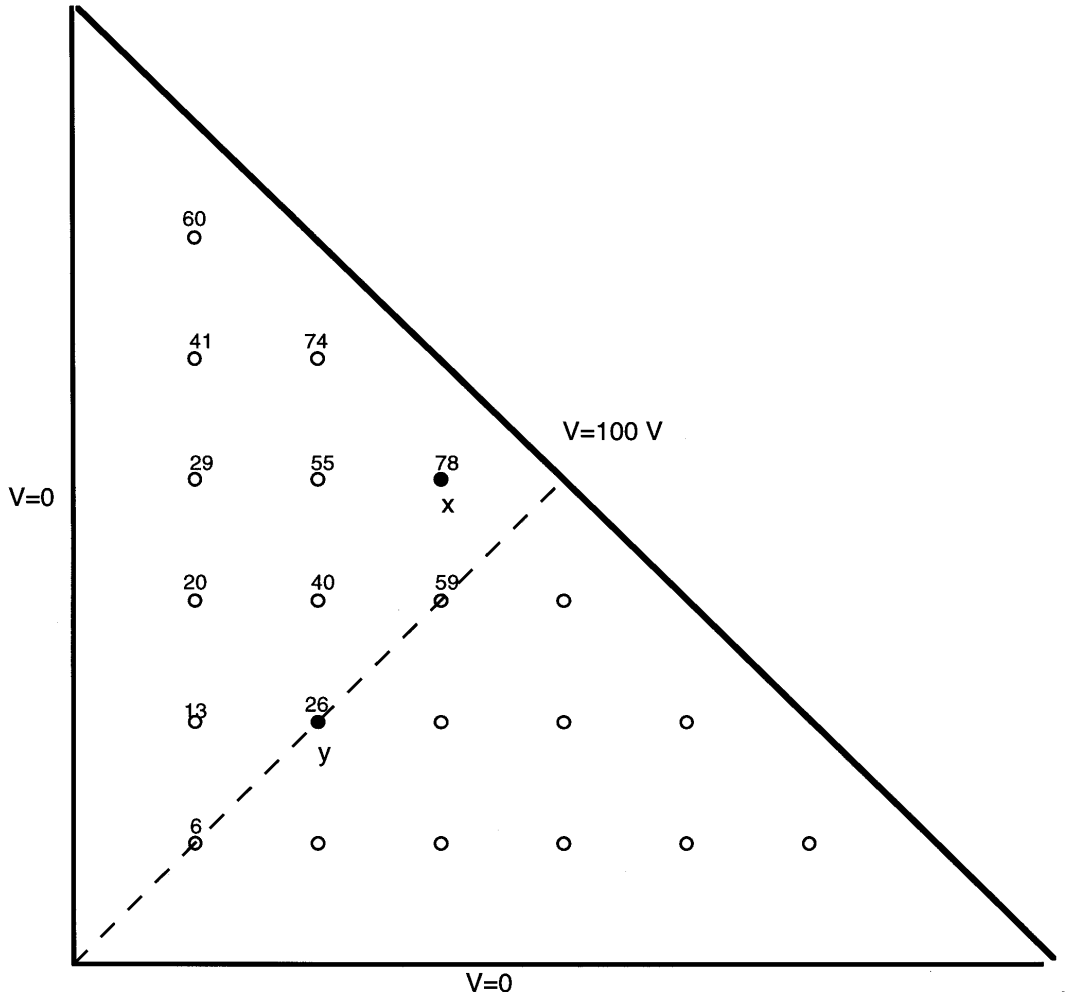
c) If *a* is changed, the result of part *b* would not change, since all dimensions retain the same relative scale.

6.6. Let the inner conductor of the transmission line shown in Fig. 6.12 be at a potential of 100V, while the outer is at zero potential. Construct a grid, $0.5a$ on a side, and use iteration to find V at a point that is a units above the upper right corner of the inner conductor. Work to the nearest volt:

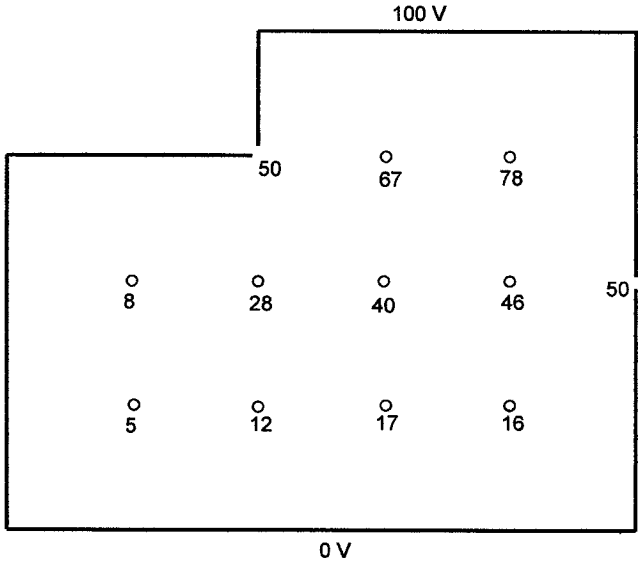
The drawing is shown below, and we identify the requested voltage as 38 V.



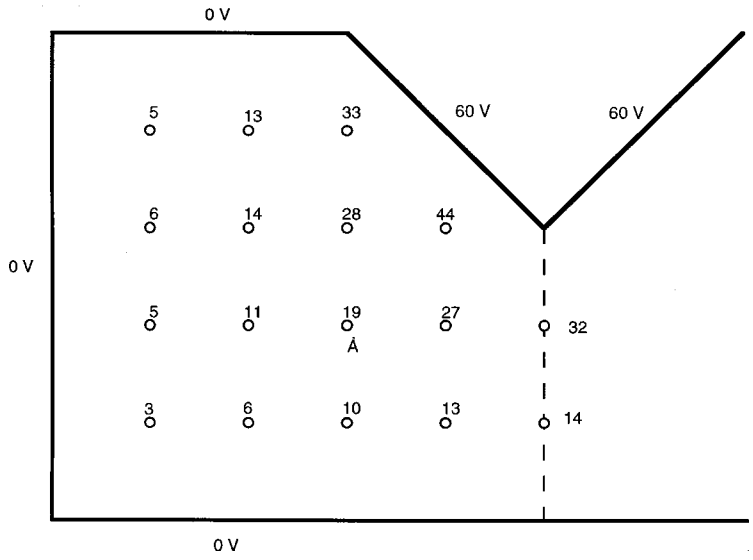
6.7. Use the iteration method to estimate the potentials at points x and y in the triangular trough of Fig. 6.13. Work only to the nearest volt: The result is shown below. The mirror image of the values shown occur at the points on the other side of the line of symmetry (dashed line). Note that $V_x = \underline{78\,\text{V}}$ and $V_y = \underline{26\,\text{V}}$.



6.8. Use iteration methods to estimate the potential at point x in the trough shown in Fig. 6.14. Working to the nearest volt is sufficient. The result is shown below, where we identify the voltage at x to be 40 V. Note that the potentials in the gaps are 50 V.



6.9. Using the grid indicated in Fig. 6.15, work to the nearest volt to estimate the potential at point A: The voltages at the grid points are shown below, where V_A is found to be 19 V. Half the figure is drawn since mirror images of all values occur across the line of symmetry (dashed line).



6.10. Conductors having boundaries that are curved or skewed usually do not permit every grid point to coincide with the actual boundary. Figure 6.16a illustrates the situation where the potential at V_0 is to be estimated in terms of V_1, V_2, V_3, and V_4, and the unequal distances h_1, h_2, h_3, and h_4.

a) Show that

$$V_0 \doteq \frac{V_1}{\left(1+\frac{h_1}{h_3}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)} + \frac{V_2}{\left(1+\frac{h_2}{h_4}\right)\left(1+\frac{h_2 h_4}{h_1 h_3}\right)} + \frac{V_3}{\left(1+\frac{h_3}{h_1}\right)\left(1+\frac{h_1 h_3}{h_4 h_2}\right)} + \frac{V_4}{\left(1+\frac{h_4}{h_2}\right)\left(1+\frac{h_4 h_2}{h_3 h_1}\right)}$$

note error, corrected here, in the equation (second term)

Referring to the figure, we write:

$$\left.\frac{\partial V}{\partial x}\right|_{M_1} \doteq \frac{V_1 - V_0}{h_1} \qquad \left.\frac{\partial V}{\partial x}\right|_{M_3} \doteq \frac{V_0 - V_3}{h_3}$$

Then

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} \doteq \frac{(V_1 - V_0)/h_1 - (V_0 - V_3)/h_3}{(h_1 + h_3)/2} = \frac{2V_1}{h_1(h_1+h_3)} + \frac{2V_3}{h_3(h_1+h_3)} - \frac{2V_0}{h_1 h_3}$$

We perform the same procedure along the y axis to obtain:

$$\left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} \doteq \frac{(V_2 - V_0)/h_2 - (V_0 - V_4)/h_4}{(h_2 + h_4)/2} = \frac{2V_2}{h_2(h_2+h_4)} + \frac{2V_4}{h_4(h_2+h_4)} - \frac{2V_0}{h_2 h_4}$$

Then, knowing that

$$\left.\frac{\partial^2 V}{\partial x^2}\right|_{V_0} + \left.\frac{\partial^2 V}{\partial y^2}\right|_{V_0} = 0$$

the two equations for the second derivatives are added to give

$$\frac{2V_1}{h_1(h_1+h_3)} + \frac{2V_2}{h_2(h_2+h_4)} + \frac{2V_3}{h_3(h_1+h_3)} + \frac{2V_4}{h_4(h_2+h_4)} = V_0\left(\frac{h_1 h_3 + h_2 h_4}{h_1 h_2 h_3 h_4}\right)$$

Solve for V_0 to obtain the given equation.

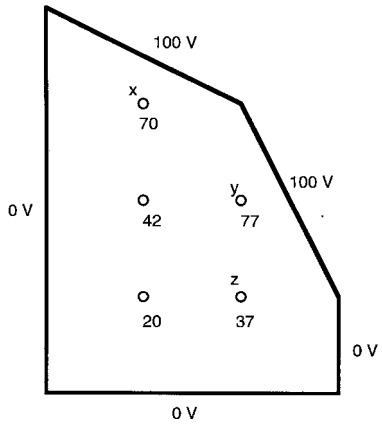
b) Determine V_0 in Fig. 6.16b: Referring to the figure, we note that $h_1 = h_2 = a$. The other two distances are found by writing equations for the circles:

$$(0.5a + h_3)^2 + a^2 = (1.5a)^2 \text{ and } (a + h_4)^2 + (0.5a)^2 = (1.5a)^2$$

These are solved to find $h_3 = 0.618a$ and $h_4 = 0.414a$. The four distances and potentials are now substituted into the given equation:

$$V_0 \doteq \frac{80}{\left(1+\frac{1}{.618}\right)\left(1+\frac{.618}{.414}\right)} + \frac{60}{\left(1+\frac{1}{.414}\right)\left(1+\frac{.414}{.618}\right)} + \frac{100}{(1+.618)\left(1+\frac{.618}{.414}\right)} + \frac{100}{(1+.414)\left(1+\frac{.414}{.618}\right)} = \underline{90\text{ V}}$$

6.11. Consider the configuration of conductors and potentials shown in Fig. 6.17. Using the method described in Problem 10, write an expression for V_x (not V_0): The result is shown below, where $V_x = \underline{70\,\text{V}}$.



6.12a) After estimating potentials for the configuation of Fig. 6.18, use the iteration method with a square grid 1 cm on a side to find better estimates at the seven grid points. Work to the nearest volt:

25	50	75	50	25
0	<u>48</u>	100	<u>48</u>	0
0	<u>42</u>	100	<u>42</u>	0
0	<u>19</u>	<u>34</u>	<u>19</u>	0
0	0	0	0	0

b) Construct a 0.5 cm grid, establish new rough estimates, and then use the iteration method on the 0.5 cm grid. Again, work to the nearest volt: The result is shown below, with values for the original grid points underlined:

25	50	50	50	75	50	50	50	25
0	32	50	68	100	68	50	32	0
0	26	<u>48</u>	72	100	72	<u>48</u>	26	0
0	23	45	70	100	70	45	23	0
0	20	<u>40</u>	64	100	64	<u>40</u>	20	0
0	15	30	44	54	44	30	15	0
0	10	<u>19</u>	26	<u>30</u>	26	<u>19</u>	10	0
0	5	9	12	14	12	9	5	0
0	0	0	0	0	0	0	0	0

6.12c. Use the computer to obtain values for a 0.25 cm grid. Work to the nearest 0.1 V: Values for the left half of the configuration are shown in the table below. Values along the vertical line of symmetry are included, and the original grid values are underlined.

25	50	50	50	50	50	50	50	75
0	26.5	38.0	44.6	49.6	54.6	61.4	73.2	100
0	18.0	31.0	40.7	49.0	57.5	67.7	81.3	100
0	14.5	27.1	38.1	48.3	58.8	70.6	84.3	100
0	12.8	24.8	36.2	<u>47.3</u>	58.8	71.4	85.2	100
0	11.7	23.1	34.4	45.8	57.8	70.8	85.0	100
0	10.8	21.6	32.5	43.8	55.8	69.0	83.8	100
0	10.0	20.0	30.2	40.9	52.5	65.6	81.2	100
0	9.0	18.1	27.4	<u>37.1</u>	47.6	59.7	75.2	100
0	7.9	15.9	24.0	32.4	41.2	50.4	59.8	67.2
0	6.8	13.6	20.4	27.3	34.2	40.7	46.3	49.2
0	5.6	11.2	16.8	22.2	27.4	32.0	35.4	36.8
0	4.4	8.8	13.2	<u>17.4</u>	21.2	24.4	26.6	<u>27.4</u>
0	3.3	6.6	9.8	12.8	15.4	17.6	19.0	19.5
0	2.2	4.4	6.4	8.4	10.0	11.4	12.2	12.5
0	1.1	2.2	3.2	4.2	5.0	5.6	6.0	6.1
0	0	0	0	0	0	0	0	0

6.13. Perfectly-conducting concentric spheres have radii of 2 and 6 cm. The region $2 < r < 3$ cm is filled with a solid conducting material for which $\sigma = 100$ S/m, while the portion for which $3 < r < 6$ cm has $\sigma = 25$ S/m. The inner sphere is held at 1 V while the outer is at $V = 0$.

a. Find E and J everywhere: From symmetry, E and J will be radially-directed, and we note the fact that the current, I, must be constant at any cross-section; i.e., through any spherical surface at radius r between the spheres. Thus we require that in both regions,

$$\mathbf{J} = \frac{I}{4\pi r^2}\mathbf{a}_r$$

The fields will thus be

$$\mathbf{E}_1 = \frac{I}{4\pi\sigma_1 r^2}\mathbf{a}_r \quad (2 < r < 3) \quad \text{and} \quad \mathbf{E}_2 = \frac{I}{4\pi\sigma_2 r^2}\mathbf{a}_r \quad (3 < r < 6)$$

where $\sigma_1 = 100$ S/m and $\sigma_2 = 25$ S/m. Since we know the voltage between spheres (1V), we can find the value of I through:

$$1\,\text{V} = -\int_{.06}^{.03} \frac{I}{4\pi\sigma_2 r^2}\,dr - \int_{.03}^{.02} \frac{I}{4\pi\sigma_1 r^2}\,dr = \frac{I}{0.24\pi}\left[\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\right]$$

and so

$$I = \frac{0.24\pi}{(1/\sigma_1 + 1/\sigma_2)} = 15.08\,\text{A}$$

Then finally, with $I = 15.08$ A substituted into the field expressions above, we find

$$\mathbf{E}_1 = \underline{\frac{.012}{r^2}\,\mathbf{a}_r\ \text{V/m}} \quad (2 < r < 3)$$

and

$$\mathbf{E}_2 = \underline{\frac{.048}{r^2}\,\mathbf{a}_r\ \text{V/m}} \quad (3 < r < 6)$$

The current density is now

$$\mathbf{J} = \sigma_1\mathbf{E}_1 = \sigma_2\mathbf{E}_2 = \underline{\frac{1.2}{r^2}\ \text{A/m}} \quad (2 < r < 6)$$

b) What resistance would be measured between the two spheres? We use

$$R = \frac{V}{I} = \frac{1\,\text{V}}{15.08\,\text{A}} = \underline{6.63 \times 10^{-2}\ \Omega}$$

c) What is V at $r = 3$ cm? This we find through

$$V = -\int_{.06}^{.03} \frac{.048}{r^2}\,dr = .048\left(\frac{1}{.03} - \frac{1}{.06}\right) = \underline{0.8\,\text{V}}$$

6.14. The cross-section of the transmission line shown in Fig. 6.12 is drawn on a sheet of conducting paper with metallic paint. The sheet resistance is 2000 Ω/sq and the dimension a is 2 cm.

a) Assuming a result for Prob. 6b of 110 pF/m, what total resistance would be measured between the metallic conductors drawn on the conducting paper? We assume a paper thickness of t m, so that the capacitance is $C = 110t$ pF, and the surface resistance is $R_s = 1/(\sigma t) = 2000$ Ω/sq. We now use

$$RC = \frac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \frac{\epsilon}{\sigma C} = \frac{\epsilon R_s t}{110 \times 10^{-12} t} = \frac{(1.6 \times 8.854 \times 10^{-12})(2000)}{110 \times 10^{-12}} = \underline{257.6\ \Omega}$$

b) What would the total resistance be if $a = 2$ cm? The result is independent of a, provided the proportions are maintained. So again, $R = \underline{257.6\ \Omega}$.

6.15. two concentric annular rings are painted on a sheet of conducting paper with a highly conducting metal paint. The four radii are 1, 1.2, 3.5, and 3.7 cm. Connections made to the two rings show a resistance of 215 ohms between them.

a) What is R_s for the conducting paper? Using the two radii (1.2 and 3.5 cm) at which the rings are at their closest separation, we first evaluate the capacitance:

$$C = \frac{2\pi\epsilon_0 t}{\ln(3.5/1.2)} = 5.19 \times 10^{-11} t \text{ F}$$

where t is the unknown paper coating thickness. Now use

$$RC = \frac{\epsilon_0}{\sigma} \quad \Rightarrow \quad R = \frac{8.85 \times 10^{-12}}{5.19 \times 10^{-11} \sigma t} = 215$$

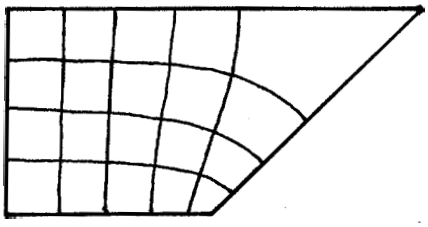
Thus

$$R_s = \frac{1}{\sigma t} = \frac{(51.9)(215)}{8.85} = \underline{1.26\,\text{k}\Omega/\text{sq}}$$

b) If the conductivity of the material used as the surface of the paper is 2 S/m, what is the thickness of the coating? We use

$$t = \frac{1}{\sigma R_s} = \frac{1}{2 \times 1.26 \times 10^3} = 3.97 \times 10^{-4} \text{ m} = \underline{0.397 \text{ mm}}$$

6.16. The square washer shown in Fig. 6.19 is 2.4 mm thick and has outer dimensions of 2.5 × 2.5 cm and inner dimensions of 1.25 × 1.25 cm. The inside and outside surfaces are perfectly-conducting. If the material has a conductivity of 6 S/m, estimate the resistance offered between the inner and outer surfaces (shown shaded in Fig. 6.19). A few curvilinear squares are suggested: First we find the surface resistance, $R_s = 1/(\sigma t) = 1/(6 \times 2.4 \times 10^{-3}) = 69.4\ \Omega$/sq. Having found this, we can construct the total resistance by using the fundamental square as a building block. Specifically, $R = R_s(N_l/N_w)$ where N_l is the number of squares between the inner and outer surfaces and N_w is the number of squares around the perimeter of the washer. These numbers are found from the curvilinear square plot shown below, which covers one-eighth the washer. The resistance is thus $R \doteq 69.4[4/(8 \times 5)] \doteq \underline{6.9\ \Omega}$.



6.17. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_R = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9}\ \text{C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9}\ \text{C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad \Rightarrow \quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\,\text{A} = \underline{206\,\text{mA}}$$

6.18. A coaxial transmission line is modelled by the use of a rubber sheet having horizontal dimensions that are 100 times those of the actual line. Let the radial coordinate of the model be ρ_m. For the line itself, let the radial dimension be designated by ρ as usual; also, let $a = 0.6$ mm and $b = 4.8$ mm. The model is 8 cm in height at the inner conductor and zero at the outer. If the potential of the inner conductor is 100 V:

a) Find the expression for $V(\rho)$: Assuming charge density ρ_s on the inner conductor, we use Gauss' Law to find $2\pi\rho D = 2\pi a\rho_s$, from which $E = D/\epsilon = a\rho_s/(\epsilon\rho)$ in the radial direction. The potential difference between inner and outer conductors is

$$V_{ab} = V_0 = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\, d\rho = \frac{a\rho_s}{\epsilon}\ln\left(\frac{b}{a}\right)$$

from which

$$\rho_s = \frac{\epsilon V_0}{a\ln(b/a)} \quad \Rightarrow \quad E = \frac{V_0}{\rho\ln(b/a)}$$

Now, as a function of radius, and assuming zero potential on the outer conductor, the potential function will be:

$$V(\rho) = -\int_b^\rho \frac{V_0}{\rho'\ln(b/a)}\, d\rho' = V_0\frac{\ln(b/\rho)}{\ln(b/a)} = 100\frac{\ln(.0048/\rho)}{\ln(.0048/.0006)} = \underline{48.1\ln\left(\frac{.0048}{\rho}\right)\ \text{V}}$$

b) Write the model height as a function of ρ_m (not ρ): We use the part a result, since the gravitational function must be the same as that for the electric potential. We replace V_0 by the maximum height, and multiply all dimensions by 100 to obtain:

$$h(\rho_m) = 0.08\frac{\ln(.48/\rho_m)}{\ln(.48/.06)} = \underline{0.038\ln\left(\frac{.48}{\rho_m}\right)\ \text{m}}$$

CHAPTER 7

7.1. Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find:

a) V at P: Substituting the coordinates into V, find $V_P = \underline{-8\,\text{V}}$.

b) $\mathbf{E}$ at P: We use $\mathbf{E} = -\nabla V = -2y^2z^3\mathbf{a}_x - 4xyz^3\mathbf{a}_y - 6xy^2z^2\mathbf{a}_z$, which, when evaluated at P, becomes $\mathbf{E}_P = \underline{8\mathbf{a}_x + 8\mathbf{a}_y - 24\mathbf{a}_z\ \text{V/m}}$

c) ρ_v at P: This is $\rho_v = \nabla \cdot \mathbf{D} = -\epsilon_0\nabla^2 V = \underline{-4xz(z^2 + 3y^2)\ \text{C/m}^3}$

d) the equation of the equipotential surface passing through P: At P, we know $V = -8\,\text{V}$, so the equation will be $\underline{xy^2z^3 = -4}$.

e) the equation of the streamline passing through P: First,

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus

$$ydy = 2xdx, \text{ and so } \frac{1}{2}y^2 = x^2 + C_1$$

Evaluating at P, we find $C_1 = 1$. Next,

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus

$$3xdx = zdz, \text{ and so } \frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations:

$$\underline{y^2 - 2x^2 = 2} \text{ and } \underline{3x^2 - z^2 = 2}$$

f) Does V satisfy Laplace's equation? No, since the charge density is not zero.

7.2. A potential field V exists in a region where $\epsilon = f(x)$. Find $\nabla^2 V$ if $\rho_v = 0$.
First, $\mathbf{D} = \epsilon(x)\mathbf{E} = -f(x)\nabla V$. Then $\nabla \cdot \mathbf{D} = \rho_v = 0 = \nabla \cdot (-f(x)\nabla V)$.

So

$$0 = \nabla \cdot (-f(x)\nabla V) = -\left[\left(\frac{df}{dx}\frac{\partial V}{\partial x} + f(x)\frac{\partial^2 V}{\partial x^2}\right) f(x)\frac{\partial^2 V}{\partial y^2} + f(x)\frac{\partial^2 V}{\partial z^2}\right]$$
$$= -\left[\frac{df}{dx}\frac{\partial V}{\partial x} + f(x)\nabla^2 V\right]$$

Therefore,

$$\nabla^2 V = \underline{-\frac{1}{f(x)}\frac{df}{dx}\frac{\partial V}{\partial x}}$$

7.3. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \;\Rightarrow\; \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \frac{df}{dx}\Big|_{x=0} = 0 \text{(as given)}$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

7.4. Given the potential field $V = A \ln \tan^2(\theta/2) + B$:

a) Show that $\nabla^2 V = 0$: Since V is a function only of θ,

$$\nabla^2 V = \frac{1}{r^2 \sin\theta)} \frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right)$$

where

$$\frac{dV}{d\theta} = \frac{d}{d\theta}\left(A \ln \tan^2(\theta/2) + B\right) = \frac{d}{d\theta}\left(2A \ln \tan(\theta/2)\right) = \frac{A}{\sin(\theta/2)\cos(\theta/2)} = \frac{2A}{\sin\theta}$$

Then

$$\nabla^2 V = \frac{1}{r^2 \sin\theta)} \frac{d}{d\theta}\left(\sin\theta \frac{2A}{\sin\theta}\right) = 0$$

b) Select A and B so that $V = 100$ V and $E_\theta = 500$ V/m at $P(r = 5, \theta = 60^\circ, \phi = 45^\circ)$:

First,

$$E_\theta = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta} = -\frac{2A}{r \sin\theta} = -\frac{2A}{5 \sin 60} = -0.462A = 500$$

So $A = \underline{-1082.5}$ V. Then

$$V_P = -(1082.5) \ln \tan^2(30^\circ) + B = 100 \;\Rightarrow\; B = \underline{-1089.3 \text{ V}}$$

Summarizing, $V(\theta) = -1082.5 \ln \tan^2(\theta/2) - 1089.3$.

7.5. Given the potential field $V = (A\rho^4 + B\rho^{-4})\sin 4\phi$:

a) Show that $\nabla^2 V = 0$: In cylindrical coordinates,

$$\begin{aligned}\nabla^2 V &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho(4A\rho^3 - 4B\rho^{-5})\right)\sin 4\phi - \frac{1}{\rho^2}16(A\rho^4 + B\rho^{-4})\sin 4\phi \\ &= \frac{16}{\rho}(A\rho^3 + B\rho^{-5})\sin 4\phi - \frac{16}{\rho^2}(A\rho^4 + B\rho^{-4})\sin 4\phi = 0\end{aligned}$$

b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$: First,

$$\begin{aligned}\mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial\rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\,\mathbf{a}_\phi \\ &= -4\left[(A\rho^3 - B\rho^{-5})\sin 4\phi\,\mathbf{a}_\rho + (A\rho^3 + B\rho^{-5})\cos 4\phi\,\mathbf{a}_\phi\right]\end{aligned}$$

and at P, $\mathbf{E}_P = -4(A - B)\,\mathbf{a}_\rho$. Thus $|\mathbf{E}_P| = \pm 4(A - B)$. Also, $V_P = A + B$. Our two equations are:

$$4(A - B) = \pm 500$$

and

$$A + B = 100$$

We thus have two pairs of values for A and B:

$$\underline{A = 112.5,\ B = -12.5} \text{ or } \underline{A = -12.5,\ B = 112.5}$$

7.6. If $V = 20\sin\theta/r^3$ V in free space, find:

a) ρ_v at $P(r = 2, \theta = 30^\circ, \phi = 0)$: We use Poisson's equation in free space, $\nabla^2 V = -\rho_v/\epsilon_0$, where, with no ϕ variation:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$$

Substituting:

$$\begin{aligned}\nabla^2 V &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(-r^2\frac{60\sin\theta}{r^4}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{20\cos\theta}{r^3}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-60\sin\theta}{r^2}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{10\sin 2\theta}{r^3}\right) \\ &= \frac{120\sin\theta}{r^5} + \frac{20\cos 2\theta}{r^5\sin\theta} = \frac{20(4\sin^2\theta + 1)}{r^5\sin\theta} = -\frac{\rho_v}{\epsilon_0}\end{aligned}$$

So

$$\rho_{vP} = -\epsilon_0\left[\frac{20(4\sin^2\theta + 1)}{r^5\sin\theta}\right]_{r=2,\theta=30} = -2.5\epsilon_0 = \underline{-22.1\ \text{pC/m}^3}$$

7.6b. the total charge within the spherical shell $1 < r < 2$ m: We integrate the charge density found in part *a* over the specified volume:

$$\begin{aligned} Q &= -\epsilon_0 \int_0^{2\pi} \int_0^{\pi} \int_1^2 \frac{20(4\sin^2\theta + 1)}{r^5 \sin\theta} r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= -2\pi(20)\epsilon_0 \int_0^{\pi} \int_1^2 \frac{(4\sin^2\theta + 1)}{r^3} \, dr \, d\theta = -40\pi\epsilon_0 \int_1^2 \frac{3\pi}{r^3} dr = 60\pi^2\epsilon_0 \frac{1}{r^2}\Big|_1^2 = -45\pi^2\epsilon_0 \\ &= \underline{-3.9 \text{ nC}} \end{aligned}$$

7.7. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$: Use Poisson's equation:

$$\begin{aligned} \rho_v &= -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} \right) \\ &= -\epsilon_0 \left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{-\cos 2\phi}{\rho} \right) - \frac{4}{\rho^2} \frac{\cos 2\phi}{\rho} \right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3} \end{aligned}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120^\circ)}{0.5^3} = -12\epsilon_0 = \underline{-106 \text{ pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30^\circ, 1)$: First, we find **E**:

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{\partial V}{\partial \rho} \mathbf{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \mathbf{a}_\phi \\ &= \frac{\cos 2\phi}{\rho^2} \mathbf{a}_\rho + \frac{2\sin 2\phi}{\rho^2} \mathbf{a}_\phi \end{aligned}$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60^\circ}{4} \mathbf{a}_\rho + \frac{2\sin 60^\circ}{4} \mathbf{a}_\phi = 0.125 \, \mathbf{a}_\rho + 0.433 \, \mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm|\mathbf{D}_B| = \pm\epsilon_0|\mathbf{E}_B| = \pm 0.451\epsilon_0 = \underline{\pm 0.399 \text{ pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

7.8. Let $V_1(r, \theta, \phi) = 20/r$ and $V_2(r, \theta, \phi) = (4/r) + 4$.

a) State whether V_1 and V_2 satisfy Laplace's equation:

$$\nabla^2 V_1 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV_1}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{-20}{r^2} \right) \right] = 0$$

$$\nabla^2 V_2 = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV_2}{dr} \right) = \frac{1}{r^2} \frac{d}{dr} \left[r^2 \left(\frac{-4}{r^2} \right) \right] = 0$$

7.8b. Evaluate V_1 and V_2 on the closed surface $r = 4$:

$$V_1(r=4) = \frac{20}{4} = 5 \quad V_2(r=4) = \frac{4}{4} + 4 = 5$$

c) Conciliate your results with the uniqueness theorem: Uniqueness specifies that there is only one potential that will satisfy all the given boundary conditions. While both potentials have the same value at $r = 4$, they do not as $r \to \infty$. So they apply to different situations.

7.9. The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \le \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$.

a) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? Yes for the first three, since Laplace's equation is linear. No for $V_1 V_2$.

b) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? At the 100 V surface ($\rho = a$), No for all. At the 0 V surfaces, yes, except for $V_1 + 3$.

c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1? Only V_2 is, since it is given as satisfying all the boundary conditions that V_1 does. Therefore, by the uniqueness theorem, $V_2 = V_1$. The others, not satisfying the boundary conditions, are not the same as V_1.

7.10. Conducting planes at $z = 2$cm and $z = 8$cm are held at potentials of -3V and 9V, respectively. The region between the plates is filled with a perfect dielectric with $\epsilon = 5\epsilon_0$. Find and sketch:

a) $V(z)$: We begin with the general solution of the one-dimensional Laplace equation in rectangular coordinates: $V(z) = Az + B$. Applying the boundary conditions, we write $-3 = A(2) + B$ and $9 = A(8) + B$. Subtracting the former equation from the latter, we find $12 = 6A$ or $A = 2$ V/cm. Using this we find $B = -7$ V. Finally, $V(z) = 2z - 7$ V (z in cm) or $V(z) = \underline{200z - 7\text{ V}}$ (z in m).

b) $E_z(z)$: We use $\mathbf{E} = -\nabla V = -(dV/dz)\mathbf{a}_z = -2\text{ V/cm} = \underline{-200\text{ V/m}}$.

c) $D_z(z)$: Working in meters, have $D_z = \epsilon E_z = -200\epsilon = \underline{-1000\epsilon_0\text{ C/m}^2}$

7.11. The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find:

a) V at $P(5, 2, 6)$: The planes are parallel, and so we expect variation in potential in the direction normal to them. Using the two boundary condtions, our general potential function can be written:

$$V(x, y) = A(2x + 3y - 12) + 100 = A(2x + 3y - 18) + 0$$

and so $A = -100/6$. We then write

$$V(x, y) = -\frac{100}{6}(2x + 3y - 18) = -\frac{100}{3}x - 50y + 300$$

and $V_P = -\frac{100}{3}(5) - 100 + 300 = \underline{33.33\text{ V}}$.

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E} = -\nabla V = \underline{\frac{100}{3}\,\mathbf{a}_x + 50\,\mathbf{a}_y\text{ V/m}}$$

7.12. Conducting cylinders at $\rho = 2$ cm and $\rho = 8$ cm in free space are held at potentials of 60mV and -30mV, respectively.

a) Find $V(\rho)$: Working in volts and meters, we write the general one-dimensional solution to the Laplace equation in cylindrical coordinates, assuming radial variation: $V(\rho) = A\ln(\rho) + B$. Applying the given boundary conditions, this becomes $V(2\text{cm}) = .060 = A\ln(.02) + B$ and $V(8\text{cm}) = -.030 = A\ln(.08) + B$. Subtracting the former equation from the latter, we find $-.090 = A\ln(.08/.02) = A\ln 4 \Rightarrow A = -.0649$. B is then found through either equation; e.g., $B = .060 + .0649\ln(.02) = -.1940$. Finally, $\underline{V(\rho) = -.0649\ln\rho - .1940}$.

b) Find $E_\rho(\rho)$: $\mathbf{E} = -\nabla V = -(dV/d\rho)\mathbf{a}_\rho = \underline{(.0649/\rho)\mathbf{a}_\rho\ \text{V/m}}$.

c) Find the surface on which $V = 30$ mV:
Use $.03 = -.0649\ln\rho - .1940 \Rightarrow \rho = .0317\,\text{m} = \underline{3.17\,\text{cm}}$.

7.13. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (16) we have

$$V(\rho) = 100\frac{\ln(.012/\rho)}{\ln(.012/.005)}\ \text{V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100\frac{\ln(.012/\rho)}{\ln(2.4)} \Rightarrow \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01\ \text{cm}}$$

b) $E_{\rho\,max}$: We have

$$E_\rho = -\frac{\partial V}{\partial\rho} = -\frac{dV}{d\rho} = \frac{100}{\rho\ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\,max} = \frac{100}{.005\ln(2.4)} = 2.28\times10^4\,\text{V/m} = \underline{22.8\,\text{kV/m}}$$

c) ϵ_R if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_R}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_R:

$$\epsilon_R = \frac{(20\times10^{-9})\ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

7.14. Two semi-infinite planes are located at $\phi = -\alpha$ and $\phi = \alpha$, where $\alpha < \pi/2$. A narrow insulating strip separates them along the z axis. The potential at $\phi = -\alpha$ is V_0, while $V = 0$ at $\phi = \alpha$.

a) Find $V(\phi)$ in terms of α and V_0: We use the one-dimensional solution form for Laplace's equation assuming variation along ϕ: $V(\phi) = A\phi + B$. The boundary conditions are then substituted: $V_0 = -A\alpha + B$ and $0 = A\alpha + B$. Subtract the latter equation from the former to obtain: $V_0 = -2A\alpha \;\Rightarrow\; A = -V_0/(2\alpha)$. Then $0 = -V_0/(2\alpha)\alpha + B \;\Rightarrow\; B = V_0/2$. Finally

$$\underline{V(\phi) = \frac{V_0}{2}\left(1 - \frac{\phi}{\alpha}\right) \text{ V}}$$

b) Find E_ϕ at $\phi = 20^\circ$, $\rho = 2$ cm, if $V_0 = 100$ V and $\alpha = 30^\circ$:

$$E_\phi = -\frac{1}{\rho}\frac{dV}{d\rho} = \frac{V_0}{2\alpha\rho} \text{ V/m} \quad \text{Then} \quad E(2\text{cm}, 20^\circ) = \frac{100}{2(30 \times 2\pi/360)(.02)} = \underline{4.8 \text{ kV/m}}$$

7.15. The two conducting planes illustrated in Fig. 7.8 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find:

a) $V(\phi)$: The general solution to Laplace's equation will be $V = C_1\phi + C_2$, and so

$$20 = C_1(.188) + C_2 \text{ and } 200 = C_1(.179) + C_2$$

Subtracting one equation from the other, we find

$$-180 = C_1(.188 - .179) \;\Rightarrow\; C_1 = -2.00 \times 10^4$$

Then

$$20 = -2.00 \times 10^4(.188) + C_2 \;\Rightarrow\; C_2 = 3.78 \times 10^3$$

Finally, $V(\phi) = \underline{(-2.00 \times 10^4)\phi + 3.78 \times 10^3 \text{ V}}$.

b) $\mathbf{E}(\rho)$: Use

$$\mathbf{E}(\rho) = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi} = \underline{\frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \text{ V/m}}$$

c) $\mathbf{D}(\rho) = \epsilon_0\mathbf{E}(\rho) = \underline{(2.00 \times 10^4\epsilon_0/\rho)\,\mathbf{a}_\phi \text{ C/m}^2}$.

d) ρ_s on the upper surface of the lower plane: We use

$$\rho_s = \mathbf{D} \cdot \mathbf{n}\Big|_{surface} = \frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi = \underline{\frac{2.00 \times 10^4}{\rho} \text{ C/m}^2}$$

e) Q on the upper surface of the lower plane: This will be

$$Q_t = \int_0^{.1}\int_{.001}^{.120} \frac{2.00 \times 10^4\epsilon_0}{\rho}\,d\rho\,dz = 2.00 \times 10^4\epsilon_0(.1)\ln(120) = 8.47 \times 10^{-8} \text{ C} = \underline{84.7 \text{ nC}}$$

f) Repeat *a*) to *c*) for region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. Back to the beginning, we use

$$20 = C_1'(.188 - 2\pi) + C_2' \text{ and } 200 = C_1'(.179) + C_2'$$

7.15f (continued) Subtracting one from the other, we find

$$-180 = C_1'(.009 - 2\pi) \;\Rightarrow\; C_1' = 28.7$$

Then $200 = 28.7(.179) + C_2' \;\Rightarrow\; C_2' = 194.9$. Thus $V(\phi) = \underline{28.7\phi + 194.9}$ in region 2. Then

$$\mathbf{E} = \underline{-\frac{28.7}{\rho}\,\mathbf{a}_\phi \text{ V/m}} \text{ and } \mathbf{D} = \underline{-\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \text{ C/m}^2}$$

ρ_s on the lower surface of the lower plane will now be

$$\rho_s = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) = \underline{\frac{28.7\epsilon_0}{\rho} \text{ C/m}^2}$$

The charge on that surface will then be $Q_b = 28.7\epsilon_0(.1)\ln(120) = \underline{122 \text{ pC}}$.

g) Find the total charge on the lower plane and the capacitance between the planes: Total charge will be $Q_{net} = Q_t + Q_b = 84.7 \text{ nC} + 0.122 \text{ nC} = \underline{84.8 \text{ nC}}$. The capacitance will be

$$C = \frac{Q_{net}}{\Delta V} = \frac{84.8}{200 - 20} = 0.471 \text{ nF} = \underline{471 \text{ pF}}$$

7.16. a) Solve Laplace's equation for the potential field in the homogeneous region between two concentric conducting spheres with radii a and b, $b > a$, if $V = 0$ at $r = b$ and $V = V_0$ at $r = a$. With radial variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

Multiply by r^2:

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \text{ or } r^2\frac{dV}{dr} = A$$

Divide by r^2:

$$\frac{dV}{dr} = \frac{A}{r^2} \;\Rightarrow\; V = \frac{A}{r} + B$$

Note that in the last integration step, I dropped the minus sign that would have otherwise occurred in front of A, since we can choose A as we wish. Next, apply the boundary conditions:

$$0 = \frac{A}{b} + B \;\Rightarrow\; B = -\frac{A}{b}$$

$$V_0 = \frac{A}{a} - \frac{A}{b} \;\Rightarrow\; A = \frac{V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

Finally,

$$V(r) = \frac{V_0}{r\left(\frac{1}{a} - \frac{1}{b}\right)} - \frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)} = \underline{V_0\frac{\left(\frac{1}{r} - \frac{1}{b}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

7.16b. Find the capacitance between them: Assume permittivity ϵ. First, the electric field will be

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}\mathbf{a}_r \text{ V/m}$$

Next, on the inner sphere, the charge density will be

$$\rho_s = \mathbf{D}\Big|_{r=a} \cdot \mathbf{a}_r = \frac{\epsilon V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \text{ C/m}^2$$

The capacitance is now

$$C = \frac{Q}{V_0} = \frac{4\pi a^2 \rho_s}{V_0} = \underline{\frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)} \text{ F}}$$

7.17. Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere at 0 V:

a) Find the location of the 20 V equipotential surface: Solving Laplace's equation gives us

$$V(r) = V_0 \frac{\frac{1}{r}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}$$

where $V_0 = 100$, $a = 5$ and $b = 20$. Setting $V(r) = 20$, and solving for r produces $r = \underline{12.5 \text{ mm}}$.

b) Find $E_{r,max}$: Use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0\,\mathbf{a}_r}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

Then

$$E_{r,max} = E(r=a) = \frac{V_0}{a(1-(a/b))} = \frac{100}{5(1-(5/20))} = 26.7 \text{ V/mm} = \underline{26.7 \text{ kV/m}}$$

c) Find ϵ_R if the surface charge density on the inner sphere is $100\,\mu\text{C/m}^2$: ρ_s will be equal in magnitude to the electric flux density at $r = a$. So $\rho_s = (2.67 \times 10^4 \text{ V/m})\epsilon_R\epsilon_0 = 10^{-4} \text{ C/m}^2$. Thus $\epsilon_R = \underline{423 \text{ !}}$ (obviously a bad choice of numbers here – possibly a misprint. A more reasonable charge on the inner sphere would have been 1 $\mu\text{C/m}^2$, leading to $\epsilon_R = 4.23$).

7.18. Concentric conducting spheres have radii of 1 and 5 cm. There is a perfect dielectric for which $\epsilon_R = 3$ between them. The potential of the inner sphere is 2V and that of the outer is -2V. Find:

a) $V(r)$: We use the general expression derived in Problem 7.16: $V(r) = (A/r) + B$. At the inner sphere, $2 = (A/.01) + B$, and at the outer sphere, $-2 = (A/.05) + B$. Subtracting the latter equation from the former gives

$$4 = A\left(\frac{1}{.01} - \frac{1}{.05}\right) = 80A$$

so $A = .05$. Substitute A into either of the two potential equations at the boundaries to find $B = -3$. Finally, $V(r) = \underline{(.05/r) - 3}$.

7.18b. $\mathbf{E}(r) = -(dV/dr)\mathbf{a}_r = \underline{(.05/r^2)\mathbf{a}_r \text{ V/m}}$.

c) V at $r = 3$ cm: $V(.03) = (.05/.03) - 3 = \underline{-1.33 \text{ V}}$.

d) the location of the 0-V equipotential surface: Use

$$0 = (.05/r_0) - 3 \;\Rightarrow\; r_0 = (.05/3) = .0167\,\text{m} = \underline{1.67\,\text{cm}}$$

e) the capacitance between the spheres:

$$C = \frac{4\pi\epsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi(3)\epsilon_0}{\left(\frac{1}{.01} - \frac{1}{.05}\right)} = \frac{12\pi\epsilon_0}{80} = \underline{4.2\,\text{pF}}$$

7.19. Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1, 0, 2)$ on its surface, while cone B has the point $B(0, 3, 2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find:

a) α for each cone: Have $\alpha_A = \tan^{-1}(1/2) = \underline{26.57^\circ}$ and $\alpha_B = \tan^{-1}(3/2) = \underline{56.31^\circ}$.

b) V at $P(1, 1, 1)$: The potential function between cones can be written as

$$V(\theta) = C_1 \ln\tan(\theta/2) + C_2$$

Then

$$20 = C_1 \ln\tan(56.31/2) + C_2 \;\text{ and }\; 100 = C_1 \ln\tan(26.57/2) + C_2$$

Solving these two equations, we find $C_1 = -97.7$ and $C_2 = -41.1$. Now at P, $\theta = \tan^{-1}(\sqrt{2}) = 54.7^\circ$. Thus

$$V_P = -97.7 \ln\tan(54.7/2) - 41.1 = \underline{23.3\,\text{V}}$$

7.20. A potential field in free space is given as $V = 100 \ln\tan(\theta/2) + 50$ V.

a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. First

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{100}{2r\tan(\theta/2)\cos^2(\theta/2)}\mathbf{a}_\theta = -\frac{100}{2r\sin(\theta/2)\cos(\theta/2)}\mathbf{a}_\theta = -\frac{100}{r\sin\theta}\mathbf{a}_\theta$$

This will maximize at the smallest value of r, or 0.1:

$$\mathbf{E}_{max}(\theta = 40^\circ) = \mathbf{E}(r = 0.1, \theta = 40^\circ) = -\frac{100}{0.1\sin(40)}\mathbf{a}_\theta = \underline{1.56\,\mathbf{a}_\theta \text{ kV/m}}$$

b) Describe the surface $V = 80$ V: Set $100 \ln\tan\theta/2 + 50 = 80$ and solve for θ: Obtain $\ln\tan\theta/2 = 0.3 \;\Rightarrow\; \tan\theta/2 = e^{.3} = 1.35 \;\Rightarrow\; \theta = \underline{107^\circ}$ (the cone surface at $\theta = 107$ degrees).

7.21. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2E_r \to 0$ when $r \to 0$, and also that $V \to 0$ as $r \to \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2\frac{dV}{dr}\right) = -\frac{200}{.6}r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2E_r \to 0$ when $r \to 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4}r^{-.4} + C_2$$

From our second boundary condition, $V \to 0$ as $r \to \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4}\text{ V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi\int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0\frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3r^{-1.4}\text{ V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4}dr' = \underline{833.3r^{-.4}\text{ V}}$$

7.22. Let the volume charge density in Fig. 7.3a be given by $\rho_v = \rho_{v0}(x/a)e^{-|x|/a}$ (note error in the exponent in the formula stated in the book).

a) Determine $\rho_{v,max}$ and $\rho_{v,min}$ and their locations: Let $x' = x/a$. Then $\rho_v = x'e^{-|x'|}$. Differentiate with respect to x' to obtain:

$$\frac{d\rho_v}{dx'} = \rho_{v0}e^{-|x'|}(1 - |x'|)$$

This derivative is zero at $x' = \pm 1$, or the minimum and maximum occur at $x = \underline{\pm a}$ respectively. The values of ρ_v at these points will be $\rho_{v,max} = \rho_{v0}e^{-1} = \underline{0.368\rho_{v0}}$, occurring at $x = a$. $\rho_{v,min} = -\rho_{v0}e^{-1} = \underline{-0.368\rho_{v0}}$, occurring at $x = -a$.

7.22b. Find E_x and $V(x)$ if $V(0) = 0$ and $E_x \to 0$ as $x \to \infty$: We use Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \Rightarrow \quad \frac{d^2V}{dx^2} = -\frac{\rho_{v0}}{\epsilon}\left(\frac{x}{a}\right)e^{-|x|/a}$$

For $x > 0$, this becomes

$$\frac{d^2V}{dx^2} = -\frac{\rho_{v0}}{\epsilon}\left(\frac{x}{a}\right)e^{-x/a}$$

Integrate once over x:

$$\frac{dV}{dx}(x > 0) = -\frac{\rho_{v0}}{\epsilon}\int\left(\frac{x}{a}\right)e^{-x/a}\,dx + C_1 = \frac{a\rho_{v0}}{\epsilon}e^{-x/a}\left(\frac{x}{a}+1\right) + C_1$$

Noting that $E_x = -dV/dx$, we use the first boundary condition, $E_x \to 0$ as $x \to \infty$, to establish that $C_1 = 0$. Over the range $x < 0$, we have

$$\frac{dV}{dx}(x < 0) = -\frac{\rho_{v0}}{\epsilon}\int\left(\frac{x}{a}\right)e^{x/a}\,dx + C_1' = \frac{a\rho_{v0}}{\epsilon}e^{x/a}\left(\frac{-x}{a}+1\right) + C_1'$$

where $C_1' = 0$, since, by symmetry, $E_x \to 0$ as $x \to -\infty$. These two equations can be unified to cover the entire range of x; the final expression for the electric field becomes:

$$E_x = -\frac{dV}{dx} = \underline{-\frac{a\rho_{v0}}{\epsilon}\left(\frac{|x|}{a}+1\right)e^{-|x|/a}\ \text{V/m}}$$

The potential function is now found by a second integration. For $x > 0$, this is

$$V(x) = \frac{a\rho_{v0}}{\epsilon}\int\left[\left(\frac{x}{a}\right)e^{-x/a} + e^{-x/a}\right]dx + C_2 = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{-x/a} - 2e^{-x/a}\right] + C_2$$

We use the second boundary condition, $V(0) = 0$, from which $C_2 = 2a^2\rho_{v0}/\epsilon$. Substituting this yields

$$V(x)\ (x > 0) = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{-x/a} + 2(1 - e^{-x/a})\right]$$

We repeat the procedure for $x < 0$ to obtain

$$V(x) = \frac{a\rho_{v0}}{\epsilon}\int\left[\left(\frac{x}{a}\right)e^{x/a} + e^{x/a}\right]dx + C_2' = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{x/a} - 2e^{x/a}\right] + C_2'$$

Again, with the $V(0) = 0$ boundary condition, we find $C_2' = -2a^2\rho_{v0}/\epsilon$, which when substituted leads to

$$V(x)\ (x < 0) = \frac{a^2\rho_{v0}}{\epsilon}\left[\frac{-x}{a}e^{x/a} - 2(1 - e^{x/a})\right]$$

Combining the results for both ranges of x, we write

$$V(x) = \underline{\frac{-a^2\rho_{v0}}{\epsilon}\left[\left(\frac{x}{a}\right)e^{-|x|/a} - \frac{2x}{|x|}\left(1 - e^{-|x|/a}\right)\right]}$$

7.22c. Use a development similar to that of Sec. 7.4 to show that $C = dQ/dV_0 = \epsilon S/8a$ (note error in problem statement): First, the overall potential difference is

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = 2 \times \frac{2a^2\rho_{v0}}{\epsilon} = \frac{4a^2\rho_{v0}}{\epsilon}$$

From this we find $a = \sqrt{(\epsilon V_0)/(4\rho_{v0})}$. Then the total charge on one side will be

$$Q = S\int_0^\infty \rho_{v0}\left(\frac{x}{a}\right)e^{-x/a}\,dx = S\rho_{v0}\,ae^{-x/a}\left[\frac{-x}{a} - 1\right]\Big|_0^\infty = S\rho_{v0}\,a = \frac{1}{2}S\sqrt{\epsilon V_0\rho_{v0}}$$

Now

$$C = \frac{dQ}{dV_0} = \frac{d}{dV_0}\left(\frac{1}{2}S\sqrt{\epsilon V_0\rho_{v0}}\right) = \frac{S}{4}\sqrt{\frac{\rho_{v0}\,\epsilon}{V_0}}$$

But $a = \sqrt{(\epsilon V_0)/(4\rho_{v0})}$, from which $(\rho_{v0}/V_0) = \epsilon/(4a^2)$. Substituting this into the capacitance expression gives

$$C = \frac{S}{4}\sqrt{\frac{\epsilon^2}{4a^2}} = \underline{\frac{\epsilon S}{8a}}$$

7.23. A rectangular trough is formed by four conducting planes located at $x = 0$ and 8 cm and $y = 0$ and 5 cm in air. The surface at $y = 5$ cm is at a potential of 100 V, the other three are at zero potential, and the necessary gaps are placed at two corners. Find the potential at $x = 3$ cm, $y = 4$ cm: This situation is the same as that of Fig. 7.6, except the non-zero boundary potential appears on the top surface, rather than the right side. The solution is found from Eq. (39) by simply interchanging x and y, and b and d, obtaining:

$$V(x,\,y) = \frac{4V_0}{\pi}\sum_{1,odd}^{\infty}\frac{1}{m}\frac{\sinh(m\pi y/d)}{\sinh(m\pi b/d)}\sin\frac{m\pi x}{d}$$

where $V_0 = 100$ V, $d = 8$ cm, and $b = 5$ cm. We will use the first three terms to evaluate the potential at (3,4):

$$V(3,\,4) \doteq \frac{400}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(5\pi/8)}\sin(3\pi/8) + \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(15\pi/8)}\sin(9\pi/8) + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(25\pi/8)}\sin(15\pi/8)\right]$$
$$= \frac{400}{\pi}\,[.609 - .040 - .011] = 71.1\text{ V}$$

Additional accuracy is found by including more terms in the expansion. Using thirteen terms, and using six significant figure accuracy, the result becomes $V(3,\,4) \doteq \underline{71.9173\text{ V}}$. The series converges rapidly enough so that terms after the sixth one produce no change in the third digit. Thus, quoting three significant figures, 71.9 V requires six terms, with subsequent terms having no effect.

7.24. The four sides of a square trough are held at potentials of 0, 20, -30, and 60 V; the highest and lowest potentials are on opposite sides. Find the potential at the center of the trough: Here we can make good use of symmetry. The solution for a single potential on the right side, for example, with all other sides at 0V is given by Eq. (39):

$$V(x, y) = \frac{4V_0}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi x/b)}{\sinh(m\pi d/b)} \sin\left(\frac{m\pi y}{b}\right)$$

In the current problem, we can account for the three voltages by superposing three solutions of the above form, suitably modified to account for the different locations of the boundary potentials. Since we want V at the center of a square trough, it no longer matters on what boundary each of the given potentials is, and we can simply write:

$$V(\text{center}) = \frac{4(0 + 20 - 30 + 60)}{\pi} \sum_{1,\text{odd}}^{\infty} \frac{1}{m} \frac{\sinh(m\pi/2)}{\sinh(m\pi)} \sin(m\pi/2) = \underline{12.5\,\text{V}}$$

The series converges to this value in three terms.

7.25. In Fig. 7.7, change the right side so that the potential varies linearly from 0 at the bottom of that side to 100 V at the top. Solve for the potential at the center of the trough: Since the potential reaches zero periodically in y and also is zero at $x = 0$, we use the form:

$$V(x, y) = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi x}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

Now, at $x = d$, $V = 100(y/b)$. Thus

$$100\frac{y}{b} = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \sin\left(\frac{m\pi y}{b}\right)$$

We then multiply by $\sin(n\pi y/b)$, where n is a fixed integer, and integrate over y from 0 to b:

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \sum_{m=1}^{\infty} V_m \sinh\left(\frac{m\pi d}{b}\right) \underbrace{\int_0^b \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{n\pi y}{b}\right) dy}_{=b/2 \text{ if } m=n, \text{ zero if } m\neq n}$$

The integral on the right hand side picks the nth term out of the series, enabling the coefficients, V_n, to be solved for individually as we vary n. We find in general,

$$V_m = \frac{2}{b \sinh(m\pi/d)} \int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy$$

The integral evaluates as

$$\int_0^b 100\frac{y}{b} \sin\left(\frac{n\pi y}{b}\right) dy = \left\{ \begin{array}{c} -100/m\pi \text{ (m even)} \\ 100/m\pi \text{ (m odd)} \end{array} \right\} = (-1)^{m+1} \frac{100}{m\pi}$$

7.25 (continued) Thus

$$V_m = \frac{200(-1)^{m+1}}{m\pi b\,\sinh(m\pi d/b)}$$

So that finally,

$$V(x,\,y) = \frac{200}{\pi b}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\frac{\sinh\,(m\pi x/b)}{\sinh\,(m\pi d/b)}\sin\left(\frac{m\pi y}{b}\right)$$

Now, with a square trough, set $b = d = 1$, and so $0 < x < 1$ and $0 < y < 1$. The potential becomes

$$V(x,\,y) = \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\frac{\sinh\,(m\pi x)}{\sinh\,(m\pi)}\sin\,(m\pi y)$$

Now at the center of the trough, $x = y = 0.5$, and, using four terms, we have

$$V(.5,\,.5) \doteq \frac{200}{\pi}\left[\frac{\sinh(\pi/2)}{\sinh(\pi)} - \frac{1}{3}\frac{\sinh(3\pi/2)}{\sinh(3\pi)} + \frac{1}{5}\frac{\sinh(5\pi/2)}{\sinh(5\pi)} - \frac{1}{7}\frac{\sinh(7\pi/2)}{\sinh(7\pi)}\right] = \underline{12.5\text{ V}}$$

where additional terms do not affect the three-significant-figure answer.

7.26. If X is a function of x and $X'' + (x-1)X - 2X = 0$, assume a solution in the form of an infinite power series and determine numerical values for a_2 to a_8 if $a_0 = 1$ and $a_1 = -1$: The series solution will be of the form:

$$X = \sum_{m=0}^{\infty} a_m x^m$$

The first 8 terms of this are substituted into the given equation to give:

$$\begin{aligned}&(2a_2 - a_1 - 2a_0) + (6a_3 + a_1 - 2a_2 - 2a_1)x + (12a_4 + 2a_2 - 3a_3 - 2a_2)x^2\\ &+ (3a_3 - 4a_4 - 2a_3 + 20a_5)x^3 + (30a_6 + 4a_4 - 5a_5 - 2a_4)x^4 + (42a_7 + 5a_5 - 6a_6 - 2a_5)x^5\\ &+ (56a_8 + 6a_6 - 7a_7 - 2a_6)x^6 + (7a_7 - 8a_8 - 2a_7)x^7 + (8a_8 - 2a_8)x^8 = 0\end{aligned}$$

For this equation to be zero, each coefficient term (in parenthesis) must be zero. The first of these is

$$2a_2 - a_1 - 2a_0 = 2a_2 + 1 - 2 = 0 \;\Rightarrow\; a_2 = \underline{1/2}$$

The second coefficient is

$$6a_3 + a_1 - 2a_2 - 2a_1 = 6a_3 - 1 - 1 + 2 = 0 \;\Rightarrow\; a_3 = \underline{0}$$

Third coefficient:

$$12a_4 + 2a_2 - 3a_3 - 2a_2 = 12a_4 + 1 - 0 - 1 = 0 \;\Rightarrow\; a_4 = \underline{0}$$

Fourth coefficient:

$$3a_3 - 4a_4 - 2a_3 + 20a_5 = 0 - 0 - 0 + 20a_5 = 0 \;\Rightarrow\; a_5 = \underline{0}$$

In a similar manner, we find $a_6 = a_7 = a_8 = \underline{0}$.

7.27. It is known that $V = XY$ is a solution of Laplace's equation, where X is a function of x alone, and Y is a function of y alone. Determine which of the following potential function are also solutions of Laplace's equation:

a) $V = 100X$: We know that $\nabla^2 XY = 0$, or

$$\frac{\partial^2}{\partial x^2}XY + \frac{\partial^2}{\partial y^2}XY = 0 \;\Rightarrow\; YX'' + XY'' = 0 \;\Rightarrow\; \frac{X''}{X} = -\frac{Y''}{Y} = \alpha^2$$

Therefore, $\nabla^2 X = 100X'' \neq 0$ – No.

b) $V = 50XY$: Would have $\nabla^2 V = 50\nabla^2 XY = 0$ – Yes.

c) $V = 2XY + x - 3y$: $\nabla^2 V = 2\nabla^2 XY + 0 - 0 = 0$ – Yes.

d) $V = xXY$: $\nabla^2 V = x\nabla^2 XY + XY\nabla^2 x = 0$ – Yes.

e) $V = X^2 Y$: $\nabla^2 V = X\nabla^2 XY + XY\nabla^2 X = 0 + XY\nabla^2 X$ – No.

7.28. Assume a product solution of Laplace's equation in cylindrical coordinates, $V = PF$, where V is not a function of z, P is a function only of ρ, and F is a function only of ϕ.

a) Obtain the two separated equations if the separation constant is n^2. Select the sign of n^2 so that the solution of the ϕ equation leads to trigonometric functions: Begin with Laplace's equation in cylindrical coordinates, in which there is no z variation:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

We substitute the product solution $V = PF$ to obtain:

$$\frac{F}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = \frac{F}{\rho}\frac{dP}{d\rho} + F\frac{d^2P}{d\rho^2} + \frac{P}{\rho^2}\frac{d^2F}{d\phi^2} = 0$$

Next, multiply by ρ^2 and divide by FP to obtain

$$\underbrace{\frac{\rho}{P}\frac{dP}{d\rho} + \frac{\rho^2}{P}\frac{d^2P}{d\rho^2}}_{n^2} + \underbrace{\frac{1}{F}\frac{d^2F}{d\phi^2}}_{-n^2} = 0$$

The equation is now grouped into two parts as shown, each a function of only one of the two variables; each is set equal to plus or minus n^2, as indicated. The ϕ equation now becomes

$$\frac{d^2F}{d\phi^2} + n^2 F = 0 \;\Rightarrow\; F = C_n\cos(n\phi) + D_n\sin(n\phi) \;\; (n \geq 1)$$

Note that n is required to be an integer, since physically, the solution must repeat itself every 2π radians in ϕ. If $n = 0$, then

$$\frac{d^2F}{d\phi^2} = 0 \;\Rightarrow\; F = C_0\phi + D_0$$

7.28b. Show that $P = A\rho^n + B\rho^{-n}$ satisfies the ρ equation: From part a, the radial equation is:

$$\rho^2 \frac{d^2P}{d\rho^2} + \rho \frac{dP}{d\rho} - n^2 P = 0$$

Substituting $A\rho^n$, we find

$$\rho^2 n(n-1)\rho^{n-2} + \rho n \rho^{n-1} - n^2\rho^n = n^2\rho^n - n\rho^n + n\rho^n - n^2\rho^n = 0$$

Substituting $B\rho^{-n}$:

$$\rho^2 n(n+1)\rho^{-(n+2)} - \rho n \rho^{-(n+1)} - n^2\rho^{-n} = n^2\rho^{-n} + n\rho^{-n} - n\rho^{-n} - n^2\rho^{-n} = 0$$

So it works.

c) Construct the solution $V(\rho, \phi)$. Functions of this form are called *circular harmonics*. To assemble the complete solution, we need the radial solution for the case in which $n = 0$. The equation to solve is

$$\rho \frac{d^2P}{d\rho^2} + \frac{dP}{d\rho} = 0$$

Let $S = dP/d\rho$, and so $dS/d\rho = d^2P/d\rho^2$. The equation becomes

$$\rho \frac{dS}{d\rho} + S = 0 \;\Rightarrow\; -\frac{d\rho}{\rho} = \frac{dS}{S}$$

Integrating, find

$$-\ln\rho + \ln A_0 = \ln S \;\Rightarrow\; \ln S = \ln\left(\frac{A_0}{\rho}\right) \;\Rightarrow\; S = \frac{A_0}{\rho} = \frac{dP}{d\rho}$$

where A_0 is a constant. So now

$$\frac{d\rho}{\rho} = \frac{dP}{A_0} \;\Rightarrow\; P_{n=0} = A_0 \ln\rho + B_0$$

We may now construct the solution in its complete form, encompassing $n \geq 0$:

$$V(\rho,\phi) = \underbrace{(A_0 \ln\rho + B_0)(C_0\phi + D_0)}_{n=0 \text{ solution}} + \sum_{n=1}^{\infty} [A_n\rho^n + B_n\rho^{-n}][C_n\cos(n\phi) + D_n\sin(n\phi)]$$

CHAPTER 8

8.1a. Find $\mathbf{H}$ in cartesian components at $P(2, 3, 4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2 - 8z + 29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y\ \mu\text{A/m}}$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2 - 8z + 26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2 - 8z + 26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y\ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts *a* and *b*, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y\ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

8.2. A current filament of $3\mathbf{a}_x$ A lies along the x axis. Find $\mathbf{H}$ in cartesian components at $P(-1, 3, 2)$: We use the Biot-Savart law,

$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where $Id\mathbf{L} = 3dx\mathbf{a}_x$, $\mathbf{a}_R = [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]/R$, and $R = \sqrt{x^2 + 2x + 14}$. Thus

$$\mathbf{H}_P = \int_{-\infty}^{\infty} \frac{3dx\mathbf{a}_x \times [-(1+x)\mathbf{a}_x + 3\mathbf{a}_y + 2\mathbf{a}_z]}{4\pi(x^2 + 2x + 14)^{3/2}} = \int_{-\infty}^{\infty} \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)\,dx}{4\pi(x^2 + 2x + 14)^{3/2}}$$
$$= \frac{(9\mathbf{a}_z - 6\mathbf{a}_y)(x+1)}{4\pi(13)\sqrt{x^2 + 2x + 14}}\Bigg|_{-\infty}^{\infty} = \frac{2(9\mathbf{a}_z - 6\mathbf{a}_y)}{4\pi(13)} = \underline{0.110\mathbf{a}_z - 0.073\mathbf{a}_y\ \text{A/m}}$$

8.3. Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ (note typographical error in problem statement) and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction.

a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$: One way to do this is to use the field from an infinite line and subtract from it that portion of the field that would arise from the current segment at $-a < z < a$, found from the Biot-Savart law. Thus,

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi - \int_{-a}^{a} \frac{I\,dz\,\mathbf{a}_z \times [\rho\,\mathbf{a}_\rho - z\,\mathbf{a}_z]}{4\pi[\rho^2 + z^2]^{3/2}}$$

The integral part simplifies and is evaluated:

$$\int_{-a}^{a} \frac{I\,dz\,\rho\,\mathbf{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} = \frac{I\rho}{4\pi}\,\mathbf{a}_\phi \left.\frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\right|_{-a}^{a} = \frac{Ia}{2\pi\rho\sqrt{\rho^2 + a^2}}\,\mathbf{a}_\phi$$

Finally,

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]\mathbf{a}_\phi \ \text{A/m}$$

b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament? We require

$$\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]_{\rho=1} = \frac{1}{2} \ \Rightarrow \ \frac{a}{\sqrt{1 + a^2}} = \frac{1}{2} \ \Rightarrow \ a = \underline{1/\sqrt{3}}$$

8.4a.) A filament is formed into a circle of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Find $\mathbf{H}$ at the origin: We use the Biot-Savart law, which in this case becomes:

$$\mathbf{H} = \int_{loop} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_0^{2\pi} \frac{Ia\,d\phi\,\mathbf{a}_\phi \times (-\mathbf{a}_\rho)}{4\pi a^2} = \underline{0.50\,\frac{I}{a}\,\mathbf{a}_z \ \text{A/m}}$$

b.) A filament of the same length is shaped into a square in the $z = 0$ plane. The sides are parallel to the coordinate axes and a current I flows in the general $\mathbf{a}_\phi$ direction. Again, find $\mathbf{H}$ at the origin: Since the loop is the same length, its perimeter is $2\pi a$, and so each of the four sides is of length $\pi a/2$. Using symmetry, we can find the magnetic field at the origin associated with each of the 8 half-sides (extending from 0 to $\pm\pi a/4$ along each coordinate direction) and multiply the result by 8: Taking one of the segments in the y direction, the Biot-Savart law becomes

$$\mathbf{H} = \int_{loop} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = 8\int_0^{\pi a/4} \frac{I dy\,\mathbf{a}_y \times \left[-(\pi a/4)\,\mathbf{a}_x - y\,\mathbf{a}_y\right]}{4\pi\left[y^2 + (\pi a/4)^2\right]^{3/2}}$$

$$= \frac{aI}{2}\int_0^{\pi a/4} \frac{dy\,\mathbf{a}_z}{\left[y^2 + (\pi a/4)^2\right]^{3/2}} = \frac{aI}{2}\left.\frac{y\,\mathbf{a}_z}{(\pi a/4)^2\sqrt{y^2 + (\pi a/4)^2}}\right|_0^{\pi a/4} = \underline{0.57\,\frac{I}{a}\,\mathbf{a}_z \ \text{A/m}}$$

8.5. The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$: We need an expression for $\mathbf{H}$ in cartesian coordinates. We can start with the known $\mathbf{H}$ in cylindrical for an infinite filament along the z axis: $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, which we transform to cartesian to obtain:

$$\mathbf{H} = \frac{-Iy}{2\pi(x^2+y^2)}\,\mathbf{a}_x + \frac{Ix}{2\pi(x^2+y^2)}\,\mathbf{a}_y$$

If we now rotate the filament so that it lies along the x axis, with current flowing in positive x, we obtain the field from the above expression by replacing x with y and y with z:

$$\mathbf{H} = \frac{-Iz}{2\pi(y^2+z^2)}\,\mathbf{a}_y + \frac{Iy}{2\pi(y^2+z^2)}\,\mathbf{a}_z$$

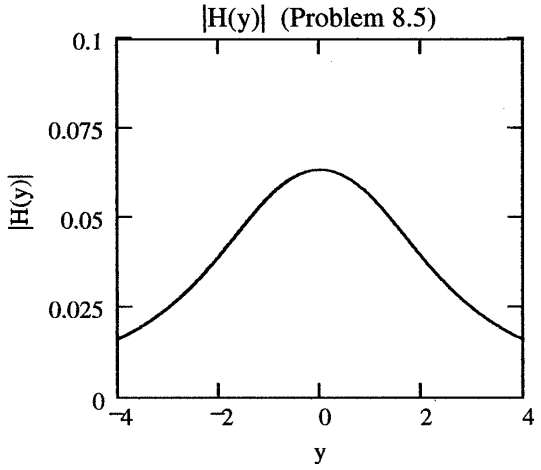
Now, with two filaments, displaced from the x axis to lie at $y = \pm 1$, and with the current directions as shown in the figure, we use the previous expression to write

$$\mathbf{H} = \left[\frac{Iz}{2\pi[(y+1)^2+z^2]} - \frac{Iz}{2\pi[(y-1)^2+z^2]}\right]\mathbf{a}_y + \left[\frac{I(y-1)}{2\pi[(y-1)^2+z^2]} - \frac{I(y+1)}{2\pi[(y+1)^2+z^2]}\right]\mathbf{a}_z$$

We now evaluate this at $z = 2$, and find the magnitude ($\sqrt{\mathbf{H}\cdot\mathbf{H}}$), resulting in

$$|\mathbf{H}| = \frac{I}{2\pi}\left[\left(\frac{2}{y^2+2y+5} - \frac{2}{y^2-2y+5}\right)^2 + \left(\frac{(y-1)}{y^2-2y+5} - \frac{(y+1)}{y^2+2y+5}\right)^2\right]^{1/2}$$

This function is plotted below



8.6a. A current filament I is formed into circle, $\rho = a$, in the $z = z'$ plane. Find H_z at $P(0, 0, z)$ if I flows in the $\mathbf{a}_\phi$ direction: Use the Biot-Savart law,

$$\mathbf{H} = \int \frac{Id\mathbf{L}\times\mathbf{a}_R}{4\pi R^2}$$

where in this case $Id\mathbf{L} = Id\phi\mathbf{a}_\phi$, $\mathbf{a}_R = [-a\mathbf{a}_\rho + (z-z')\mathbf{a}_z]/R$, and $R = \sqrt{a^2+(z-z')^2}$. The setup becomes

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times [-a\mathbf{a}_\rho + (z-z')\mathbf{a}_z]}{4\pi[a^2+(z-z')^2]^{3/2}} = \int_0^{2\pi} \frac{Ia[a\mathbf{a}_z + (z-z')\mathbf{a}_\rho]\,d\phi}{4\pi[a^2+(z-z')^2]^{3/2}}$$

At this point we need to be especially careful. Note that we are integrating a vector with an $\mathbf{a}_\rho$ component around a complete circle, where the vector has no ϕ dependence. This sum of all $\mathbf{a}_\rho$ components will be zero – even though this doesn't happen when we go ahead with the integration without this knowledge. The problem is that the integral "interprets" $\mathbf{a}_\rho$ as a constant direction, when in fact – as we know – $\mathbf{a}_\rho$ continually changes direction as ϕ varies. We drop the $\mathbf{a}_\rho$ component in the integral to give

$$\mathbf{H} = \int_0^{2\pi} \frac{Ia^2\mathbf{a}_z\, d\phi}{4\pi[a^2+(z-z')^2]^{3/2}} = \frac{\pi a^2 I\mathbf{a}_z}{2\pi[a^2+(z-z')^2]^{3/2}} = \underline{\frac{\mathbf{m}}{2\pi[a^2+(z-z')^2]^{3/2}}\ \text{A/m}}$$

where $\mathbf{m} = \pi a^2 I\mathbf{a}_z$ is the *magnetic moment* of the loop.

b) Find H_z at P caused by a uniform surface current density $\mathbf{K} = K_0\mathbf{a}_\phi$, flowing on the cylindrical surface, $\rho = a$, $0 < z < h$. The results of part a should help: Using part a, we can write down the differential field at P arising from a circular current ribbon of differential height, dz', at location z'. The ribbon is of radius a and carries current $K_0 dz'\mathbf{a}_\phi$ A:

$$d\mathbf{H} = \frac{\pi a^2 K_0 dz'\mathbf{a}_z}{2\pi[a^2+(z-z')^2]^{3/2}}\ \text{A/m}$$

The total magnetic field at P is now the sum of the contributions of all differential rings that comprise the cylinder:

$$\begin{aligned} H_z &= \int_0^h \frac{\pi a^2 K_0 dz'}{2\pi[a^2+(z-z')^2]^{3/2}} = \frac{a^2K_0}{2}\int_0^h \frac{dz'}{[a^2+z^2-2zz'+(z')^2]^{3/2}} \\ &= \frac{a^2K_0}{2}\frac{2(2z'-2z)}{4a^2\sqrt{a^2+z^2-2zz'+(z')^2}}\Big|_0^h = \frac{K_0(z'-z)}{2\sqrt{a^2+(z'-z)^2}}\Big|_0^h \\ &= \underline{\frac{K_0}{2}\left[\frac{(h-z)}{\sqrt{a^2+(h-z)^2}} + \frac{z}{\sqrt{a^2+z^2}}\right]\ \text{A/m}} \end{aligned}$$

8.7. Given points $C(5,-2,3)$ and $P(4,-1,2)$; a current element $Id\mathbf{L} = 10^{-4}(4,-3,1)$ A · m at C produces a field $d\mathbf{H}$ at P.

a) Specify the direction of $d\mathbf{H}$ by a unit vector $\mathbf{a}_H$: Using the Biot-Savart law, we find

$$d\mathbf{H} = \frac{Id\mathbf{L}\times\mathbf{a}_{CP}}{4\pi R_{CP}^2} = \frac{10^{-4}[4\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z]\times[-\mathbf{a}_x+\mathbf{a}_y-\mathbf{a}_z]}{4\pi 3^{3/2}} = \frac{[2\mathbf{a}_x+3\mathbf{a}_y+\mathbf{a}_z]\times 10^{-4}}{65.3}$$

from which

$$\mathbf{a}_H = \frac{2\mathbf{a}_x+3\mathbf{a}_y+\mathbf{a}_z}{\sqrt{14}} = \underline{0.53\mathbf{a}_x + 0.80\mathbf{a}_y + 0.27\mathbf{a}_z}$$

b) Find $|d\mathbf{H}|$.

$$|d\mathbf{H}| = \frac{\sqrt{14}\times 10^{-4}}{65.3} = 5.73\times 10^{-6}\ \text{A/m} \quad = \underline{5.73\ \mu\text{A/m}}$$

c) What direction $\mathbf{a}_l$ should $Id\mathbf{L}$ have at C so that $d\mathbf{H} = 0$? $Id\mathbf{L}$ should be collinear with $\mathbf{a}_{CP}$, thus rendering the cross product in the Biot-Savart law equal to zero. Thus the answer is $\mathbf{a}_l = \underline{\pm(-\mathbf{a}_x+\mathbf{a}_y-\mathbf{a}_z)/\sqrt{3}}$

8.8. For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1: The Biot-Savart law reads:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} \frac{I dz \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2 + z^2)^{3/2}} = \int_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} \frac{I\rho \mathbf{a}_\phi \, dz}{4\pi(\rho^2 + z^2)^{3/2}}$$

The integral is evaluated (using tables) and gives the desired result:

$$\mathbf{H} = \frac{Iz\mathbf{a}_\phi}{4\pi\rho\sqrt{\rho^2 + z^2}}\Big|_{\rho \tan \alpha_1}^{\rho \tan \alpha_2} = \frac{I}{4\pi\rho}\left[\frac{\tan \alpha_2}{\sqrt{1 + \tan^2 \alpha_2}} - \frac{\tan \alpha_1}{\sqrt{1 + \tan^2 \alpha_1}}\right]\mathbf{a}_\phi = \frac{I}{4\pi\rho}(\sin \alpha_2 - \sin \alpha_1)\mathbf{a}_\phi$$

8.9. A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0, 0, 3)$: Using the Biot-Savart law, we write

$$\mathbf{H}_P = \int\int \frac{\mathbf{K} \times \mathbf{a}_R \, dx \, dy}{4\pi R^2} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\mathbf{a}_x \times (-x\mathbf{a}_x - y\mathbf{a}_y + 3\mathbf{a}_z)}{4\pi(x^2 + y^2 + 9)^{3/2}} \, dx \, dy$$

Taking the cross product gives:

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\mathbf{a}_z - 3\mathbf{a}_y) \, dx \, dy}{4\pi(x^2 + y^2 + 9)^{3/2}}$$

We note that the z component is anti-symmetric in y about the origin (odd parity). Since the limits are symmetric, the integral of the z component over y is zero. We are left with

$$\begin{aligned}\mathbf{H}_P &= \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\mathbf{a}_y \, dx \, dy}{4\pi(x^2 + y^2 + 9)^{3/2}} = -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{x}{(y^2 + 9)\sqrt{x^2 + y^2 + 9}}\Big|_{-\infty}^{\infty} dy \\ &= -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{2}{y^2 + 9} \, dy = -\frac{12}{\pi}\mathbf{a}_y \frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\Big|_{-2}^{2} = -\frac{4}{\pi}(2)(0.59)\,\mathbf{a}_y = \underline{-1.50\,\mathbf{a}_y \text{ A/m}}\end{aligned}$$

8.10. Let a filamentary current of 5 mA be directed from infinity to the origin on the positive z axis and then back out to infinity on the positive x axis. Find $\mathbf{H}$ at $P(0, 1, 0)$: The Biot-Savart law is applied to the two wire segments using the following setup:

$$\begin{aligned}\mathbf{H}_P &= \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_0^\infty \frac{-I dz \mathbf{a}_z \times (-z\mathbf{a}_z + \mathbf{a}_y)}{4\pi(z^2 + 1)^{3/2}} + \int_0^\infty \frac{I dx \mathbf{a}_x \times (-x\mathbf{a}_x + \mathbf{a}_y)}{4\pi(x^2 + 1)^{3/2}} \\ &= \int_0^\infty \frac{I dz \mathbf{a}_x}{4\pi(z^2 + 1)^{3/2}} + \int_0^\infty \frac{I dx \mathbf{a}_z}{4\pi(x^2 + 1)^{3/2}} = \frac{I}{4\pi}\left[\frac{z\mathbf{a}_x}{\sqrt{z^2 + 1}}\Big|_0^\infty + \frac{x\mathbf{a}_z}{\sqrt{x^2 + 1}}\Big|_0^\infty\right] \\ &= \frac{I}{4\pi}(\mathbf{a}_x + \mathbf{a}_z) = \underline{0.40(\mathbf{a}_x + \mathbf{a}_z) \text{ mA/m}}\end{aligned}$$

8.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad \Rightarrow \quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

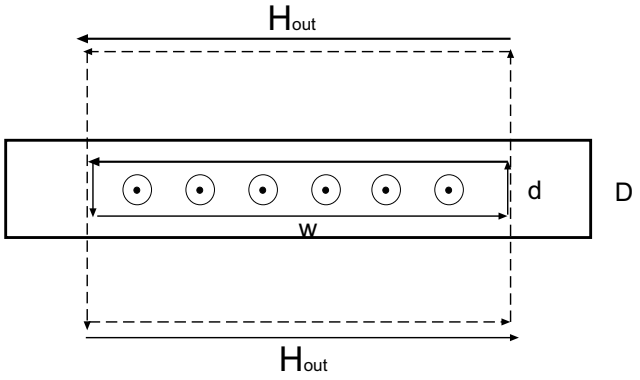
Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

8.12. In Fig. 8.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. The problem asks you to find **H** at various positions. Before continuing, we need to know how to find **H** for this type of current configuration. The sketch below shows one of the slabs (of thickness D) oriented with the current coming out of the page. The problem statement implies that both slabs are of infinite length and width. To find the magnetic field *inside* a slab, we apply Ampere's circuital law to the rectangular path of height d and width w, as shown, since by symmetry, **H** should be oriented horizontally. For example, if the sketch below shows the upper slab in Fig. 8.22, current will be in the positive y direction. Thus **H** will be in the positive x direction above the slab midpoint, and will be in the negative x direction below the midpoint.



8.12 (continued) In taking the line integral in Ampere's law, the two vertical path segments will cancel each other. Ampere's circuital law for the interior loop becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2H_{in} \times w = I_{encl} = J \times w \times d \;\; \Rightarrow \;\; H_{in} = \frac{Jd}{2}$$

The field outside the slab is found similarly, but with the enclosed current now bounded by the slab thickness, rather than the integration path height:

$$2H_{out} \times w = J \times w \times D \;\; \Rightarrow \;\; H_{out} = \frac{JD}{2}$$

where H_{out} is directed from right to left below the slab and from left to right above the slab (right hand rule). Reverse the current, and the fields, of course, reverse direction. We are now in a position to solve the problem.

Find $\mathbf{H}$ at:

a) $z = -0.2$m: Here the fields from the top and bottom slabs (carrying opposite currents) will cancel, and so $\mathbf{H} = \underline{0}$.

b) $z = 0.2$m. This point lies within the lower slab above its midpoint. Thus the field will be oriented in the negative x direction. Referring to Fig. 8.22 and to the sketch on the previous page, we find that $d = 0.1$. The total field will be this field plus the contribution from the upper slab current:

$$\mathbf{H} = \underbrace{\frac{-10(0.1)}{2}\mathbf{a}_x}_{\text{lower slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{upper slab}} = \underline{-2\mathbf{a}_x \text{ A/m}}$$

c) $z = 0.4$m: Here the fields from both slabs will add constructively in the negative x direction:

$$\mathbf{H} = -2\frac{10(0.3)}{2}\mathbf{a}_x = \underline{-3\mathbf{a}_x \text{ A/m}}$$

d) $z = 0.75$m: This is in the interior of the upper slab, whose midpoint lies at $z = 0.85$. Therefore $d = 0.2$. Since 0.75 lies below the midpoint, magnetic field from the upper slab will lie in the negative x direction. The field from the lower slab will be negative x-directed as well, leading to:

$$\mathbf{H} = \underbrace{\frac{-10(0.2)}{2}\mathbf{a}_x}_{\text{upper slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{lower slab}} = \underline{-2.5\mathbf{a}_x \text{ A/m}}$$

e) $z = 1.2$m: This point lies above both slabs, where again fields cancel completely: Thus $\mathbf{H} = \underline{0}$.

8.13. A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a \mathbf{a}_\phi$.

a) Show that H is not a function of ϕ or z: Consider this situation as illustrated in Fig. 8.11. There (sec. 8.2) it was stated that the field will be entirely z-directed. We can see this by applying Ampere's circuital law to a closed loop path whose orientation we choose such that current is enclosed by the path. The only way to enclose current is to set up the loop (which we choose to be rectangular) such that it is oriented with two parallel opposing segments lying in the z direction; one of these lies inside the cylinder, the other outside. The other two parallel segments lie in the ρ direction. The loop is now cut by the current sheet, and if we assume a length of the loop in z of d, then the enclosed current will be given by Kd A. There will be no ϕ variation in the field because where we position the loop around the circumference of the cylinder does not affect the result of Ampere's law. If we assume an infinite cylinder length, there will be no z dependence in the field, since as we lengthen the loop in the z direction, the path length (over which the integral is taken) increases, but then so does the enclosed current – by the same factor. Thus H would not change with z. There would also be no change if the loop was simply moved along the z direction.

b) Show that H_ϕ and H_ρ are everywhere zero. First, if H_ϕ were to exist, then we should be able to find a closed loop path *that encloses current*, in which all or or portion of the path lies in the ϕ direction. This we cannot do, and so H_ϕ must be zero. Another argument is that when applying the Biot-Savart law, there is no current element that would produce a ϕ component. Again, using the Biot-Savart law, we note that radial field components will be produced by individual current elements, but such components will cancel from two elements that lie at symmetric distances in z on either side of the observation point.

c) Show that $H_z = 0$ for $\rho > a$: Suppose the rectangular loop was drawn such that the outside z-directed segment is moved further and further away from the cylinder. We would expect H_z outside to decrease (as the Biot-Savart law would imply) but the same amount of current is always enclosed no matter how far away the outer segment is. We therefore must conclude that the field outside is zero.

d) Show that $H_z = K_a$ for $\rho < a$: With our rectangular path set up as in part a, we have no path integral contributions from the two radial segments, and no contribution from the outside z-directed segment. Therefore, Ampere's circuital law would state that

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z d = I_{encl} = K_a d \quad \Rightarrow \quad H_z = K_a$$

where d is the length of the loop in the z direction.

e) A second shell, $\rho = b$, carries a current $K_b \mathbf{a}_\phi$. Find $\mathbf{H}$ everywhere: For $\rho < a$ we would have both cylinders contributing, or $H_z(\rho < a) = K_a + K_b$. Between the cylinders, we are outside the inner one, so its field will not contribute. Thus $H_z(a < \rho < b) = K_b$. Outside ($\rho > b$) the field will be zero.

8.14. A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find $\mathbf{H}$ at the point $P(\rho, \phi, z)$: The construction is similar to that of the toroid of round cross section as done on p.239. Again, magnetic field exists only inside the toroid cross section, and is given by

$$\mathbf{H} = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi \quad (2 < \rho < 3) \text{ cm}, \quad (1 < z < 2.5) \text{ cm}$$

where I_{encl} is found from the given current density: On the outer radius, the current is

$$I_{outer} = -50(2\pi \times 3 \times 10^{-2}) = -3\pi \text{ A}$$

This current is directed along negative z, which means that the current on the *inner* radius ($\rho = 2$) is directed along *positive* z. Inner and outer currents have the same magnitude. It is the inner current that is enclosed by the circular integration path in $\mathbf{a}_\phi$ within the toroid that is used in Ampere's law. So $I_{encl} = +3\pi$ A. We can now proceed with what is requested:

a) $P_A(1.5\text{cm}, 0, 2\text{cm})$: The radius, $\rho = 1.5$ cm, lies outside the cross section, and so $\mathbf{H}_A = \underline{0}$.

b) $P_B(2.1\text{cm}, 0, 2\text{cm})$: This point does lie inside the cross section, and the ϕ and z values do not matter. We find

$$\mathbf{H}_B = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi = \frac{3\mathbf{a}_\phi}{2(2.1 \times 10^{-2})} = \underline{71.4\,\mathbf{a}_\phi \text{ A/m}}$$

c) $P_C(2.7\text{cm}, \pi/2, 2\text{cm})$: again, ϕ and z values make no difference, so

$$\mathbf{H}_C = \frac{3\mathbf{a}_\phi}{2(2.7 \times 10^{-2})} = \underline{55.6\,\mathbf{a}_\phi \text{ A/m}}$$

d) $P_D(3.5\text{cm}, \pi/2, 2\text{cm})$. This point lies outside the cross section, and so $\mathbf{H}_D = \underline{0}$.

8.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\,\mathbf{a}_z$ V/m is present.

a) Find $\mathbf{J}$: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z \text{ kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\,\rho\,d\rho\,d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\,[-150\rho - 1]\Big|_0^{\rho_0} \text{ kA}$$
$$= \underline{12.6\left[1 - (1 + 150\rho_0)e^{-150\rho_0}\right] \text{ A}}$$

c) Make use of Ampere's circuital law to find $\mathbf{H}$: Symmetry suggests that $\mathbf{H}$ will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part *b*, except with ρ_0 replaced by the variable, ρ. We obtain

$$H_\phi = \underline{\frac{2.00}{\rho}\left[1 - (1 + 150\rho)e^{-150\rho}\right] \text{ A/m}}$$

8.16. The cylindrical shell, 2mm $< \rho <$ 3mm, carries a uniformly-distributed total current of 8A in the $-\mathbf{a}_z$ direction, and a filament on the z axis carries 8A in the $\mathbf{a}_z$ direction. Find $\mathbf{H}$ everywhere: We use Ampere's circuital law, noting that from symmetry, $\mathbf{H}$ will be $\mathbf{a}_\phi$ directed. Inside the shell ($\rho <$ 2mm), A circular integration path centered on the z axis encloses only the filament current along z: Therefore

$$\mathbf{H}(\rho < 2\text{mm}) = \frac{8}{2\pi\rho}\,\mathbf{a}_\phi = \underline{\frac{4}{\pi\rho}\,\mathbf{a}_\phi \text{ A/m}} \quad (\rho \text{ in m})$$

With the circular integration path within (2 $< \rho <$ 3mm), the enclosed current will consist of the filament plus that portion of the shell current that lies inside ρ. Ampere's circuital law applied to a loop of radius ρ is:

$$\oint \mathbf{H} \cdot d\mathbf{L} = I_{filament} + \int\int_{shell\,area} \mathbf{J} \cdot d\mathbf{S}$$

where the current density is

$$\mathbf{J} = -\frac{8}{\pi(3\times10^{-3})^2 - \pi(2\times10^{-3})^2}\,\mathbf{a}_z = -\frac{8\times10^6}{5\pi}\mathbf{a}_z \text{ A/m}^2$$

So

$$2\pi\rho H_\phi = 8 + \int_0^{2\pi}\int_{2\times10^{-3}}^{\rho}\left(\frac{-8}{5\pi}\times10^6\right)\mathbf{a}_z\cdot\mathbf{a}_z\,\rho'\,d\rho' d\phi = 8 - 1.6\times10^6\,(\rho')^2\Big|_{2\times10^{-3}}^{\rho}$$

Solve for H_ϕ to find:

$$\mathbf{H}(2 < \rho < 3\,\text{mm}) = \underline{\frac{4}{\pi\rho}\left[1 - (2\times10^5)(\rho^2 - 4\times10^{-6})\right]\mathbf{a}_\phi \text{ A/m}} \quad (\rho \text{ in m})$$

Outside ($\rho >$ 3mm), the total enclosed current is zero, and so $\mathbf{H}(\rho >$ 3mm$) = \underline{0}$.

8.17. A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of $0.5\,\mathbf{a}_z$ A/m and $-0.2\,\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at:

a) $\rho = 0.5$ cm: Here, we are either just inside or just outside the first current sheet, so both we will calculate $\mathbf{H}$ for both cases. Just inside, applying Ampere's circuital law to a circular path centered on the z axis produces:

$$2\pi\rho H_\phi = 7\times 10^{-3} \;\Rightarrow\; \mathbf{H}(\text{just inside}) = \frac{7\times 10^{-3}}{2\pi(0.5\times 10^{-2}}\mathbf{a}_\phi = \underline{2.2\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

Just outside the current sheet at .5 cm, Ampere's law becomes

$$2\pi\rho H_\phi = 7\times 10^{-3} - 2\pi(0.5\times 10^{-2})(0.2)$$
$$\Rightarrow\; \mathbf{H}(\text{just outside}) = \frac{7.2\times 10^{-4}}{2\pi(0.5\times 10^{-2})}\mathbf{a}_\phi = \underline{2.3\times 10^{-2}\mathbf{a}_\phi \text{ A/m}}$$

b) $\rho = 1.5$ cm: Here, all three currents are enclosed, so Ampere's law becomes

$$2\pi(1.5\times 10^{-2})H_\phi = 7\times 10^{-3} - 6.28\times 10^{-3} + 2\pi(10^{-2})(0.5)$$
$$\Rightarrow\; \mathbf{H}(\rho = 1.5) = \underline{3.4\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

c) $\rho = 4$ cm: Ampere's law as used in part b applies here, except we replace $\rho = 1.5$ cm with $\rho = 4$ cm on the left hand side. The result is $\mathbf{H}(\rho = 4) = \underline{1.3\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$.

d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm? We require that the total enclosed current be zero, and so the net current in the proposed cylinder at 4 cm must be negative the right hand side of the first equation in part b. This will be -3.2×10^{-2}, so that the surface current density at 4 cm must be

$$\mathbf{K} = \frac{-3.2\times 10^{-2}}{2\pi(4\times 10^{-2})}\mathbf{a}_z = \underline{-1.3\times 10^{-1}\,\mathbf{a}_z \text{ A/m}}$$

8.18. Current density is distributed as follows: $\mathbf{J} = 0$ for $|y| > 2$ m, $\mathbf{J} = 8y\mathbf{a}_z$ A/m^2 for $|y| < 1$ m, $\mathbf{J} = 8(2-y)\,\mathbf{a}_z$ A/m^2 for $1 < y < 2$ m, $\mathbf{J} = -8(2+y)\,\mathbf{a}_z$ A/m^2 for $-2 < y < -1$ m. Use symmetry and Ampere's law to find $\mathbf{H}$ everywhere.

Symmetry does help significantly in this problem. The current densities in the regions $0 < y < 1$ and $-1 < y < 0$ are mirror images of each other across the plane $y = 0$ – this in addition to being of opposite sign. This is also true of the current densities in the regions $1 < y < 2$ and $-2 < y < -1$. As a consequence of this, we find that the net current in region 1, I_1 (see the diagram on the next page), is equal and opposite to the net current in region 4, I_4. Also, I_2 is equal and opposite to I_3. This means that when applying Ampere's law to the path $a-b-c-d-a$, as shown in the figure, zero current is enclosed, so that $\oint \mathbf{H}\cdot d\mathbf{L} = 0$ over the path. In addition, the symmetry of the current configuration implies that $\mathbf{H} = 0$ outside the slabs along the vertical paths $a-b$ and $c-d$. $\mathbf{H}$ from all sources should completely cancel along the two vertical paths, as well as along the two horizontal paths.

8.18. (continued) To find the magnetic field in region 1, we apply Ampere's circuital law to the path $c - d - e - f - c$, again noting that **H** will be zero along the two horizontal segments and along the right vertical segment. This leaves only the left vertical segment, $e - f$, pointing in the $+x$ direction, and along which is field, H_{x1}. The counter-clockwise direction of the path integral is chosen using the right-hand convention, where we take the normal to the path in the $+z$ direction, which is the same as the current direction. Assuming the height of the path is Δx, we find

$$H_{x1}\Delta x = \Delta x \int_{y_1}^{2} 8(2-y)dy = \Delta x \left[16y - 4y^2\right]_{y_1}^{2} = \Delta x \left[16(2-y_1) - 4(4-y_1^2)\right]$$

Replacing y_1 with y, we find

$$H_{x1} = 4[8 - 4y - 4 + y^2] \;\Rightarrow\; \mathbf{H}_1(1 < y < 2) = \underline{4(y-2)^2\,\mathbf{a}_x \text{ A/m}}$$

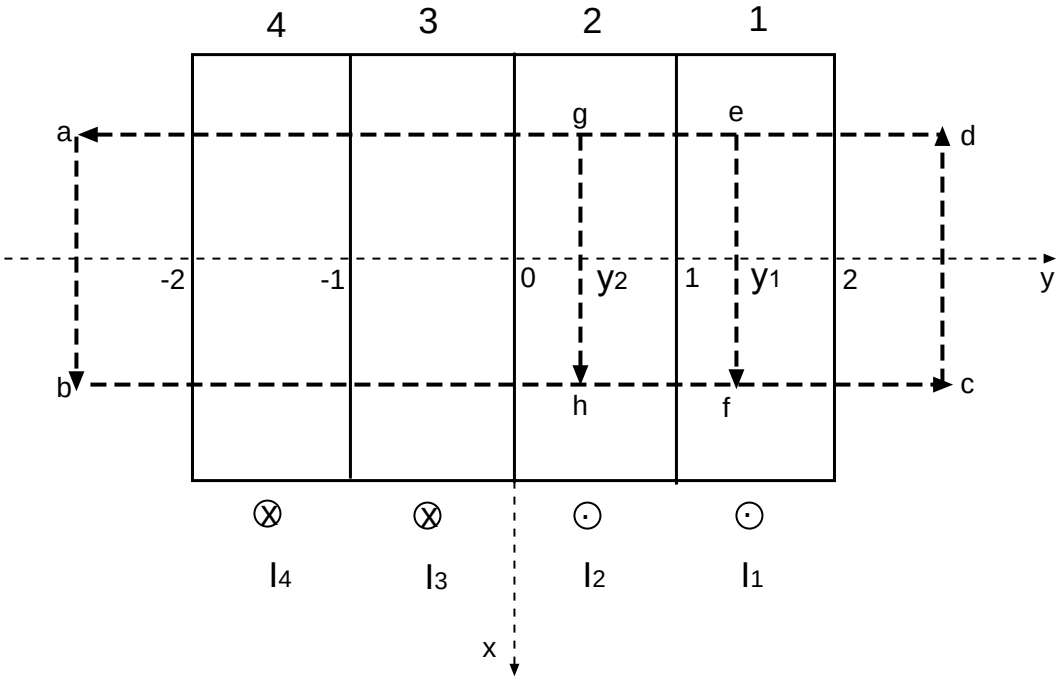
$\mathbf{H}_1$ lies in the positive x direction, since the result of the integration is net positive.

H in region 2 is now found through the line integral over the path $d - g - h - c$, enclosing all of region 1 within Δx and part of region 2 from $y = y_2$ to 1:

$$H_{x2}\Delta x = \Delta x \int_{1}^{2} 8(2-y)\,dy + \Delta x \int_{y_2}^{1} 8y\,dy = \Delta x \left[4(1-2)^2 + 4(1-y_2^2)\right] = 4(2-y_2^2)\Delta x$$

so that in terms of y,

$$\mathbf{H}_2(0 < y < 1) = \underline{4(2-y^2)\mathbf{a}_x \text{ A/m}}$$



8.18. (continued) The procedure is repeated for the remaining two regions, $-2 < y < -1$ and $-1 < y < 0$, by taking the integration path with its right vertical segment within each of these two regions, while the left vertical path is $a - b$. Again the integral is taken counter-clockwise, which means that the right vertical path will be directed along $-x$. But the current is now in the opposite direction of that for $y > 0$, making the enclosed current net negative. Therefore, $\mathbf{H}$ will be in the opposite direction from that of the right vertical path, which is the positive x direction. The magnetic field will therefore be symmetric about the $y = 0$ plane. We can use the results for regions 1 and 2 to construct the field everywhere:

$$\mathbf{H} = \underline{0 \ \ (y > 2) \ \text{ and } (y < -2)}$$

$$\mathbf{H} = \underline{4(2 - |y|^2)\mathbf{a}_x \text{ A/m} \ \ (0 < |y| < 1)}$$

$$\mathbf{H} = \underline{4(|y| - 2)^2\mathbf{a}_x \text{ A/m} \ \ (1 < |y| < 2)}$$

8.19. Calculate $\nabla \times [\nabla(\nabla \cdot \mathbf{G})]$ if $\mathbf{G} = 2x^2yz\,\mathbf{a}_x - 20y\,\mathbf{a}_y + (x^2 - z^2)\,\mathbf{a}_z$: Proceding, we first find $\nabla \cdot \mathbf{G} = 4xyz - 20 - 2z$. Then $\nabla(\nabla \cdot \mathbf{G}) = 4yz\,\mathbf{a}_x + 4xz\,\mathbf{a}_y + (4xy - 2)\,\mathbf{a}_z$. Then

$$\nabla \times [\nabla(\nabla \cdot \mathbf{G})] = (4x - 4x)\,\mathbf{a}_x - (4y - 4y)\,\mathbf{a}_y + (4z - 4z)\,\mathbf{a}_z = \underline{0}$$

8.20. The magnetic field intensity is given in the square region $x = 0$, $0.5 < y < 1$, $1 < z < 1.5$ by $\mathbf{H} = z^2\mathbf{a}_x + x^3\mathbf{a}_y + y^4\mathbf{a}_z$ A/m.

a) evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ about the perimeter of the square region: Using $d\mathbf{L} = dx\mathbf{a}_x + dy\mathbf{a}_y + dz\mathbf{a}_z$, and using the given field, we find, in the $x = 0$ plane:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_{.5}^{1} 0\,dy + \int_{1}^{1.5} (1)^4\,dz + \int_{1}^{.5} 0\,dy + \int_{1.5}^{1} (.5)^4\,dz = \underline{0.46875}$$

b) Find $\nabla \times \mathbf{H}$:

$$\nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$
$$= \underline{4y^3\mathbf{a}_x + 2z\mathbf{a}_y + 3x^2\mathbf{a}_z}$$

c) Calculate $(\nabla \times \mathbf{H})_x$ at the center of the region: Here, $y = 0.75$ and so $(\nabla \times \mathbf{H})_x = 4(.75)^3 = \underline{1.68750}$.

d) Does $(\nabla \times \mathbf{H})_x = [\oint \mathbf{H} \cdot d\mathbf{L}]/\text{Area Enclosed}$? Using the part a result, $[\oint \mathbf{H} \cdot d\mathbf{L}]/\text{Area Enclosed} = 0.46875/0.25 = \underline{1.8750}$, which is off the value found in part c. Answer: No. Reason: the limit of the area shrinking to zero must be taken before the results will be equal.

8.21. Points A, B, C, D, E, and F are each 2 mm from the origin on the coordinate axes indicated in Fig. 8.23. The value of $\mathbf{H}$ at each point is given. Calculate an approximate value for $\nabla \times \mathbf{H}$ at the origin: We use the approximation:

$$\text{curl}\,\mathbf{H} \doteq \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta a}$$

where no limit as $\Delta a \to 0$ is taken (hence the approximation), and where $\Delta a = 4\,\text{mm}^2$. Each curl component is found by integrating $\mathbf{H}$ over a square path that is normal to the component in question.

8.21. (continued) Each of the four segments of the contour passes through one of the given points. Along each segment, the field is assumed constant, and so the integral is evaluated by summing the products of the field and segment length (4 mm) over the four segments. The x component of the curl is thus:

$$(\nabla \times \mathbf{H})_x \doteq \frac{(H_{z,C} - H_{y,E} - H_{z,D} + H_{y,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.69 + 13.88 - 14.35 - 13.10)(250) = 530 \text{ A/m}^2$$

The other components are:

$$(\nabla \times \mathbf{H})_y \doteq \frac{(H_{z,B} + H_{x,E} - H_{z,A} - H_{x,F})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (15.82 + 11.11 - 14.21 - 10.88)(250) = 460 \text{ A/m}^2$$

and

$$(\nabla \times \mathbf{H})_z \doteq \frac{(H_{y,A} - H_{x,C} - H_{y,B} H_{x,D})(4 \times 10^{-3})}{(4 \times 10^{-3})^2}$$
$$= (-13.78 - 10.49 + 12.19 + 11.49)(250) = -148 \text{ A/m}^2$$

Finally we assemble the results and write:

$$\nabla \times \mathbf{H} \doteq \underline{530\,\mathbf{a}_x + 460\,\mathbf{a}_y - 148\,\mathbf{a}_z}$$

8.22. In the cylindrical region $\rho \leq 0.6$ mm, $H_\phi = (2/\rho) + (\rho/2)$ A/m, while $H_\phi = (3/\rho)$ A/m for $\rho > 0.6$ mm.

a) Determine $\mathbf{J}$ for $\rho < 0.6$mm: We have only a ϕ component that varies with ρ. Therefore

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left[2 + \frac{\rho^2}{2}\right]\mathbf{a}_z = \mathbf{J} = \underline{1\mathbf{a}_z \text{ A/m}^2}$$

b) Determine $\mathbf{J}$ for $\rho > 0.6$ mm: In this case

$$\mathbf{J} = \frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{3}{\rho}\right]\mathbf{a}_z = \underline{0}$$

c) Is there a filamentary current at $\rho = 0$? If so, what is its value? As $\rho \to 0$, $H_\phi \to \infty$, which implies the existence of a current filament along the z axis: So, <u>YES</u>. The value is found by through Ampere's circuital law, by integrating H_ϕ around a circular path of vanishingly-small radius. The current enclosed is therefore $I = 2\pi\rho(2/\rho) = \underline{4\pi \text{ A}}$.

d) What is $\mathbf{J}$ at $\rho = 0$? Since a filament current lies along z at $\rho = 0$, this forms a singularity, and so the current density there is <u>infinite</u>.

8.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density $\mathbf{J}$: This is found through the curl of $\mathbf{H}$, which simplifies to a single term, since $\mathbf{H}$ varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z \text{ A/m}^2}$$

8.23. (continued)

b) Integrate **J** over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 60\rho \mathbf{a}_z \cdot \rho \, d\rho \, d\phi \mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2 \, \mathbf{a}_\phi \big|_{\rho=1} \cdot (1) d\phi \mathbf{a}_\phi = \int_0^{2\pi} 20 \, d\phi = \underline{40\pi \text{ A}}$$

8.24. Evaluate both sides of Stokes' theorem for the field $\mathbf{G} = 10 \sin\theta \, \mathbf{a}_\phi$ and the surface $r = 3$, $0 \le \theta \le 90^\circ$, $0 \le \phi \le 90^\circ$. Let the surface have the $\mathbf{a}_r$ direction: Stokes' theorem reads:

$$\oint_C \mathbf{G} \cdot d\mathbf{L} = \int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} \, da$$

Considering the given surface, the contour, C, that forms its perimeter consists of three joined arcs of radius 3 that sweep out 90° in the xy, xz, and zy planes. Their centers are at the origin. Of these three, only the arc in the xy plane (which lies along $\mathbf{a}_\phi$) is in the direction of **G**; the other two (in the $-\mathbf{a}_\theta$ and $\mathbf{a}_\theta$ directions respectively) are perpendicular to it, and so will not contribute to the path integral. The left-hand side therefore consists of only the xy plane portion of the closed path, and evaluates as

$$\oint \mathbf{G} \cdot d\mathbf{L} = \int_0^{\pi/2} 10 \sin\theta \big|_{\pi/2} \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, 3 \sin\theta \big|_{\pi/2} \, d\phi = \underline{15\pi}$$

To evaluate the right-hand side, we first find

$$\nabla \times \mathbf{G} = \frac{1}{r \sin\theta} \frac{d}{d\theta} \left[(\sin\theta) 10 \sin\theta \right] \mathbf{a}_r = \frac{20 \cos\theta}{r} \mathbf{a}_r$$

The surface over which we integrate this is the one-eighth spherical shell of radius 3 in the first octant, bounded by the three arcs described earlier. The right-hand side becomes

$$\int\int_S (\nabla \times \mathbf{G}) \cdot \mathbf{n} \, da = \int_0^{\pi/2} \int_0^{\pi/2} \frac{20 \cos\theta}{3} \, \mathbf{a}_r \cdot \mathbf{a}_r \, (3)^2 \sin\theta \, d\theta \, d\phi = \underline{15\pi}$$

It would appear that the theorem works.

8.25. (This problem was discovered to be flawed – I will proceed with it and show how). Given the field

$$\mathbf{H} = \frac{1}{2}\cos\frac{\phi}{2}\,\mathbf{a}_\rho - \sin\frac{\phi}{2}\,\mathbf{a}_\phi \text{ A/m}$$

evaluate both sides of Stokes' theorem for the path formed by the intersection of the cylinder $\rho = 3$ and the plane $z = 2$, and for the surface defined by $\rho = 3$, $0 \le \phi \le 2\pi$, and $z = 0$, $0 \le \rho \le 3$: This surface resembles that of an open tin can whose bottom lies in the $z = 0$ plane, and whose open circular edge, at $z = 2$, defines the line integral contour. We first evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ over the circular contour, where we take the integration direction as clockwise, looking down on the can. We do this because the outward normal from the bottom of the can will be in the $-\mathbf{a}_z$ direction.

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} \mathbf{H} \cdot 3d\phi(-\mathbf{a}_\phi) = \int_0^{2\pi} 3\sin\frac{\phi}{2}d\phi = \underline{12 \text{ A}}$$

With our choice of contour direction, this indicates that the current will flow in the negative z direction. Note for future reference that only the ϕ component of the given field contributed here. Next, we evalute $\int\int \nabla \times \mathbf{H} \cdot d\mathbf{S}$, over the surface of the tin can. We find

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\phi}\right)\mathbf{a}_z = \frac{1}{\rho}\left(-\sin\frac{\phi}{2} + \frac{1}{4}\sin\frac{\phi}{2}\right)\mathbf{a}_z = -\frac{3}{4\rho}\sin\frac{\phi}{2}\,\mathbf{a}_z \text{ A/m}$$

Note that *both* field components contribute here. The integral over the tin can is now only over the bottom surface, since $\nabla \times \mathbf{H}$ has only a z component. We use the outward normal, $-\mathbf{a}_z$, and find

$$\int\int \nabla \times \mathbf{H} \cdot d\mathbf{S} = -\frac{3}{4}\int_0^{2\pi}\int_0^3 \frac{1}{\rho}\sin\frac{\phi}{2}\mathbf{a}_z \cdot (-\mathbf{a}_z)\rho\, d\rho\, d\phi = \frac{9}{4}\int_0^{2\pi}\sin\frac{\phi}{2}\,d\phi = \underline{9 \text{ A}}$$

Note that if the radial component of $\mathbf{H}$ were not included in the computation of $\nabla \times \mathbf{H}$, then the factor of 3/4 in front of the above integral would change to a factor of 1, and the result would have been 12 A. What would appear to be a violation of Stokes' theorem is likely the result of a missing term in the ϕ component of $\mathbf{H}$, having zero curl, which would have enabled the original line integral to have a value of 9A. The reader is invited to explore this further.

8.26. Let $\mathbf{G} = 15r\,\mathbf{a}_\phi$.

a) Determine $\oint \mathbf{G} \cdot d\mathbf{L}$ for the circular path $r = 5$, $\theta = 25^\circ$, $0 \le \phi \le 2\pi$:

$$\oint \mathbf{G} \cdot d\mathbf{L} = \int_0^{2\pi} 15(5)\mathbf{a}_\phi \cdot \mathbf{a}_\phi(5)\sin(25^\circ)\,d\phi = 2\pi(375)\sin(25^\circ) = \underline{995.8}$$

b) Evaluate $\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$ over the spherical cap $r = 5$, $0 \le \theta \le 25^\circ$, $0 \le \phi \le 2\pi$: When evaluating the curl of $\mathbf{G}$ using the formula in spherical coordinates, only one of the six terms survives:

$$\nabla \times \mathbf{G} = \frac{1}{r\sin\theta}\frac{\partial(G_\phi \sin\theta)}{\partial\theta}\,\mathbf{a}_r = \frac{1}{r\sin\theta}\,15r\cos\theta\,\mathbf{a}_r = 15\cot\theta\,\mathbf{a}_r$$

Then

$$\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{25^\circ} 15\cot\theta\,\mathbf{a}_r \cdot \mathbf{a}_r\,(5)^2\sin\theta\,d\theta\,d\phi$$

$$= 2\pi\int_0^{25^\circ} 15\cos\theta(25)\,d\theta = 2\pi(15)(25)\sin(25^\circ) = \underline{995.8}$$

8.27. The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x+2y}{z^2}\,\mathbf{a}_y + \frac{2}{z}\,\mathbf{a}_z \text{ A/m}$$

a) Find $\nabla \times \mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x + \frac{\partial H_y}{\partial x}\,\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\,\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \text{ A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla \times \mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 < x < 2$, $3 < y < 5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\int \mathbf{J}\big|_{z=4} \cdot \mathbf{a}_z\, dx\, dy = \int_3^5 \int_1^2 \frac{1}{4^2} dx\, dy = \underline{1/8 \text{ A}}$$

d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint \mathbf{H} \cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint \mathbf{H}\big|_{z=4} \cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}\, dy + \int_5^3 \frac{1+2y}{16}\, dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

8.28. Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\,\mathbf{a}_\phi$ A/m in free space:

a) find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like best. I chose the line integral side, where the integration path is the circular path in ϕ around the top edge of the cone, at $r = 5$. The path direction is chosen to be *clockwise* looking down on the xy plane. This, by convention, leads to the normal from the cone surface that points in the positive $\mathbf{a}_\theta$ direction (right hand rule). We find

$$\begin{aligned}\oint \mathbf{H} \cdot d\mathbf{L} &= \int_0^{2\pi} \Big[(3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\,\mathbf{a}_\phi\Big]_{r=5,\theta=20} \cdot 5\sin(20^\circ)\, d\phi\, (-\mathbf{a}_\phi) \\ &= -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ) = \underline{-2.73 \times 10^3 \text{ A}}\end{aligned}$$

This result means that there is a component of current that enters the cone surface in the $-\mathbf{a}_\theta$ direction, to which is associated a component of $\mathbf{H}$ in the positive $\mathbf{a}_\phi$ direction.

b) Check the result by using the other side of Stokes' theorem: We first find the current density through the curl of the magnetic field, where three of the six terms in the spherical coordinate formula survive:

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(54r\cos\theta\sin\theta\right)\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(54r^2\cos\theta\right)\mathbf{a}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3r^3}{\sin\theta}\right)\mathbf{a}_\phi = \mathbf{J}$$

Thus

$$\mathbf{J} = 54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi$$

8.28b. (continued)

The calculation of the other side of Stokes' theorem now involves integrating **J** over the surface of the cone, where the outward normal is positive $\mathbf{a}_\theta$, as defined in part *a*:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^5 \left[54 \cot\theta \, \mathbf{a}_r - 108 \cos\theta \, \mathbf{a}_\theta + \frac{9r}{\sin\theta} \, \mathbf{a}_\phi \right]_{\theta=20^\circ} \cdot \mathbf{a}_\theta \, r \sin(20^\circ) \, dr \, d\phi$$

$$= -\int_0^{2\pi} \int_0^5 108 \cos(20^\circ) \sin(20^\circ) r \, dr \, d\phi = -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ)$$

$$= \underline{-2.73 \times 10^3 \text{ A}}$$

8.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2 \times 10^{-3})^2} \mathbf{a}_z = \underline{1.59 \times 10^7 \, \mathbf{a}_z \text{ A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad \Rightarrow \quad \mathbf{H} = \frac{\rho J}{2} \mathbf{a}_\phi = \underline{7.96 \times 10^6 \rho \, \mathbf{a}_\phi \text{ A/m}}$$

Then $\mathbf{B} = \mu_0 \mathbf{H} = (4\pi \times 10^{-7})(7.96 \times 10^6)\rho \mathbf{a}_\phi = \underline{10\rho \, \mathbf{a}_\phi \text{ Wb/m}^2}$.

c) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part *b*, we find,

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) \, \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} \left(\frac{1.59 \times 10^7 \rho^2}{2} \right) \mathbf{a}_z = \underline{1.59 \times 10^7 \, \mathbf{a}_z \text{ A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho} \mathbf{a}_\phi = \underline{\frac{1}{\pi\rho} \, \mathbf{a}_\phi \text{ A/m}}$$

Now $\mathbf{B} = \mu_0 \mathbf{H} = \underline{\mu_0/(\pi\rho) \, \mathbf{a}_\phi \text{ Wb/m}^2}$.

e) Show that $\nabla \times \mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla \times \mathbf{H} = \frac{1}{\rho} \frac{d}{d\rho} (\rho H_\phi) \, \mathbf{a}_z = \frac{1}{\rho} \frac{d}{d\rho} \left(\rho \frac{1}{\pi\rho} \right) \mathbf{a}_z = \underline{0} \text{ (as expected)}$$

8.30. A solid nonmagnetic conductor of circular cross-section has a radius of 2mm. The conductor is inhomogeneous, with $\sigma = 10^6(1+10^6\rho^2)$ S/m. If the conductor is 1m in length and has a voltage of 1mV between its ends, find:

a) **H** inside: With current along the cylinder length (along $\mathbf{a}_z$, and with ϕ symmetry, **H** will be ϕ-directed only. We find $\mathbf{E} = (V_0/d)\mathbf{a}_z = 10^{-3}\mathbf{a}_z$ V/m. Then $\mathbf{J} = \sigma\mathbf{E} = 10^3(1+10^6\rho^2)\mathbf{a}_z$ A/m^2. Next we apply Ampere's circuital law to a circular path of radius ρ, centered on the z axis and normal to the axis:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi\rho H_\phi = \int\int_S \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho} 10^3(1+10^6(\rho')^2)\mathbf{a}_z\cdot\mathbf{a}_z\rho' d\rho' d\phi$$

Thus

$$H_\phi = \frac{10^3}{\rho}\int_0^{\rho}\rho' + 10^6(\rho')^3 d\rho' = \frac{10^3}{\rho}\left[\frac{\rho^2}{2} + \frac{10^6}{4}\rho^4\right]$$

Finally, $\mathbf{H} = \underline{500\rho(1+5\times10^5\rho^3)\mathbf{a}_\phi \text{ A/m } (0<\rho<2\text{mm})}$.

b) the total magnetic flux inside the conductor: With field in the ϕ direction, a plane normal to **B** will be that in the region $0<\rho<2$ mm, $0<z<1$ m. The flux will be

$$\Phi = \int\int_S \mathbf{B}\cdot d\mathbf{S} = \mu_0\int_0^1\int_0^{2\times10^{-3}}\left(500\rho + 2.5\times10^8\rho^3\right)d\rho dz = 8\pi\times10^{-10}\text{ Wb} = \underline{2.5\,\text{nWb}}$$

8.31. The cylindrical shell defined by 1 cm $<\rho<$ 1.4 cm consists of a non-magnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0<z<1$:

a) $0<\rho<1.2$ cm: We first need to find **J**, **H**, and **B**: The current density will be:

$$\mathbf{J} = \frac{50}{\pi[(1.4\times10^{-2})^2 - (1.0\times10^{-2})^2]}\,\mathbf{a}_z = 1.66\times10^5\,\mathbf{a}_z \text{ A/m}^2$$

Next we find H_ϕ at radius ρ between 1.0 and 1.4 cm, by applying Ampere's circuital law, and noting that the current density is zero at radii less than 1 cm:

$$2\pi\rho H_\phi = I_{encl} = \int_0^{2\pi}\int_{10^{-2}}^{\rho} 1.66\times10^5\rho'\,d\rho'\,d\phi$$

$$\Rightarrow\quad H_\phi = 8.30\times10^4\frac{(\rho^2 - 10^{-4})}{\rho}\text{ A/m}\quad(10^{-2}\text{ m}<\rho<1.4\times10^{-2}\text{ m})$$

Then $\mathbf{B} = \mu_0\mathbf{H}$, or

$$\mathbf{B} = 0.104\frac{(\rho^2 - 10^{-4})}{\rho}\,\mathbf{a}_\phi \text{ Wb/m}^2$$

Now,

$$\Phi_a = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^1\int_{10^{-2}}^{1.2\times10^{-2}} 0.104\left[\rho - \frac{10^{-4}}{\rho}\right]d\rho\,dz$$

$$= 0.104\left[\frac{(1.2\times10^{-2})^2 - 10^{-4}}{2} - 10^{-4}\ln\left(\frac{1.2}{1.0}\right)\right] = 3.92\times10^{-7}\text{ Wb} = \underline{0.392\,\mu\text{Wb}}$$

8.31b) $1.0\,\text{cm} < \rho < 1.4\,\text{cm}$ (note typo in book): This is part *a* over again, except we change the upper limit of the radial integration:

$$\Phi_b = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.4\times 10^{-2}} 0.104\left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\, dz$$
$$= 0.104\left[\frac{(1.4\times 10^{-2})^2 - 10^{-4}}{2} - 10^{-4}\ln\left(\frac{1.4}{1.0}\right)\right] = 1.49\times 10^{-6}\,\text{Wb} = \underline{1.49\,\mu\text{Wb}}$$

c) $1.4\,\text{cm} < \rho < 20\,\text{cm}$: This is entirely outside the current distribution, so we need **B** there: We modify the Ampere's circuital law result of part *a* to find:

$$\mathbf{B}_{out} = 0.104\frac{[(1.4\times 10^{-2})^2 - 10^{-4}]}{\rho}\,\mathbf{a}_\phi = \frac{10^{-5}}{\rho}\,\mathbf{a}_\phi\ \text{Wb/m}^2$$

We now find

$$\Phi_c = \int_0^1 \int_{1.4\times 10^{-2}}^{20\times 10^{-2}} \frac{10^{-5}}{\rho}\, d\rho\, dz = 10^{-5}\ln\left(\frac{20}{1.4}\right) = 2.7\times 10^{-5}\,\text{Wb} = \underline{27\,\mu\text{Wb}}$$

8.32. The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross-section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m.

a) Specify the current densities on the surfaces at $\rho = 2$ cm, $z = 1$cm, and $z = 4$cm. All surfaces must carry equal currents. With this requirement, we find: $\mathbf{K}(\rho = 2) = \underline{-3\,\mathbf{a}_z\ \text{kA/m}}$. Next, the current densities on the $z = 1$ and $z = 4$ surfaces must transistion between the current density values at $\rho = 2$ and $\rho = 3$. Knowing the the radial current density will vary as $1/\rho$, we find $\mathbf{K}(z = 1) = \underline{(60/\rho)\mathbf{a}_\rho\ \text{A/m}}$ with ρ in meters. Similarly, $\mathbf{K}(z = 4) = \underline{-(60/\rho)\mathbf{a}_\rho\ \text{A/m}}$.

b) Find **H** everywhere: Outside the toroid, $\mathbf{H} = 0$. Inside, we apply Ampere's circuital law in the manner of Problem 8.14:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi\rho H_\phi = \int_0^{2\pi} \mathbf{K}(\rho = 2)\cdot\mathbf{a}_z\,(2\times 10^{-2})\,d\phi$$
$$\Rightarrow\quad \mathbf{H} = -\frac{2\pi(3000)(.02)}{\rho}\mathbf{a}_\phi = \underline{-60/\rho\,\mathbf{a}_\phi\ \text{A/m (inside)}}$$

c) Calculate the total flux within the toriod: We have $\mathbf{B} = -(60\mu_0/\rho)\mathbf{a}_\phi\ \text{Wb/m}^2$. Then

$$\Phi = \int_{.01}^{.04}\int_{.02}^{.03} \frac{-60\mu_0}{\rho}\,\mathbf{a}_\phi\cdot(-\mathbf{a}_\phi)\,d\rho\,dz = (.03)(60)\mu_0\ln\left(\frac{3}{2}\right) = \underline{0.92\,\mu\text{Wb}}$$

8.33. Use an expansion in cartesian coordinates to show that the curl of the gradient of any scalar field *G* is identically equal to zero. We begin with

$$\nabla G = \frac{\partial G}{\partial x}\,\mathbf{a}_x + \frac{\partial G}{\partial y}\,\mathbf{a}_y + \frac{\partial G}{\partial z}\,\mathbf{a}_z$$

and

$$\nabla\times\nabla G = \left[\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial y}\right)\right]\mathbf{a}_x + \left[\frac{\partial}{\partial z}\left(\frac{\partial G}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial z}\right)\right]\mathbf{a}_y$$
$$+\left[\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right)\right]\mathbf{a}_z = \underline{0}\ \text{ for any } G$$

8.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

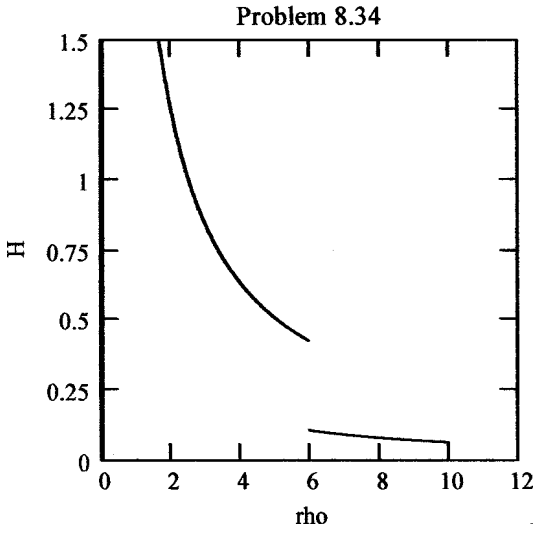
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: This will be

$$\Phi = \int_0^1 \int_1^6 \frac{16\mu_0}{2\pi\rho}\, d\rho\, dz + \int_0^1 \int_6^7 \frac{4\mu_0}{2\pi\rho}\, d\rho\, dz = \frac{2\mu_0}{\pi}\,[4\ln 6 + \ln(7/6)] = \underline{5.9\ \mu\text{Wb}}$$

8.35. A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\,\mathbf{a}_z$ A/m is located at $\rho = 4$.

a.) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$: Since the current is cylindrically-symmetric, we know that $\mathbf{H} = I/(2\pi\rho)\ \mathbf{a}_\phi$, where I is the current enclosed, equal in this case to $2\pi(2)K = 80\pi$ A. Thus, using the result of Section 8.6, we find

$$V_m = -\frac{I}{2\pi}\,\phi = -\frac{80\pi}{2\pi}\,\phi = \underline{-40\phi\ \text{A}}$$

which is valid over the region $2 < \rho < 4$, $-\pi < \phi < \pi$, and $-\infty < z < \infty$. For $\rho > 4$, the outer current contributes, leading to a total enclosed current of

$$I_{net} = 2\pi(2)(20) - 2\pi(4)(10) = 0$$

With zero enclosed current, $H_\phi = 0$, and the magnetic potential is zero as well.

8.35b. Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$: Again, we know that $\mathbf{H} = H_\phi(\rho)$, since the current is cylindrically symmetric. With the current only in the z direction, and again using symmmetry, we expect only a z component of $\mathbf{A}$ which varies only with ρ. We can then write:

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

Thus

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho} \quad \Rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + C$$

We require that $A_z = 0$ at $\rho = 3$. Therefore $C = [(\mu_0 I)/(2\pi)]\ln(3)$, Then, with $I = 80\pi$, we finally obtain

$$\mathbf{A} = -\frac{\mu_0(80\pi)}{2\pi}[\ln(\rho) - \ln(3)]\,\mathbf{a}_z = \underline{40\mu_0 \ln\left(\frac{3}{\rho}\right)\mathbf{a}_z \text{ Wb/m}}$$

8.36. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space.

a) Show that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(3y - z) + \frac{\partial}{\partial y}2xz = \underline{0}$$

b) At $P(2, -1, 3)$, find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$: First $\mathbf{A}_P = \underline{-6\mathbf{a}_x + 12\mathbf{a}_y}$. Then, using the curl formula in cartesian coordinates,

$$\mathbf{B} = \nabla \times \mathbf{A} = -2x\mathbf{a}_x - \mathbf{a}_y + (2z - 3)\mathbf{a}_z \quad \Rightarrow \quad \mathbf{B}_P = \underline{-4\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}$$

Now

$$\mathbf{H}_P = (1/\mu_0)\mathbf{B}_P = \underline{-3.2 \times 10^6\mathbf{a}_x - 8.0 \times 10^5\mathbf{a}_y + 2.4 \times 10^6\mathbf{a}_z \text{ A/m}}$$

Then $\mathbf{J} = \nabla \times \mathbf{H} = (1/\mu_0)\nabla \times \mathbf{B} = \underline{0}$, as the curl formula in cartesian coordinates shows.

8.37. Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Fig. 8.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$: Well-within the toroid, we have

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi = -\nabla V_m = -\frac{1}{\rho}\frac{dV_m}{d\phi}\mathbf{a}_\phi$$

Thus

$$V_m = -\frac{NI\phi}{2\pi} + C$$

Then,

$$0 = -\frac{1000(0.8)(0.3\pi)}{2\pi} + C$$

or C = 120. Finally

$$V_m = \underline{\left[120 - \frac{400}{\pi}\phi\right] \text{ A} \quad (0 < \phi < 2\pi)}$$

8.38. The solenoid shown in Fig. 8.11b contains 400 turns, carries a current $I = 5$ A, has a length of 8cm, and a radius $a = 1.2$ cm (hope it doesn't blow up!).

a) Find $\mathbf{H}$ within the solenoid. Assuming the current flows in the $\mathbf{a}_\phi$ direction, $\mathbf{H}$ will then be along the positive z direction, and will be given by

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z = \frac{(400)(5)}{.08}\mathbf{a}_z = \underline{2.5 \times 10^4 \text{ A/m}}$$

b) If $V_m = 0$ at the origin, specify $V_m(\rho, \phi, z)$ inside the solenoid: Since $\mathbf{H}$ is only in the z direction, V_m should vary with z only. Use

$$\mathbf{H} = -\nabla V_m = -\frac{dV_m}{dz}\mathbf{a}_z \quad \Rightarrow \quad V_m = -H_z z + C$$

At $z = 0$, $V_m = 0$, so $C = 0$. Therefore $V_m(z) = \underline{-2.5 \times 10^4 z \text{ A}}$

c) Let $\mathbf{A} = 0$ at the origin, and specify $\mathbf{A}(\rho, \phi, z)$ inside the solenoid if the medium is free space. $\mathbf{A}$ should be in the same direction as the current, and so would have a ϕ component only. Furthermore, since $\nabla \times \mathbf{A} = \mathbf{B}$, the curl will be z-directed only. Therefore

$$\nabla \times \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\mathbf{a}_z = \mu_0 H_z \mathbf{a}_z$$

Then

$$\frac{\partial}{\partial \rho}(\rho A_\phi) = \mu_0 H_z \rho \quad \Rightarrow \quad A_\phi = \frac{\mu_0 H_z \rho}{2} + C$$

$A_\phi = 0$ at the origin, so $C = 0$. Finally,

$$\mathbf{A} = \frac{(4\pi \times 10^{-7})(2.5 \times 10^4)\rho}{2}\mathbf{a}_\phi = \underline{15.7\mathbf{a}_\phi \text{ mWb/m}}$$

8.39. Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$:

a) Find $\mathbf{H}$: Since we have parallel current sheets carrying equal and opposite currents, we use Eq. (12), $\mathbf{H} = \mathbf{K} \times \mathbf{a}_N$, where $\mathbf{a}_N$ is the unit normal directed into the region between currents, and where either one of the two currents are used. Choosing the sheet at $x = 0.2$, we find

$$\mathbf{H} = 30\mathbf{a}_z \times -\mathbf{a}_x = \underline{-30\mathbf{a}_y \text{ A/m}}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: Use

$$\mathbf{H} = -30\mathbf{a}_y = -\nabla V_m = -\frac{dV_m}{dy}\mathbf{a}_y$$

So

$$\frac{dV_m}{dy} = 30 \quad \Rightarrow \quad V_m = 30y + C_1$$

Then

$$0 = 30(0.2) + C_1 \quad \Rightarrow \quad C_1 = -6 \quad \Rightarrow \quad V_m = \underline{30y - 6 \text{ A}}$$

8.39c) Find $\mathbf{B}$: $\mathbf{B} = \mu_0\mathbf{H} = \underline{-30\mu_0\mathbf{a}_y \text{ Wb/m}^2}$.

d) Obtain an expression for $\mathbf{A}$ if $\mathbf{A} = 0$ at P: We expect $\mathbf{A}$ to be z-directed (with the current), and so from $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is y-directed, we set up

$$-\frac{dA_z}{dx} = -30\mu_0 \quad \Rightarrow \quad A_z = 30\mu_0 x + C_2$$

Then

$$0 = 30\mu_0(0.1) + C_2 \quad \Rightarrow \quad C_2 = -3\mu_0$$

So finally

$$\mathbf{A} = \underline{\mu_0(30x - 3)\mathbf{a}_z \text{ Wb/m}}$$

8.40. Let $\mathbf{A} = (3y^2 - 2z)\mathbf{a}_x - 2x^2z\mathbf{a}_y + (x + 2y)\mathbf{a}_z$ Wb/m in free space. Find $\nabla \times \nabla \times \mathbf{A}$ at $P(-2, 3, -1)$:

First $\nabla \times \mathbf{A} =$

$$\left(\frac{\partial(x+2y)}{\partial y} - \frac{\partial(-2x^2z)}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial(3y^2-2z)}{\partial z} - \frac{\partial(x+2y)}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial(-2x^2z)}{\partial x} - \frac{\partial(3y^2-2z)}{\partial y}\right)\mathbf{a}_z$$
$$= (2 + 2x^2)\mathbf{a}_x - 3\mathbf{a}_y - (4xz + 6y)\mathbf{a}_z$$

Then

$$\nabla \times \nabla \times \mathbf{A} = \frac{\partial(4xz+6y)}{\partial x}\mathbf{a}_y - \frac{\partial(4xz+6y)}{\partial y}\mathbf{a}_x = -6\mathbf{a}_x + 4z\mathbf{a}_y$$

At P this becomes $\nabla \times \nabla \times \mathbf{A}|_P = \underline{-6\mathbf{a}_x - 4\mathbf{a}_y \text{ Wb/m}^3}$.

8.41. Assume that $\mathbf{A} = 50\rho^2\mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find $\mathbf{H}$ and $\mathbf{B}$: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \underline{-100\rho\,\mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0\,\mathbf{a}_\phi \text{ A/m}}$.

b) Find $\mathbf{J}$: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\mathbf{a}_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{-100\rho^2}{\mu_0}\right)\mathbf{a}_z = \underline{-\frac{200}{\mu_0}\mathbf{a}_z \text{ A/m}^2}$$

c) Use $\mathbf{J}$ to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{-200}{\mu_0}\mathbf{a}_z \cdot \mathbf{a}_z\,\rho\,d\rho\,d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0}\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,(1)d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ kA}}$$

8.42. Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$. First

$$\nabla_2\left(\frac{1}{R_{12}}\right) = \nabla_2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{-1/2}$$
$$= -\frac{1}{2}\left[\frac{2(x_2-x_1)\mathbf{a}_x + 2(y_2-y_1)\mathbf{a}_y + 2(z_2-z_1)\mathbf{a}_z}{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]^{3/2}}\right] = \frac{-\mathbf{R}_{12}}{R_{12}^3} = \frac{\mathbf{R}_{21}}{R_{12}^3}$$

Also note that $\nabla_1(1/R_{12})$ would give the same result, but of opposite sign.

8.43. Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure: We do this by first finding **B** within the outer conductor and then "uncurling" the result to find **A**. With $-z$-directed current I in the outer conductor, the current density is

$$\mathbf{J}_{out} = -\frac{I}{\pi(7a)^2 - \pi(5a)^2}\mathbf{a}_z = -\frac{I}{24\pi a^2}\mathbf{a}_z$$

Since current I flows in both conductors, but in opposite directions, Ampere's circuital law inside the outer conductor gives:

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_{5a}^{\rho}\frac{I}{24\pi a^2}\rho'\,d\rho'\,d\phi \quad\Rightarrow\quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Now, with $\mathbf{B} = \mu_0\mathbf{H}$, we note that $\nabla\times\mathbf{A}$ will have a ϕ component only, and from the direction and symmetry of the current, we expect **A** to be z-directed, and to vary only with ρ. Therefore

$$\nabla\times\mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mu_0\mathbf{H}$$

and so

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Then by direct integration,

$$A_z = \int\frac{-\mu_0 I(49)}{48\pi\rho}\,d\rho + \int\frac{\mu_0 I\rho}{48\pi a^2}\,d\rho + C = \frac{\mu_0 I}{96\pi}\left[\frac{\rho^2}{a^2} - 98\ln\rho\right] + C$$

As per Fig. 8.20, we establish a zero reference at $\rho = 5a$, enabling the evaluation of the integration constant:

$$C = -\frac{\mu_0 I}{96\pi}\left[25 - 98\ln(5a)\right]$$

Finally,

$$A_z = \underline{\frac{\mu_0 I}{96\pi}\left[\left(\frac{\rho^2}{a^2} - 25\right) + 98\ln\left(\frac{5a}{\rho}\right)\right]\ \text{Wb/m}}$$

A plot of this continues the plot of Fig. 8.20, in which the curve goes negative at $\rho = 5a$, and then approaches a minimum of $-.09\mu_0 I/\pi$ at $\rho = 7a$, at which point the slope becomes zero.

8.44. By expanding Eq.(58), Sec. 8.7 in cartesian coordinates, show that (59) is correct. Eq. (58) can be rewritten as

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$

We begin with

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Then the x component of $\nabla(\nabla\cdot\mathbf{A})$ is

$$[\nabla(\nabla\cdot\mathbf{A})]_x = \underline{\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x\partial y} + \frac{\partial^2 A_z}{\partial x\partial z}}$$

Now

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z$$

and the x component of $\nabla\times\nabla\times\mathbf{A}$ is

$$[\nabla\times\nabla\times\mathbf{A}]_x = \underline{\frac{\partial^2 A_y}{\partial x\partial y} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z\partial y}}$$

Then, using the underlined results

$$[\nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}]_x = \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2} = \nabla^2 A_x$$

Similar results will be found for the other two components, leading to

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A} = \nabla^2 A_x\mathbf{a}_x + \nabla^2 A_y\mathbf{a}_y + \nabla^2 A_z\mathbf{a}_z \equiv \nabla^2\mathbf{A} \quad \text{QED}$$

CHAPTER 9

9.1. A point charge, $Q = -0.3\,\mu$C and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{E} = 30\,\mathbf{a}_z$ V/m. Use Eq. (1) and Newton's laws to develop the appropriate differential equations and solve them, subject to the initial conditions at $t = 0$: $\mathbf{v} = 3 \times 10^5\,\mathbf{a}_x$ m/s at the origin. At $t = 3\,\mu$s, find:

a) the position $P(x, y, z)$ of the charge: The force on the charge is given by $\mathbf{F} = q\mathbf{E}$, and Newton's second law becomes:

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2\mathbf{z}}{dt^2} = q\mathbf{E} = (-0.3 \times 10^{-6})(30\,\mathbf{a}_z)$$

describing motion of the charge in the z direction. The initial velocity in x is constant, and so no force is applied in that direction. We integrate once:

$$\frac{dz}{dt} = v_z = \frac{qE}{m}t + C_1$$

The initial velocity along z, $v_z(0)$ is zero, and so $C_1 = 0$. Integrating a second time yields the z coordinate:

$$z = \frac{qE}{2m}t^2 + C_2$$

The charge lies at the origin at $t = 0$, and so $C_2 = 0$. Introducing the given values, we find

$$z = \frac{(-0.3 \times 10^{-6})(30)}{2 \times 3 \times 10^{-16}}t^2 = -1.5 \times 10^{10}t^2 \text{ m}$$

At $t = 3\ \mu$s, $z = -(1.5 \times 10^{10})(3 \times 10^{-6})^2 = -.135$ cm. Now, considering the initial constant velocity in x, the charge in $3\ \mu$s attains an x coordinate of $x = vt = (3 \times 10^5)(3 \times 10^{-6}) = .90$ m. In summary, at $t = 3\ \mu$s we have $P(x, y, z) = \underline{(.90, 0, -.135)}$.

b) the velocity, $\mathbf{v}$: After the first integration in part a, we find

$$v_z = \frac{qE}{m}t = -(3 \times 10^{10})(3 \times 10^{-6}) = -9 \times 10^4 \text{ m/s}$$

Including the intial x-directed velocity, we finally obtain $\mathbf{v} = \underline{3 \times 10^5\,\mathbf{a}_x - 9 \times 10^4\mathbf{a}_z \text{ m/s}}$.

c) the kinetic energy of the charge: Have

$$\text{K.E.} = \frac{1}{2}m|v|^2 = \frac{1}{2}(3 \times 10^{-16})(1.13 \times 10^5)^2 = \underline{1.5 \times 10^{-5} \text{ J}}$$

9.2. A point charge, $Q = -0.3\,\mu$C and $m = 3 \times 10^{-16}$ kg, is moving through the field $\mathbf{B} = 30\mathbf{a}_z$ mT. Make use of Eq. (2) and Newton's laws to develop the appropriate differential equations, and solve them, subject to the initial condition at $t = 0$, $\mathbf{v} = 3 \times 10^5$ m/s at the origin. Solve these equations (perhaps with the help of an example given in Section 7.5) to evaluate at $t = 3\mu$s: a) the position $P(x, y, z)$ of the charge; b) its velocity; c) and its kinetic energy:

We begin by visualizing the problem. Using $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, we find that a *positive* charge moving along positive $\mathbf{a}_x$, would encounter the z-directed $\mathbf{B}$ field and be deflected into the *negative* y direction.

9.2 (continued) Motion along negative y through the field would cause further deflection into the *negative* x direction. We can construct the differential equations for the forces in x and in y as follows:

$$F_x\mathbf{a}_x = m\frac{dv_x}{dt}\mathbf{a}_x = qv_y\mathbf{a}_y \times B\mathbf{a}_z = qBv_y\mathbf{a}_x$$

$$F_y\mathbf{a}_y = m\frac{dv_y}{dt}\mathbf{a}_y = qv_x\mathbf{a}_x \times B\mathbf{a}_z = -qBv_x\mathbf{a}_y$$

or

$$\frac{dv_x}{dt} = \frac{qB}{m}v_y \tag{1}$$

and

$$\frac{dv_y}{dt} = -\frac{qB}{m}v_x \tag{2}$$

To solve these equations, we first differentiate (2) with time and substitute (1), obtaining:

$$\frac{d^2v_y}{dt^2} = -\frac{qB}{m}\frac{dv_x}{dt} = -\left(\frac{qB}{m}\right)^2 v_y$$

Therefore, $v_y = A\sin(qBt/m) + A'\cos(qBt/m)$. However, at $t = 0$, $v_y = 0$, and so $A' = 0$, leaving $v_y = A\sin(qBt/m)$. Then, using (2),

$$v_x = -\frac{m}{qB}\frac{dv_y}{dt} = -A\cos\left(\frac{qBt}{m}\right)$$

Now at $t = 0$, $v_x = v_{x0} = 3\times10^5$. Therefore $A = -v_{x0}$, and so $v_x = v_{x0}\cos(qBt/m)$, and $v_y = -v_{x0}\sin(qBt/m)$. The positions are then found by integrating v_x and v_y over time:

$$x(t) = \int v_{x0}\cos\left(\frac{qBt}{m}\right)dt + C = \frac{mv_{x0}}{qB}\sin\left(\frac{qBt}{m}\right) + C$$

where $C = 0$, since $x(0) = 0$. Then

$$y(t) = \int -v_{x0}\sin\left(\frac{qBt}{m}\right)dt + D = \frac{mv_{x0}}{qB}\cos\left(\frac{qBt}{m}\right) + D$$

We require that $y(0) = 0$, so $D = -(mv_{x0})/(qB)$, and finally $y(t) = -mv_{x0}/qB\,[1 - \cos(qBt/m)]$. Summarizing, we have, using $q = -3\times10^{-7}$ C, $m = 3\times10^{-16}$ kg, $B = 30\times10^{-3}$ T, and $v_{x0} = 3\times10^5$ m/s:

$$x(t) = \frac{mv_{x0}}{qB}\sin\left(\frac{qBt}{m}\right) = -10^{-2}\sin(-3\times10^{-7}t)\text{ m}$$

$$y(t) = -\frac{mv_{x0}}{qB}\left[1 - \cos\left(\frac{qBt}{m}\right)\right] = 10^{-2}[1 - \cos(-3\times10^7 t)]\text{ m}$$

$$v_x(t) = v_{x0}\cos\left(\frac{qBt}{m}\right) = 3\times10^5\cos(-3\times10^7 t)\text{ m/s}$$

$$v_y(t) = -v_{x0}\sin\left(\frac{qBt}{m}\right) = -3\times10^5\sin(-3\times10^7 t)\text{ m/s}$$

9.2 (continued) The answers are now:

a) At $t = 3 \times 10^{-6}$ s, $x = \underline{8.9 \text{ mm}}$, $y = \underline{14.5 \text{ mm}}$, and $z = \underline{0}$.

b) At $t = 3 \times 10^{-6}$ s, $v_x = -1.3 \times 10^5$ m/s, $v_y = 2.7 \times 10^5$ m/s, and so

$$\mathbf{v}(t = 3\,\mu\text{s}) = \underline{-1.3 \times 10^5 \mathbf{a}_x + 2.7 \times 10^5 \mathbf{a}_y \text{ m/s}}$$

whose magnitude is $v = 3 \times 10^5$ m/s as would be expected.

c) Kinetic energy is K.E. $= (1/2)mv^2 = \underline{1.35\ \mu\text{J}}$ at all times.

9.3. A point charge for which $Q = 2 \times 10^{-16}$ C and $m = 5 \times 10^{-26}$ kg is moving in the combined fields $\mathbf{E} = 100\mathbf{a}_x - 200\mathbf{a}_y + 300\mathbf{a}_z$ V/m and $\mathbf{B} = -3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z$ mT. If the charge velocity at $t = 0$ is $\mathbf{v}(0) = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z) \times 10^5$ m/s:

a) give the unit vector showing the direction in which the charge is accelerating at $t = 0$: Use $\mathbf{F}(t = 0) = q[\mathbf{E} + (\mathbf{v}(0) \times \mathbf{B})]$, where

$$\mathbf{v}(0) \times \mathbf{B} = (2\mathbf{a}_x - 3\mathbf{a}_y - 4\mathbf{a}_z)10^5 \times (-3\mathbf{a}_x + 2\mathbf{a}_y - \mathbf{a}_z)10^{-3} = 1100\mathbf{a}_x + 1400\mathbf{a}_y - 500\mathbf{a}_z$$

So the force in newtons becomes

$$\mathbf{F}(0) = (2\times10^{-16})[(100+1100)\mathbf{a}_x+(1400-200)\mathbf{a}_y+(300-500)\mathbf{a}_z] = 4\times10^{-14}[6\mathbf{a}_x+6\mathbf{a}_y-\mathbf{a}_z]$$

The unit vector that gives the acceleration direction is found from the force to be

$$\mathbf{a}_F = \frac{6\mathbf{a}_x + 6\mathbf{a}_y - \mathbf{a}_z}{\sqrt{73}} = \underline{.70\mathbf{a}_x + .70\mathbf{a}_y - .12\mathbf{a}_z}$$

b) find the kinetic energy of the charge at $t = 0$:

$$\text{K.E.} = \frac{1}{2}m|\mathbf{v}(0)|^2 = \frac{1}{2}(5 \times 10^{-26}\,\text{kg})(5.39 \times 10^5\,\text{m/s})^2 = 7.25 \times 10^{-15}\,\text{J} = \underline{7.25\ \text{fJ}}$$

9.4. An electron ($q_e = -1.60219 \times 10^{-19}$ C, $m = 9.10956 \times 10^{-31}$ kg) is moving at a constant velocity $\mathbf{v} = 4.5 \times 10^7 \mathbf{a_y}$ m/s along the negative y axis. At the origin it encounters the uniform magnetic field $\mathbf{B} = 2.5\mathbf{a}_z$ mT, and remains in it up to $y = 2.5$ cm. If we assume (with good accuracy) that the electron remains on the y axis while it is in the magnetic field, find its x-, y-, and z-coordinate values when $y = 50$ cm: The procedure is to find the electron velocity as it leaves the field, and then determine its coordinates at the time corresponding to $y = 50$ cm. The force it encounters while in the field is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = (-1.60219 \times 10^{-19})(4.5 \times 10^7)(2.5 \times 10^{-3})(\mathbf{a}_y \times \mathbf{a}_z) = -1.80 \times 10^{-14}\mathbf{a}_x\,\text{N}$$

This force will be constant during the time the electron traverses the field. It establishes a negative x-directed velocity as it leaves the field, given by the acceleration times the transit time, t_t:

$$v_x = \frac{Ft_t}{m} = \left(\frac{-1.80 \times 10^{14}\,\text{N}}{9.10956 \times 10^{-31}\,\text{kg}}\right)\left(\frac{2.5 \times 10^{-2}\,\text{m}}{4.5 \times 10^7\,\text{m/s}}\right) = -1.09 \times 10^7\,\text{m/s}$$

9.4 (continued) The time for the electron to travel along y between 2.5 and 50 cm is

$$t_{50} = \frac{(50 - 2.5) \times 10^{-2}}{4.5 \times 10^{7}} = 1.06 \times 10^{-8}\,\text{s}$$

In that time, the electron moves to an x coordinate given by

$$x = v_x t_{50} = -(1.09 \times 10^{7})(1.06 \times 10^{-8}) = -.115\,\text{m}$$

The coordinates at the time the electron reaches $y = 50$ cm are then:

$$\underline{x = -11.5\,\text{cm},\ y = 50\,\text{cm},\ z = 0}$$

9.5. A rectangular loop of wire in free space joins points $A(1, 0, 1)$ to $B(3, 0, 1)$ to $C(3, 0, 4)$ to $D(1, 0, 4)$ to A. The wire carries a current of 6 mA, flowing in the $\mathbf{a}_z$ direction from B to C. A filamentary current of 15 A flows along the entire z axis in the $\mathbf{a}_z$ direction.

a) Find $\mathbf{F}$ on side BC:

$$\mathbf{F}_{BC} = \int_B^C I_{\text{loop}} d\mathbf{L} \times \mathbf{B}_{\text{from wire at BC}}$$

Thus

$$\mathbf{F}_{BC} = \int_1^4 (6 \times 10^{-3})\, dz\, \mathbf{a}_z \times \frac{15\mu_0}{2\pi(3)}\, \mathbf{a}_y = -1.8 \times 10^{-8} \mathbf{a}_x\ \text{N} = \underline{-18\mathbf{a}_x\ \text{nN}}$$

b) Find $\mathbf{F}$ on side AB: The field from the long wire now varies with position along the loop segment. We include that dependence and write

$$\mathbf{F}_{AB} = \int_1^3 (6 \times 10^{-3})\, dx\, \mathbf{a}_x \times \frac{15\mu_0}{2\pi x}\, \mathbf{a}_y = \frac{45 \times 10^{-3}}{\pi} \mu_0 \ln 3\, \mathbf{a}_z = \underline{19.8\mathbf{a}_z\ \text{nN}}$$

c) Find $\mathbf{F}_{\text{total}}$ on the loop: This will be the vector sum of the forces on the four sides. Note that by symmetry, the forces on sides AB and CD will be equal and opposite, and so will cancel. This leaves the sum of forces on sides BC (part a) and DA, where

$$\mathbf{F}_{DA} = \int_1^4 -(6 \times 10^{-3})\, dz\, \mathbf{a}_z \times \frac{15\mu_0}{2\pi(1)}\, \mathbf{a}_y = 54\mathbf{a}_x\ \text{nN}$$

The total force is then $\mathbf{F}_{\text{total}} = \mathbf{F}_{DA} + \mathbf{F}_{BC} = (54 - 18)\mathbf{a}_x = \underline{36\,\mathbf{a}_x\ \text{nN}}$

9.6 The magnetic flux density in a region of free space is given by $\mathbf{B} = -3x\mathbf{a}_x + 5y\mathbf{a}_y - 2z\mathbf{a}_z$ T. Find the total force on the rectangular loop shown in Fig. 9.15 if it lies in the plane $z = 0$ and is bounded by $x = 1$, $x = 3$, $y = 2$, and $y = 5$, all dimensions in cm: First, note that in the plane $z = 0$, the z component of the given field is zero, so will not contribute to the force. We use

$$\mathbf{F} = \int_{loop} I d\mathbf{L} \times \mathbf{B}$$

which in our case becomes, with $I = 30$ A:

$$\mathbf{F} = \int_{.01}^{.03} 30 dx \mathbf{a}_x \times (-3x\mathbf{a}_x + 5y|_{y=.02}\, \mathbf{a}_y) + \int_{.02}^{.05} 30 dy \mathbf{a}_y \times (-3x|_{x=.03}\, \mathbf{a}_x + 5y\mathbf{a}_y)$$
$$+ \int_{.03}^{.01} 30 dx \mathbf{a}_x \times (-3x\mathbf{a}_x + 5y|_{y=.05}\, \mathbf{a}_y) + \int_{.05}^{.02} 30 dy \mathbf{a}_y \times (-3x|_{x=.01}\, \mathbf{a}_x + 5y\mathbf{a}_y)$$

9.6. (continued) Simplifying, this becomes

$$\mathbf{F} = \int_{.01}^{.03} 30(5)(.02)\,\mathbf{a}_z\,dx + \int_{.02}^{.05} -30(3)(.03)(-\mathbf{a}_z)\,dy$$
$$+ \int_{.03}^{.01} 30(5)(.05)\,\mathbf{a}_z\,dx + \int_{.05}^{.02} -30(3)(.01)(-\mathbf{a}_z)\,dy = (.060 + .081 - .150 - .027)\mathbf{a}_z \text{ N}$$
$$= \underline{-36\,\mathbf{a}_z \text{ mN}}$$

9.7. Uniform current sheets are located in free space as follows: $8\mathbf{a}_z$ A/m at $y = 0$, $-4\mathbf{a}_z$ A/m at $y = 1$, and $-4\mathbf{a}_z$ A/m at $y = -1$. Find the vector force per meter length exerted on a current filament carrying 7 mA in the $\mathbf{a}_L$ direction if the filament is located at:

a) $x = 0$, $y = 0.5$, and $\mathbf{a}_L = \mathbf{a}_z$: We first note that within the region $-1 < y < 1$, the magnetic fields from the two outer sheets (carrying $-4\mathbf{a}_z$ A/m) cancel, leaving only the field from the center sheet. Therefore, $\mathbf{H} = -4\mathbf{a}_x$ A/m ($0 < y < 1$) and $\mathbf{H} = 4\mathbf{a}_x$ A/m ($-1 < y < 0$). Outside ($y > 1$ and $y < -1$) the fields from all three sheets cancel, leaving $\mathbf{H} = 0$ ($y > 1$, $y < -1$). So at $x = 0$, $y = .5$, the force per meter length will be

$$\mathbf{F}/\text{m} = I\mathbf{a}_z \times \mathbf{B} = (7 \times 10^{-3})\mathbf{a}_z \times -4\mu_0\mathbf{a}_x = \underline{-35.2\mathbf{a}_y \text{ nN/m}}$$

b.) $y = 0.5$, $z = 0$, and $\mathbf{a}_L = \mathbf{a}_x$: $\mathbf{F}/\text{m} = I\mathbf{a}_x \times -4\mu_0\mathbf{a}_x = \underline{0}$.

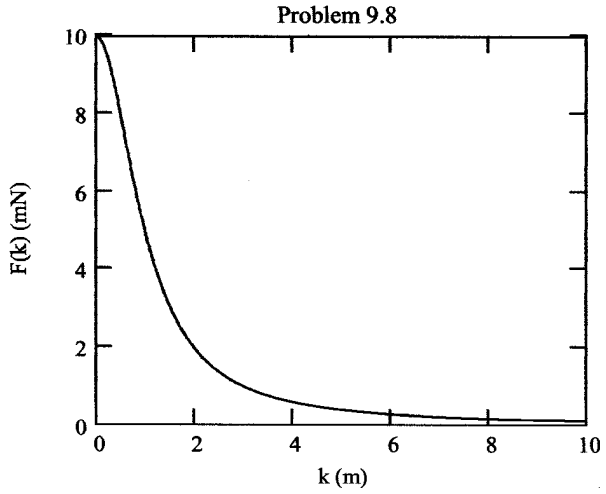
c) $x = 0$, $y = 1.5$, $\mathbf{a}_L = \mathbf{a}_z$: Since $y = 1.5$, we are in the region in which $\mathbf{B} = 0$, and so the force is zero.

9.8. Filamentary currents of $-25\mathbf{a}_z$ and $25\mathbf{a}_z$ A are located in the $x = 0$ plane in free space at $y = -1$ and $y = 1$m respectively. A third filamentary current of $10^{-3}\mathbf{a}_z$ A is located at $x = k$, $y = 0$. Find the vector force on a 1-m length of the 1-mA filament and plot $|\mathbf{F}|$ versus k: The total $\mathbf{B}$ field arising from the two 25A filaments evaluated at the location of the 1-mA filament is, in cartesian components:

$$\mathbf{B} = \underbrace{\frac{25\mu_0}{2\pi(1+k^2)}(k\mathbf{a}_y + \mathbf{a}_x)}_{\text{line at } y=+1} + \underbrace{\frac{25\mu_0}{2\pi(1+k^2)}(-k\mathbf{a}_y + \mathbf{a}_x)}_{\text{line at } y=-1} = \frac{25\mu_0\mathbf{a}_x}{\pi(1+k^2)}$$

The force on the 1m length of 1-mA line is now

$$\mathbf{F} = 10^{-3}(1)\mathbf{a}_z \times \frac{25\mu_0\mathbf{a}_x}{\pi(1+k^2)} = \frac{(2.5 \times 10^{-2})(4 \times 10^{-7})}{(1+k^2)}\mathbf{a}_y = \frac{10^{-8}\mathbf{a}_y}{(1+k^2)}\mathbf{a}_y \text{ N} = \underline{\frac{10\mathbf{a}_y}{(1+k^2)} \text{ nN}}$$



9.9. A current of $-100\mathbf{a}_z$ A/m flows on the conducting cylinder $\rho = 5$ mm and $+500\mathbf{a}_z$ A/m is present on the conducting cylinder $\rho = 1$ mm. Find the magnitude of the total force acting to split the outer cylinder apart along its length: The differential force acting on the outer cylinder arising from the field of the inner cylinder is $d\mathbf{F} = \mathbf{K}_{\text{outer}} \times \mathbf{B}$, where $\mathbf{B}$ is the field from the inner cylinder, evaluated at the outer cylinder location:

$$\mathbf{B} = \frac{2\pi(1)(500)\mu_0}{2\pi(5)}\mathbf{a}_\phi = 100\mu_0\,\mathbf{a}_\phi \text{ T}$$

Thus $d\mathbf{F} = -100\mathbf{a}_z \times 100\mu_0\mathbf{a}_\phi = 10^4\mu_0\mathbf{a}_\rho$ N/m^2. We wish to find the force acting to split the outer cylinder, which means we need to evaluate the net force in one cartesian direction on one half of the cylinder. We choose the "upper" half ($0 < \phi < \pi$), and integrate the y component of $d\mathbf{F}$ over this range, and over a unit length in the z direction:

$$F_y = \int_0^1 \int_0^\pi 10^4\mu_0\mathbf{a}_\rho \cdot \mathbf{a}_y(5\times10^{-3})\,d\phi\,dz = \int_0^\pi 50\mu_0 \sin\phi\,d\phi = 100\mu_0 = \underline{4\pi \times 10^{-5} \text{ N/m}}$$

Note that we did not include the "self force" arising from the outer cylinder's $\mathbf{B}$ field on itself. Since the outer cylinder is a two-dimensional current sheet, its field exists only just outside the cylinder, and so no force exists. If this cylinder possessed a finite thickness, then we would need to include its self-force, since there would be an interior field and a volume current density that would spatially overlap.

9.10. Two infinitely-long parallel filaments each carry 50 A in the $\mathbf{a}_z$ direction. If the filaments lie in the plane $y = 0$ at $x = 0$ and $x = 5$mm (note bad wording in problem statement in book), find the vector force per meter length on the filament passing through the origin: The force will be

$$\mathbf{F} = \int_0^1 Id\mathbf{L} \times \mathbf{B}$$

where $Id\mathbf{L}$ is that of the filament at the origin, and $\mathbf{B}$ is that arising from the filament at $x = 5$mm evaluated at the location of the other filament (along the z axis). We obtain

$$\mathbf{F} = \int_0^1 50\,dz\mathbf{a}_z \times \frac{-50\mu_0\mathbf{a}_y}{2\pi(5\times10^{-3})} = \underline{0.10\,\mathbf{a}_x \text{ N/m}}$$

9.11. a) Use Eq. (14), Sec. 9.3, to show that the force of attraction per unit length between two filamentary conductors in free space with currents $I_1\mathbf{a}_z$ at $x = 0$, $y = d/2$, and $I_2\mathbf{a}_z$ at $x = 0$, $y = -d/2$, is $\mu_0 I_1 I_2/(2\pi d)$: The force on I_2 is given by

$$\mathbf{F}_2 = \mu_0\frac{I_1 I_2}{4\pi}\oint\left[\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2}\right] \times d\mathbf{L}_2$$

Let z_1 indicate the z coordinate along I_1, and z_2 indicate the z coordinate along I_2. We then have $R_{12} = \sqrt{(z_2 - z_1)^2 + d^2}$ and

$$\mathbf{a}_{R12} = \frac{(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y}{\sqrt{(z_2 - z_1)^2 + d^2}}$$

Also, $d\mathbf{L}_1 = dz_1\mathbf{a}_z$ and $d\mathbf{L}_2 = dz_2\mathbf{a}_z$ The "inside" integral becomes:

$$\oint \frac{\mathbf{a}_{R12} \times d\mathbf{L}_1}{R_{12}^2} = \oint \frac{[(z_2 - z_1)\mathbf{a}_z - d\mathbf{a}_y] \times dz_1\mathbf{a}_z}{[(z_2 - z_1)^2 + d^2]^{1.5}} = \int_{-\infty}^{\infty} \frac{-d\,dz_1\,\mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

9.11a. (continued) The force expression now becomes

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\int_{-\infty}^{\infty} \frac{-d\, dz_1\, \mathbf{a}_x}{[(z_2 - z_1)^2 + d^2]^{1.5}} \times dz_2 \mathbf{a}_z \right] = \mu_0 \frac{I_1 I_2}{4\pi} \int_0^1 \int_{-\infty}^{\infty} \frac{d\, dz_1\, dz_2\, \mathbf{a}_y}{[(z_2 - z_1)^2 + d^2]^{1.5}}$$

Note that the "outside" integral is taken over a unit length of current I_2. Evaluating, obtain,

$$\mathbf{F}_2 = \mu_0 \frac{I_1 I_2 d\, \mathbf{a}_y}{4\pi d^2}(2) \int_0^1 dz_2 = \frac{\mu_0 I_1 I_2}{2\pi d} \mathbf{a}_y \text{ N/m}$$

as expected.

b) Show how a simpler method can be used to check your result: We use $d\mathbf{F}_2 = I_2 d\mathbf{L}_2 \times \mathbf{B}_{12}$, where the field from current 1 at the location of current 2 is

$$\mathbf{B}_{12} = \frac{\mu_0 I_1}{2\pi d} \mathbf{a}_x \text{ T}$$

so over a unit length of I_2, we obtain

$$\mathbf{F}_2 = I_2 \mathbf{a}_z \times \frac{\mu_0 I_1}{2\pi d} \mathbf{a}_x = \mu_0 \frac{I_1 I_2}{2\pi d} \mathbf{a}_y \text{ N/m}$$

This second method is really just the first over again, since we recognize the inside integral of the first method as the Biot-Savart law, used to find the field from current 1 at the current 2 location.

9.12. A conducting current strip carrying $\mathbf{K} = 12\mathbf{a}_z$ A/m lies in the $x = 0$ plane between $y = 0.5$ and $y = 1.5$ m. There is also a current filament of $I = 5$ A in the $\mathbf{a}_z$ direction on the z axis. Find the force exerted on the:

a) filament by the current strip: We first need to find the field from the current strip at the filament location. Consider the strip as made up of many adjacent strips of width dy, each carrying current $dI\mathbf{a}_z = \mathbf{K}dy$. The field along the z axis from each differential strip will be $d\mathbf{B} = [(Kdy\mu_0)/(2\pi y)]\mathbf{a}_x$. The total $\mathbf{B}$ field from the strip evaluated along the z axis is therefore

$$\mathbf{B} = \int_{0.5}^{1.5} \frac{12\mu_0 \mathbf{a}_x}{2\pi y}\, dy = \frac{6\mu_0}{\pi} \ln\left(\frac{1.5}{0.5}\right) \mathbf{a}_x = 2.64 \times 10^{-6} \mathbf{a}_x \text{ Wb/m}^2$$

Now

$$\mathbf{F} = \int_0^1 Id\mathbf{L} \times \mathbf{B} = \int_0^1 5dz\, \mathbf{a}_z \times 2.64 \times 10^{-6}\, \mathbf{a}_x\, dz = \underline{13.2\, \mathbf{a}_y\ \mu\text{N/m}}$$

b) strip by the filament: In this case we integrate $\mathbf{K} \times \mathbf{B}$ over a unit length in z of the strip area, where $\mathbf{B}$ is the field from the filament evaluated on the strip surface:

$$\mathbf{F} = \int_{Area} \mathbf{K} \times \mathbf{B}\, da = \int_0^1 \int_{0.5}^{1.5} 12\mathbf{a}_z \times \frac{-5\mu_0 \mathbf{a}_x}{2\pi y}\, dy = \frac{-30\mu_0}{\pi} \ln(3)\, \mathbf{a}_y = \underline{-13.2\, \mathbf{a}_y\ \mu\text{N/m}}$$

9.13. A current of 6A flows from $M(2, 0, 5)$ to $N(5, 0, 5)$ in a straight solid conductor in free space. An infinite current filament lies along the z axis and carries 50A in the $\mathbf{a}_z$ direction. Compute the vector torque on the wire segment using:

a) an origin at (0, 0, 5): The $\mathbf{B}$ field from the long wire at the short wire is $\mathbf{B} = (\mu_0 I_z \mathbf{a_y})/(2\pi x)$ T. Then the force acting on a differential length of the wire segment is

$$d\mathbf{F} = I_w d\mathbf{L} \times \mathbf{B} = I_w dx\, \mathbf{a}_x \times \frac{\mu_0 I_z}{2\pi x}\mathbf{a}_y = \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z \text{ N}$$

Now the differential torque about (0, 0, 5) will be

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\mathbf{a}_x \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

The net torque is now found by integrating the differential torque over the length of the wire segment:

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y = -\frac{3\mu_0(6)(50)}{2\pi}\, \mathbf{a}_y = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \ \text{N} \cdot \text{m}}$$

b) an origin at (0, 0, 0): Here, the only modification is in $\mathbf{R}_T$, which is now $\mathbf{R}_T = x\,\mathbf{a}_x + 5\,\mathbf{a}_z$ So now

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = \left[x\mathbf{a}_x + 5\mathbf{a}_z\right] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z}{2\pi}\, dx\, \mathbf{a}_y$$

Everything from here is the same as in part *a*, so again, $\mathbf{T} = \underline{-1.8 \times 10^{-4}\, \mathbf{a}_y \ \text{N} \cdot \text{m}}$.

c) an origin at (3, 0, 0): In this case, $\mathbf{R}_T = (x - 3)\mathbf{a}_x + 5\mathbf{a}_z$, and the differential torque is

$$d\mathbf{T} = \left[(x-3)\mathbf{a}_x + 5\mathbf{a}_z\right] \times \frac{\mu_0 I_w I_z}{2\pi x} dx\, \mathbf{a}_z = -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y$$

Thus

$$\mathbf{T} = \int_2^5 -\frac{\mu_0 I_w I_z (x-3)}{2\pi x}\, dx\, \mathbf{a}_y = -6.0 \times 10^{-5}\left[3 - 3\ln\left(\frac{5}{2}\right)\right] \mathbf{a}_y = \underline{-1.5 \times 10^{-5}\, \mathbf{a}_y \ \text{N} \cdot \text{m}}$$

9.14. The rectangular loop of Prob. 6 is now subjected to the $\mathbf{B}$ field produced by two current sheets, $\mathbf{K}_1 = 400\,\mathbf{a}_y$ A/m at $z = 2$, and $\mathbf{K}_2 = 300\,\mathbf{a}_z$ A/m at $y = 0$ in free space. Find the vector torque on the loop, referred to an origin:

a) at (0,0,0): The fields from both current sheets, at the loop location, will be negative x-directed. They will add together to give, in the loop plane:

$$\mathbf{B} = -\mu_0\left(\frac{K_1}{2} + \frac{K_2}{2}\right)\mathbf{a}_x = -\mu_0(200 + 150)\,\mathbf{a}_x = -350\mu_0\,\mathbf{a}_x \text{ Wb/m}^2$$

With this field, forces will be acting only on the wire segments that are parallel to the y axis. The force on the segment nearer to the y axis will be

$$\mathbf{F}_1 = I\mathbf{L} \times \mathbf{B} = -30(3 \times 10^{-2})\mathbf{a}_y \times -350\mu_0\mathbf{a}_x = -315\mu_0\,\mathbf{a}_z \text{ N}$$

9.14a (continued) The force acting on the segment farther from the y axis will be

$$\mathbf{F}_2 = I\mathbf{L} \times \mathbf{B} = 30(3 \times 10^{-2})\mathbf{a}_y \times -350\mu_0\mathbf{a}_x = 315\mu_0\,\mathbf{a}_z \text{ N}$$

The torque about the origin is now $\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$, where $\mathbf{R}_1$ is the vector directed from the origin to the midpoint of the nearer y-directed segment, and $\mathbf{R}_2$ is the vector joining the origin to the midpoint of the farther y-directed segment. So $\mathbf{R}_1(\text{cm}) = \mathbf{a}_x + 3.5\mathbf{a}_y$ and $\mathbf{R}_2(\text{cm}) = 3\mathbf{a}_x + 3.5\mathbf{a}_y$. Therefore

$$\begin{aligned}\mathbf{T}_{0,0,0} &= [(\mathbf{a}_x + 3.5\mathbf{a}_y) \times 10^{-2}] \times -315\mu_0\,\mathbf{a}_z + [(3\mathbf{a}_x + 3.5\mathbf{a}_y) \times 10^{-2}] \times 315\mu_0\,\mathbf{a}_z \\ &= -6.30\mu_0\mathbf{a}_y = \underline{-7.92 \times 10^{-6}\,\mathbf{a}_y \text{ N}-\text{m}}\end{aligned}$$

b) at the center of the loop: Use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ where $\mathbf{S} = (2 \times 3) \times 10^{-4}\,\mathbf{a}_z$ m^2. So

$$\mathbf{T} = 30(6 \times 10^{-4}\mathbf{a}_z) \times (-350\mu_0\,\mathbf{a}_x) = \underline{-7.92 \times 10^{-6}\,\mathbf{a}_y \text{ N}-\text{m}}$$

9.15. A solid conducting filament extends from $x = -b$ to $x = b$ along the line $y = 2$, $z = 0$. This filament carries a current of 3 A in the $\mathbf{a}_x$ direction. An infinite filament on the z axis carries 5 A in the $\mathbf{a}_z$ direction. Obtain an expression for the torque exerted on the finite conductor about an origin located at (0, 2, 0): The differential force on the wire segment arising from the field from the infinite wire is

$$d\mathbf{F} = 3\,dx\,\mathbf{a}_x \times \frac{5\mu_0}{2\pi\rho}\,\mathbf{a}_\phi = -\frac{15\mu_0\cos\phi\,dx}{2\pi\sqrt{x^2+4}}\,\mathbf{a}_z = -\frac{15\mu_0 x\,dx}{2\pi(x^2+4)}\,\mathbf{a}_z$$

So now the differential torque about the (0, 2, 0) origin is

$$d\mathbf{T} = \mathbf{R}_T \times d\mathbf{F} = x\,\mathbf{a}_x \times -\frac{15\mu_0 x\,dx}{2\pi(x^2+4)}\,\mathbf{a}_z = \frac{15\mu_0 x^2\,dx}{2\pi(x^2+4)}\,\mathbf{a}_y$$

The torque is then

$$\begin{aligned}\mathbf{T} &= \int_{-b}^{b} \frac{15\mu_0 x^2\,dx}{2\pi(x^2+4)}\,\mathbf{a}_y = \frac{15\mu_0}{2\pi}\,\mathbf{a}_y\left[x - 2\tan^{-1}\left(\frac{x}{2}\right)\right]_{-b}^{b} \\ &= \underline{(6 \times 10^{-6})\left[b - 2\tan^{-1}\left(\frac{b}{2}\right)\right]\mathbf{a}_y \text{ N}\cdot\text{m}}\end{aligned}$$

9.16. Assume that an electron is describing a circular orbit of radius a about a positively-charged nucleus.

a) By selecting an appropriate current and area, show that the equivalent orbital dipole moment is $ea^2\omega/2$, where ω is the electron's angular velocity: The current magnitude will be $I = \frac{e}{T}$, where e is the electron charge and T is the orbital period. The latter is $T = 2\pi/\omega$, and so $I = e\omega/(2\pi)$. Now the dipole moment magnitude will be $m = IA$, where A is the loop area. Thus

$$m = \frac{e\omega}{2\pi}\,\pi a^2 = \frac{1}{2}ea^2\omega \;\; //$$

b) Show that the torque produced by a magnetic field parallel to the plane of the orbit is $ea^2\omega B/2$: With B assumed constant over the loop area, we would have $\mathbf{T} = \mathbf{m} \times \mathbf{B}$. With $\mathbf{B}$ parallel to the loop plane, $\mathbf{m}$ and $\mathbf{B}$ are orthogonal, and so $T = mB$. So, using part a, $T = ea^2\omega B/2$.

9.16. (continued)

c) by equating the Coulomb and centrifugal forces, show that ω is $(4\pi\epsilon_0 m_e a^3/e^2)^{-1/2}$, where m_e is the electron mass: The force balance is written as

$$\frac{e^2}{4\pi\epsilon_0 a^2} = m^e\omega^2 a \;\;\Rightarrow\;\; \omega = \left(\frac{4\pi\epsilon_0 m_e a^3}{e^2}\right)^{-1/2} \quad //$$

d) Find values for the angular velocity, torque, and the orbital magnetic moment for a hydrogen atom, where a is about 6×10^{-11} m; let $B = 0.5$ T: First

$$\omega = \left[\frac{(1.60\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.1\times10^{-31})(6\times10^{-11})^3}\right]^{1/2} = \underline{3.42\times10^{16}\ \text{rad/s}}$$

$$T = \frac{1}{2}(3.42\times10^{16})(1.60\times10^{-19})(0.5)(6\times10^{-11})^2 = \underline{4.93\times10^{-24}\ \text{N}\cdot\text{m}}$$

Finally,

$$m = \frac{T}{B} = \underline{9.86\times10^{-24}\ \text{A}\cdot\text{m}^2}$$

9.17. The hydrogen atom described in Problem 16 is now subjected to a magnetic field having the same direction as that of the atom. Show that the forces caused by B result in a decrease of the angular velocity by $eB/(2m_e)$ and a decrease in the orbital moment by $e^2a^2B/(4m_e)$. What are these decreases for the hydrogen atom in parts per million for an external magnetic flux density of 0.5 T? We first write down all forces on the electron, in which we equate its coulomb force toward the nucleus to the sum of the centrifugal force and the force associated with the applied B field. With the field applied in the same direction as that of the atom, this would yield a Lorentz force that is radially outward – in the same direction as the centrifugal force.

$$F_e = F_{cent} + F_B \;\;\Rightarrow\;\; \frac{e^2}{4\pi\epsilon_0 a^2} = m_e\omega^2 a + \underbrace{e\omega a B}_{QvB}$$

With $B = 0$, we solve for ω to find:

$$\omega = \omega_0 = \sqrt{\frac{e^2}{4\pi\epsilon_0 m_e a^3}}$$

Then with B present, we find

$$\omega^2 = \frac{e^2}{4\pi\epsilon_0 m_e a^3} - \frac{e\omega B}{m_e} = \omega_0^2 - \frac{e\omega B}{m_e}$$

Therefore

$$\omega = \omega_0\sqrt{1 - \frac{e\omega B}{\omega_0^2 m_e}} \doteq \omega_0\left(1 - \frac{e\omega B}{2\omega_0^2 m_e}\right)$$

But $\omega \doteq \omega_0$, and so

$$\omega \doteq \omega_0\left(1 - \frac{eB}{2\omega_0 m_e}\right) = \omega_0 - \frac{eB}{2m_e} \quad //$$

9.17. (continued) As for the magnetic moment, we have

$$m = IS = \frac{e\omega}{2\pi}\pi a^2 = \frac{1}{2}\omega e a^2 \doteq \frac{1}{2}ea^2\left(\omega_0 - \frac{eB}{2m_e}\right) = \frac{1}{2}\omega_0 e a^2 - \frac{1}{4}\frac{e^2a^2B}{m_e} \quad //$$

Finally, for $a = 6 \times 10^{-11}$ m, $B = 0.5$ T, we have

$$\frac{\Delta\omega}{\omega} = \frac{eB}{2m_e}\frac{1}{\omega} \doteq \frac{eB}{2m_e}\frac{1}{\omega_0} = \frac{1.60 \times 10^{-19} \times 0.5}{2 \times 9.1 \times 10^{-31} \times 3.4 \times 10^{16}} = \underline{1.3 \times 10^{-6}}$$

where $\omega_0 = 3.4 \times 10^{16}$ sec^{-1} is found from Problem 16. Finally,

$$\frac{\Delta m}{m} = \frac{e^2a^2B}{4m_e} \times \frac{2}{\omega e a^2} \doteq \frac{eB}{2m_e\omega_0} = \underline{1.3 \times 10^{-6}}$$

9.18. Calculate the vector torque on the square loop shown in Fig. 9.16 about an origin at A in the field $\mathbf{B}$, given:

a) $A(0, 0, 0)$ and $\mathbf{B} = 100\mathbf{a}_y$ mT: The field is uniform and so does not produce any translation of the loop. Therefore, we may use $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$ about any origin, where $I = 0.6$ A and $\mathbf{S} = 16\mathbf{a}_z$ m^2. We find $\mathbf{T} = 0.6(16)\mathbf{a}_z \times 0.100\mathbf{a}_y = \underline{-0.96\,\mathbf{a}_x\ \text{N}-\text{m}}$.

b) $A(0, 0, 0)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y$ mT: Using the same reasoning as in part a, we find

$$\mathbf{T} = 0.6(16)\mathbf{a}_z \times (0.200\mathbf{a}_x + 0.100\mathbf{a}_y) = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y\ \text{N}-\text{m}}$$

c) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT: We observe two things here: 1) The field is again uniform and so again the torque is independent of the origin chosen, and 2) The field differs from that of part b only by the addition of a z component. With $\mathbf{S}$ in the z direction, this new component of $\mathbf{B}$ will produce no torque, so the answer is the *same as part b*, or $\mathbf{T} = \underline{-0.96\mathbf{a}_x + 1.92\mathbf{a}_y\ \text{N}-\text{m}}$.

d) $A(1, 2, 3)$ and $\mathbf{B} = 200\mathbf{a}_x + 100\mathbf{a}_y - 300\mathbf{a}_z$ mT for $x \geq 2$ and $\mathbf{B} = 0$ elsewhere: Now, force is acting only on the y-directed segment at $x = +2$, so we need to be careful, since translation will occur. So we must use the given origin. The differential torque acting on the differential wire segment at location (2,y) is $d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F}$, where

$$d\mathbf{F} = Id\mathbf{L} \times \mathbf{B} = 0.6\,dy\,\mathbf{a}_y \times [0.2\mathbf{a}_x + 0.1\mathbf{a}_y - 0.3\mathbf{a}_z] = [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy$$

and $\mathbf{R}(y) = (2, y, 0) - (1, 2, 3) = \mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z$. We thus find

$$\begin{aligned} d\mathbf{T} = \mathbf{R}(y) \times d\mathbf{F} &= \left[\mathbf{a}_x + (y-2)\mathbf{a}_y - 3\mathbf{a}_z\right] \times [-0.18\mathbf{a}_x - 0.12\mathbf{a}_z]\,dy \\ &= \left[-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z\right] dy \end{aligned}$$

The net torque is now

$$\mathbf{T} = \int_{-2}^{2}\left[-0.12(y-2)\mathbf{a}_x + 0.66\mathbf{a}_y + 0.18(y-2)\mathbf{a}_z\right] dy = \underline{0.96\mathbf{a}_x + 2.64\mathbf{a}_y - 1.44\mathbf{a}_z\ \text{N}-\text{m}}$$

9.19. Given a material for which $\chi_m = 3.1$ and within which $\mathbf{B} = 0.4y\mathbf{a}_z$ T, find:

a) $\mathbf{H}$: We use $\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H}$, or

$$\mathbf{H} = \frac{0.4y\mathbf{a}_y}{(1+3.1)\mu_0} = \underline{77.6y\mathbf{a}_z \text{ kA/m}}$$

b) $\mu = (1+3.1)\mu_0 = \underline{5.15\times10^{-6} \text{ H/m}}$.

c) $\mu_R = (1+3.1) = \underline{4.1}$.

d) $\mathbf{M} = \chi_m\mathbf{H} = (3.1)(77.6y\mathbf{a}_y) = \underline{241y\mathbf{a}_z \text{ kA/m}}$

e) $\mathbf{J} = \nabla\times\mathbf{H} = (dH_z)/(dy)\,\mathbf{a}_x = \underline{77.6\,\mathbf{a}_x \text{ kA/m}^2}$.

f) $\mathbf{J}_b = \nabla\times\mathbf{M} = (dM_z)/(dy)\,\mathbf{a}_x = \underline{241\,\mathbf{a}_x \text{ kA/m}^2}$.

g) $\mathbf{J}_T = \nabla\times\mathbf{B}/\mu_0 = \underline{318\mathbf{a}_x \text{ kA/m}^2}$.

9.20. Find $\mathbf{H}$ in a material where:

a) $\mu_R = 4.2$, there are 2.7×10^{29} atoms/m^3, and each atom has a dipole moment of $2.6\times10^{-30}\,\mathbf{a}_y$ A $\cdot$ m^2. Since all dipoles are identical, we may write $\mathbf{M} = N\mathbf{m} = (2.7\times10^{29})(2.6\times10^{-30}\mathbf{a}_y) = 0.70\mathbf{a}_y$ A/m. Then

$$\mathbf{H} = \frac{\mathbf{M}}{\mu_R - 1} = \frac{0.70\,\mathbf{a}_y}{4.2-1} = \underline{0.22\,\mathbf{a}_y \text{ A/m}}$$

b) $\mathbf{M} = 270\,\mathbf{a}_z$ A/m and $\mu = 2$ μH/m: Have $\mu_R = \mu/\mu_0 = (2\times10^{-6})/(4\pi\times10^{-7}) = 1.59$. Then $\mathbf{H} = 270\mathbf{a}_z/(1.59-1) = \underline{456\,\mathbf{a}_z \text{ A/m}}$.

c) $\chi_m = 0.7$ and $\mathbf{B} = 2\mathbf{a}_z$ T: Use

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0(1+\chi_m)} = \frac{2\mathbf{a}_z}{(4\pi\times10^{-7})(1.7)} = \underline{936\,\mathbf{a}_z \text{ kA/m}}$$

d) Find $\mathbf{M}$ in a material where bound surface current densities of $12\,\mathbf{a}_z$ A/m and $-9\,\mathbf{a}_z$ A/m exist at $\rho = 0.3$ m and $\rho = 0.4$ m, respectively: We use $\oint \mathbf{M}\cdot d\mathbf{L} = I_b$, where, since currents are in the z direction and are symmetric about the z axis, we chose the path integrals to be circular loops centered on and normal to z. From the symmetry, $\mathbf{M}$ will be ϕ-directed and will vary only with radius. Note first that for $\rho < 0.3$ m, no bound current will be enclosed by a path integral, so we conclude that $\mathbf{M} = \underline{0 \text{ for } \rho < 0.3\text{m}}$. At radii between the currents the path integral will enclose only the inner current so,

$$\oint \mathbf{M}\cdot d\mathbf{L} = 2\pi\rho M_\phi = 2\pi(0.3)12 \;\Rightarrow\; \mathbf{M} = \underline{\frac{3.6}{\rho}\,\mathbf{a}_\phi \text{ A/m} \;\; (0.3<\rho<0.4\text{m})}$$

Finally, for $\rho > 0.4$ m, the total enclosed bound current is $I_{b,tot} = 2\pi(0.3)(12) - 2\pi(0.4)(9) = 0$, so therefore $\mathbf{M} = \underline{0\ (\rho > 0.4\text{m})}$.

9.21. Find the magnitude of the magnetization in a material for which:

a) the magnetic flux density is 0.02 Wb/m^2 *and the magnetic susceptibility is 0.003* (note that this latter quantity is missing in the original problem statement): From $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$ and from $\mathbf{M} = \chi_m\mathbf{H}$, we write

$$M = \frac{B}{\mu_0}\left(\frac{1}{\chi_m}+1\right)^{-1} = \frac{B}{\mu_0(334)} = \frac{0.02}{(4\pi\times10^{-7})(334)} = \underline{47.7 \text{ A/m}}$$

9.21b) the magnetic field intensity is 1200 A/m and the relative permeability is 1.005: From $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0\mu_R\mathbf{H}$, we write

$$M = (\mu_R - 1)H = (.005)(1200) = \underline{6.0 \text{ A/m}}$$

c) there are 7.2×10^{28} atoms per cubic meter, each having a dipole moment of 4×10^{-30} A $\cdot$ m^2 in the same direction, and the magnetic susceptibility is 0.0003: With all dipoles identical the dipole moment density becomes

$$M = n\,m = (7.2 \times 10^{28})(4 \times 10^{-30}) = \underline{0.288 \text{ A/m}}$$

9.22. Three current sheets are located as follows: $160\mathbf{a}_z$ A/m at $x = 1$cm, $-40\mathbf{a}_z$ A/m at $x = 5$cm, and $50\mathbf{a}_z$ A/m at $x = 8$cm. Let $\mu = \mu_0$ for $x < 1$cm and $x > 8$cm; for $1 < x < 5$ cm, $\mu = 3\mu_0$, and for $5 < x < 8$cm, $\mu = 2\mu_0$. Find $\mathbf{B}$ everywhere: We know that the $\mathbf{H}$ field from an infinite current sheet will be given in magnitude by $H = K/2$, and will be directed parallel to the sheet and perpendicular to the current, with the directions on either side of the sheet determined by the right hand rule. With this in mind, we can construct the following expressions for the $\mathbf{B}$ field in all four regions:

$$\mathbf{B}(x < 1) = \frac{1}{2}\mu_0(-160 + 40 - 50) = \underline{-1.07 \times 10^{-4}\,\mathbf{a}_y \text{ T}}$$

$$\mathbf{B}(1 < x < 5) = \frac{1}{2}(3\mu_0)(160 + 40 - 50) = \underline{2.83 \times 10^{-4}\,\mathbf{a}_y \text{ T}}$$

$$\mathbf{B}(5 < x < 8) = \frac{1}{2}(2\mu_0)(160 - 40 - 50) = \underline{8.80 \times 10^{-5}\,\mathbf{a}_y \text{ T}}$$

$$\mathbf{B}(x > 8) = \frac{1}{2}\mu_0(160 - 40 + 50) = \underline{1.07 \times 10^{-4}\,\mathbf{a}_y \text{ T}}$$

9.23. Calculate values for H_ϕ, B_ϕ, and M_ϕ at $\rho = c$ for a coaxial cable with $a = 2.5$ mm and $b = 6$ mm if it carries current $I = 12$ A in the center conductor, and $\mu = 3$ μH/m for $2.5 < \rho < 3.5$ mm, $\mu = 5$ μH/m for $3.5 < \rho < 4.5$ mm, and $\mu = 10$ μH/m for $4.5 < \rho < 6$ mm. Compute for:

a) $c = 3$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(3 \times 10^{-3})} = \underline{637 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (3 \times 10^{-6})(637) = \underline{1.91 \times 10^{-3} \text{ Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{884 \text{ A/m}}$.

b) $c = 4$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(4 \times 10^{-3})} = \underline{478 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (5 \times 10^{-6})(478) = \underline{2.39 \times 10^{-3} \text{ Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{1.42 \times 10^3 \text{ A/m}}$.

c) $c = 5$ mm: Have

$$H_\phi = \frac{I}{2\pi\rho} = \frac{12}{2\pi(5 \times 10^{-3})} = \underline{382 \text{ A/m}}$$

Then $B_\phi = \mu H_\phi = (10 \times 10^{-6})(382) = \underline{3.82 \times 10^{-3} \text{ Wb/m}^2}$.

Finally, $M_\phi = (1/\mu_0)B_\phi - H_\phi = \underline{2.66 \times 10^3 \text{ A/m}}$.

9.24. A coaxial transmission line has $a = 5$ mm and $b = 20$ mm. Let its center lie on the z axis and let a dc current I flow in the $\mathbf{a}_z$ direction in the center conductor. The volume between the conductors contains a magnetic material for which $\mu_R = 2.5$, as well as air. Find **H**, **B**, and **M** everywhere between conductors if $H_\phi = 600/\pi$ A/m at $\rho = 10$ mm, $\phi = \pi/2$, and the magnetic material is located where:

a) $a < \rho < 3a$; First, we know that $H_\phi = I/2\pi\rho$, from which we construct:

$$\frac{I}{2\pi(10^{-2})} = \frac{600}{\pi} \quad \Rightarrow \quad I = 12\,\text{A}$$

Since the interface between the two media lies in the $\mathbf{a}_\phi$ direction, we use the boundary condition of continuity of tangential **H** and write

$$\mathbf{H}(5 < \rho < 20) = \frac{12}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{6}{\pi\rho}\mathbf{a}_\phi\ \text{A/m}}$$

In the magnetic material, we find

$$\mathbf{B}(5 < \rho < 15) = \mu\mathbf{H} = \frac{(2.5)(4\pi \times 10^{-7})(12)}{2\pi\rho}\mathbf{a}_\phi = \underline{(6/\rho)\mathbf{a}_\phi\ \mu\text{T}}$$

Then, in the free space region, $\mathbf{B}(15 < \rho < 20) = \mu_0\mathbf{H} = \underline{(2.4/\rho)\mathbf{a}_\phi\ \mu\text{T}}$.

b) $0 < \phi < \pi$; Again, we are given $\mathbf{H} = 600/\pi\ \mathbf{a}_\phi$ A/m at $\rho = 10$ and at $\phi = \pi/2$. Now, since the interface between media lies in the $\mathbf{a}_\rho$ direction, and noting that magnetic field will be normal to this ($\mathbf{a}_\phi$ directed), we use the boundary condition of continuity of **B** normal to an interface, and write $\mathbf{B}(0 < \phi < \pi) = \mathbf{B}_1 = \mathbf{B}(\pi < \phi < 2\pi) = \mathbf{B}_2$, or $2.5\mu_0\mathbf{H}_1 = \mu_0\mathbf{H}_2$. Now, using Ampere's circuital law, we write

$$\oint \mathbf{H} \cdot d\mathbf{L} = \pi\rho H_1 + \pi\rho H_2 = 3.5\pi\rho H_1 = I$$

Using the given value for H_1 at $\rho = 10$ mm, $I = 3.5(600/\pi)(\pi \times 10^{-2}) = 21$ A. Therefore, $H_1 = 21/(3.5\pi\rho) = 6/(\pi\rho)$, or $\mathbf{H}(0 < \phi < \pi) = \underline{6/(\pi\rho)\,\mathbf{a}_\phi\ \text{A/m}}$. Then $H_2 = 2.5H_1$, or $\mathbf{H}(\pi < \phi < 2\pi) = \underline{15/(\pi\rho)\,\mathbf{a}_\phi\ \text{A/m}}$. Now $\mathbf{B}(0 < \phi < 2\pi) = 2.5\mu_0(6/(\pi\rho))\mathbf{a}_\phi = \underline{6/\rho\,\mathbf{a}_\phi\ \mu\text{T}}$. Now, in general, $\mathbf{M} = (\mu_R - 1)\mathbf{H}$, and so $\mathbf{M}(0 < \phi < \pi) = (2.5-1)6/(\pi\rho)\mathbf{a}_\phi = \underline{9/(\pi\rho)\,\mathbf{a}_\phi\ \text{A/m}}$ and $\mathbf{M}(\pi < \phi < 2\pi) = \underline{0}$.

9.25. A conducting filament at $z = 0$ carries 12 A in the $\mathbf{a}_z$ direction. Let $\mu_R = 1$ for $\rho < 1$ cm, $\mu_R = 6$ for $1 < \rho < 2$ cm, and $\mu_R = 1$ for $\rho > 2$ cm. Find

a) **H** everywhere: This result will depend on the current and not the materials, and is:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1.91}{\rho}\ \text{A/m}\ \ (0 < \rho < \infty)}$$

b) **B** everywhere: We use $\mathbf{B} = \mu_R\mu_0\mathbf{H}$ to find:

$\mathbf{B}(\rho < 1\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi\ \text{T}}$

$\mathbf{B}(1 < \rho < 2\,\text{cm}) = (6)\mu_0(1.91/\rho) = \underline{(1.4 \times 10^{-5}/\rho)\mathbf{a}_\phi\ \text{T}}$

$\mathbf{B}(\rho > 2\,\text{cm}) = (1)\mu_0(1.91/\rho) = \underline{(2.4 \times 10^{-6}/\rho)\mathbf{a}_\phi\ \text{T}}$ where ρ is in meters.

9.26. Point $P(2, 3, 1)$ lies on the planar boundary boundary separating region 1 from region 2. The unit vector $\mathbf{a}_{N12} = 0.6\mathbf{a}_x + 0.48\mathbf{a}_y + 0.64\mathbf{a}_z$ is directed from region 1 to region 2. Let $\mu_{R1} = 2$, $\mu_{R2} = 8$, and $\mathbf{H}_1 = 100\mathbf{a}_x - 300\mathbf{a}_y + 200\mathbf{a}_z$ A/m. Find $\mathbf{H}_2$: First $\mathbf{B}_1 = 200\mu_0\mathbf{a}_x - 600\mu_0\mathbf{a}_y + 400\mu_0\mathbf{a}_z$. Then its normal component at the boundary will be $\mathbf{B}_{1N} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = (52.8\mathbf{a}_x + 42.24\mathbf{a}_y + 56.32\mathbf{a}_z)\mu_0 = \mathbf{B}_{2N}$. Then $\mathbf{H}_{2N} = \mathbf{B}_{2N}/(8\mu_0) = 6.60\mathbf{a}_x + 5.28\mathbf{a}_y + 7.04\mathbf{a}_z$, and $\mathbf{H}_{1N} = \mathbf{B}_{1N}/2\mu_0 = 26.40\mathbf{a}_x + 21.12\mathbf{a}_y + 28.16\mathbf{a}_z$. Now $\mathbf{H}_{1T} = \mathbf{H}_1 - \mathbf{H}_{1N} = (100\mathbf{a}_x - 300\mathbf{a}_y + 200\mathbf{a}_z) - (26.40\mathbf{a}_x + 21.12\mathbf{a}_y + 28.16\mathbf{a}_z) = 73.60\mathbf{a}_x - 321.12\mathbf{a}_y + 171.84\mathbf{a}_z = \mathbf{H}_{2T}$.

Finally, $\mathbf{H}_2 = \mathbf{H}_{2N} + \mathbf{H}_{2T} = \underline{80.2\mathbf{a}_x - 315.8\mathbf{a}_y + 178.9\mathbf{a}_z \text{ A/m}}$.

9.27. Let $\mu_{R1} = 2$ in region 1, defined by $2x+3y-4z > 1$, while $\mu_{R2} = 5$ in region 2 where $2x+3y-4z < 1$. In region 1, $\mathbf{H}_1 = 50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z$ A/m. Find:

a) $\mathbf{H}_{N1}$ (normal component of $\mathbf{H}_1$ at the boundary): We first need a unit vector normal to the surface, found through

$$\mathbf{a}_N = \frac{\nabla(2x + 3y - 4z)}{|\nabla(2x + 3y - 4z)|} = \frac{2\mathbf{a}_x + 3\mathbf{a}_y - 4\mathbf{a}_z}{\sqrt{29}} = .37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z$$

Since this vector is found through the gradient, it will point in the direction of increasing values of $2x + 3y - 4z$, and so will be directed into region 1. Thus we write $\mathbf{a}_N = \mathbf{a}_{N21}$. The normal component of $\mathbf{H}_1$ will now be:

$$\begin{aligned}\mathbf{H}_{N1} &= (\mathbf{H}_1 \cdot \mathbf{a}_{N21})\mathbf{a}_{N21} \\ &= \left[(50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z) \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z)\right](.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) \\ &= \underline{-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z \text{ A/m}}\end{aligned}$$

b) $\mathbf{H}_{T1}$ (tangential component of $\mathbf{H}_1$ at the boundary):

$$\begin{aligned}\mathbf{H}_{T1} &= \mathbf{H}_1 - \mathbf{H}_{N1} \\ &= (50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z) - (-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) \\ &= \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}\end{aligned}$$

c) $\mathbf{H}_{T2}$ (tangential component of $\mathbf{H}_2$ at the boundary): Since tangential components of $\mathbf{H}$ are continuous across a boundary between two media of different permeabilities, we have

$$\mathbf{H}_{T2} = \mathbf{H}_{T1} = \underline{54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z \text{ A/m}}$$

d) $\mathbf{H}_{N2}$ (normal component of $\mathbf{H}_2$ at the boundary): Since normal components of $\mathbf{B}$ are continuous across a boundary between media of different permeabilities, we write $\mu_1\mathbf{H}_{N1} = \mu_2\mathbf{H}_{N2}$ or

$$\mathbf{H}_{N2} = \frac{\mu_{R1}}{\mu_R 2}\mathbf{H}_{N1} = \frac{2}{5}(-4.83\mathbf{a}_x - 7.24\mathbf{a}_y + 9.66\mathbf{a}_z) = \underline{-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z \text{ A/m}}$$

e) θ_1, the angle between $\mathbf{H}_1$ and $\mathbf{a}_{N21}$: This will be

$$\cos\theta_1 = \frac{\mathbf{H}_1}{|\mathbf{H}_1|} \cdot \mathbf{a}_{N21} = \left[\frac{50\mathbf{a}_x - 30\mathbf{a}_y + 20\mathbf{a}_z}{(50^2 + 30^2 + 20^2)^{1/2}}\right] \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.21$$

Therefore $\theta_1 = \cos^{-1}(-.21) = \underline{102^\circ}$.

9.27f) θ_2, the angle between $\mathbf{H}_2$ and $\mathbf{a}_{N21}$: First,

$$\begin{aligned}\mathbf{H}_2 &= \mathbf{H}_{T2} + \mathbf{H}_{N2} = (54.83\mathbf{a}_x - 22.76\mathbf{a}_y + 10.34\mathbf{a}_z) + (-1.93\mathbf{a}_x - 2.90\mathbf{a}_y + 3.86\mathbf{a}_z) \\ &= 52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z \text{ A/m}\end{aligned}$$

Now

$$\cos\theta_2 = \frac{\mathbf{H}_2}{|\mathbf{H}_2|} \cdot \mathbf{a}_{N21} = \left[\frac{52.90\mathbf{a}_x - 25.66\mathbf{a}_y + 14.20\mathbf{a}_z}{60.49}\right] \cdot (.37\mathbf{a}_x + .56\mathbf{a}_y - .74\mathbf{a}_z) = -0.09$$

Therefore $\theta_2 = \cos^{-1}(-.09) = \underline{95^\circ}$.

9.28. For values of B below the knee on the magnetization curve for silicon steel, approximate the curve by a straight line with $\mu = 5$ mH/m. The core shown in Fig. 9.17 has areas of 1.6 cm^2 and lengths of 10 cm in each outer leg, and an area of 2.5 cm^2 and a length of 3 cm in the central leg. A coil of 1200 turns carrying 12 mA is placed around the central leg. Find B in the:

a) center leg: We use $mmf = \Phi R$, where, in the central leg,

$$R_c = \frac{L_{in}}{\mu A_{in}} = \frac{3 \times 10^{-2}}{(5 \times 10^{-3})(2.5 \times 10^{-4})} = 2.4 \times 10^4 \text{ H}$$

In each outer leg, the reluctance is

$$R_o = \frac{L_{out}}{\mu A_{out}} = \frac{10 \times 10^{-2}}{(5 \times 10^{-3})(1.6 \times 10^{-4})} = 1.25 \times 10^5 \text{ H}$$

The magnetic circuit is formed by the center leg in series with the parallel combination of the two outer legs. The total reluctance seen at the coil location is $R_T = R_c + (1/2)R_o = 8.65 \times 10^4$ H. We now have

$$\Phi = \frac{mmf}{R_T} = \frac{14.4}{8.65 \times 10^4} = 1.66 \times 10^{-4} \text{ Wb}$$

The flux density in the center leg is now

$$B = \frac{\Phi}{A} = \frac{1.66 \times 10^{-4}}{2.5 \times 10^{-4}} = \underline{0.666 \text{ T}}$$

b) center leg, if a 0.3-mm air gap is present in the center leg: The air gap reluctance adds to the total reluctance already calculated, where

$$R_{air} = \frac{0.3 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 9.55 \times 10^5 \text{ H}$$

Now the total reluctance is $R_{net} = R_T + R_{air} = 8.56 \times 10^4 + 9.55 \times 10^5 = 1.04 \times 10^6$. The flux in the center leg is now

$$\Phi = \frac{14.4}{1.04 \times 10^6} = 1.38 \times 10^{-5} \text{ Wb}$$

and

$$B = \frac{1.38 \times 10^{-5}}{2.5 \times 10^{-4}} = \underline{55.3 \text{ mT}}$$

9.29. In Problem 9.28, the linear approximation suggested in the statement of the problem leads to a flux density of 0.666 T in the center leg. Using this value of B and the magnetization curve for silicon steel, what current is required in the 1200-turn coil? With $B = 0.666$ T, we read $H_{in} \doteq 120$ A · t/m in Fig. 9.11. The flux in the center leg is $\Phi = 0.666(2.5 \times 10^{-4}) = 1.66 \times 10^{-4}$ Wb. This divides equally in the two outer legs, so that the flux density in each outer leg is

$$B_{out} = \left(\frac{1}{2}\right) \frac{1.66 \times 10^{-4}}{1.6 \times 10^{-4}} = 0.52 \text{ Wb/m}^2$$

Using Fig. 9.11 with this result, we find $H_{out} \doteq 90$ A · t/m We now use

$$\oint \mathbf{H} \cdot d\mathbf{L} = NI$$

to find

$$I = \frac{1}{N}\left(H_{in}L_{in} + H_{out}L_{out}\right) = \frac{(120)(3 \times 10^{-2}) + (90)(10 \times 10^{-2})}{1200} = \underline{10.5 \text{ mA}}$$

9.30. A toroidal core has a circular cross section of 4 cm^2 area. The mean radius of the toroid is 6 cm. The core is composed of two semi-circular segments, one of silicon steel and the other of a linear material with $\mu_R = 200$. There is a 4mm air gap at each of the two joints, and the core is wrapped by a 4000-turn coil carrying a dc current I_1.

a) Find I_1 if the flux density in the core is 1.2 T: I will use the reluctance method here. Reluctances of the steel and linear materials are respectively,

$$R_s = \frac{\pi(6 \times 10^{-2})}{(3.0 \times 10^{-3})(4 \times 10^{-4})} = 1.57 \times 10^5 \text{ H}^{-1}$$

$$R_l = \frac{\pi(6 \times 10^{-2})}{(200)(4\pi \times 10^{-7})(4 \times 10^{-4})} = 1.88 \times 10^6 \text{ H}^{-1}$$

where μ_s is found from Fig. 9.11, using $B = 1.2$, from which $H = 400$, and so $B/H = 3.0$ mH/m. The reluctance of each gap is now

$$R_g = \frac{0.4 \times 10^{-3}}{(4\pi \times 10^{-7})(4 \times 10^{-4})} = 7.96 \times 10^5 \text{ H}^{-1}$$

We now construct

$$NI_1 = \Phi R = 1.2(4 \times 10^{-4})\left[R_s + R_l + 2R_g\right] = 1.74 \times 10^3$$

Thus $I_1 = (1.74 \times 10^3)/4000 = \underline{435 \text{ mA}}$.

9.30b. Find the flux density in the core if $I_1 = 0.3$ A: We are not sure what to use for the permittivity of steel in this case, so we use the iterative approach. Since the current is down from the value obtained in part a, we can try $B = 1.0$ T and see what happens. From Fig. 9.11, we find $H = 200$ A/m. Then, in the linear material,

$$H_l = \frac{1.0}{200(4\pi \times 10^{-7})} = 3.98 \times 10^3 \text{ A/m}$$

and in each gap,

$$H_g = \frac{1.0}{4\pi \times 10^{-7}} = 7.96 \times 10^5 \text{ A/m}$$

Now Ampere's circuital law around the toroid becomes

$$NI_1 = \pi(.06)(200 + 3.98 \times 10^3) + 2(7.96 \times 10^5)(4 \times 10^{-4}) = 1.42 \times 10^3 \text{ A}-\text{t}$$

Then $I_1 = (1.42 \times 10^3)/4000 = .356$ A. This is still larger than the given value of .3A, so we can extrapolate down to find a better value for B:

$$B = 1.0 - (1.2 - 1.0)\left[\frac{.356 - .300}{.435 - .356}\right] = \underline{0.86 \text{ T}}$$

Using this value in the procedure above to evaluate Ampere's circuital law leads to a value of I_1 of 0.306 A. The result of 0.86 T for B is probably good enough for this problem, considering the limited resolution of Fig. 9.11.

9.31. A toroid is constructed of a magnetic material having a cross-sectional area of 2.5 cm^2 and an effective length of 8 cm. There is also a short air gap 0.25 mm length and an effective area of 2.8 cm^2. An mmf of 200 A $\cdot$ t is applied to the magnetic circuit. Calculate the total flux in the toroid if:

a) the magnetic material is assumed to have infinite permeability: In this case the core reluctance, $R_c = l/(\mu A)$, is zero, leaving only the gap reluctance. This is

$$R_g = \frac{d}{\mu_0 A_g} = \frac{0.25 \times 10^{-3}}{(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 7.1 \times 10^5 \text{ H}$$

Now

$$\Phi = \frac{mmf}{\mathcal{R}_g} = \frac{200}{7.1 \times 10^5} = \underline{2.8 \times 10^{-4} \text{ Wb}}$$

b) the magnetic material is assumed to be linear with $\mu_R = 1000$: Now the core reluctance is no longer zero, but

$$R_c = \frac{8 \times 10^{-2}}{(1000)(4\pi \times 10^{-7})(2.5 \times 10^{-4})} = 2.6 \times 10^5 \text{ H}$$

The flux is then

$$\Phi = \frac{mmf}{R_c + R_g} = \frac{200}{9.7 \times 10^5} = \underline{2.1 \times 10^{-4} \text{ Wb}}$$

c) the magnetic material is silicon steel: In this case we use the magnetization curve, Fig. 9.11, and employ an iterative process to arrive at the final answer. We can begin with the value of Φ found in part a, assuming infinite permeability: $\Phi^{(1)} = 2.8 \times 10^{-4}$ Wb. The flux density in the core is then $B_c^{(1)} = (2.8 \times 10^{-4})/(2.5 \times 10^{-4}) = 1.1$ Wb/m^2. From Fig. 9.11, this corresponds to

magnetic field strength $H_c^{(1)} \doteq 270$ A/m. We check this by applying Ampere's circuital law to the magnetic circuit:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(1)} L_c + H_g^{(1)} d$$

where $H_c^{(1)} L_c = (270)(8 \times 10^{-2}) = 22$, and where $H_g^{(1)} d = \Phi^{(1)} \cdot_g = (2.8 \times 10^{-4})(7.1 \times 10^5) = 199$. But we require that

$$\oint \mathbf{H} \cdot d\mathbf{L} = 200 \text{ A} \cdot \text{t}$$

whereas the actual result in this first calculation is $199 + 22 = 221$, which is too high. So, for a second trial, we reduce B to $B_c^{(2)} = 1$ Wb/m^2. This yields $H_c^{(2)} = 200$ A/m from Fig. 9.11, and thus $\Phi^{(2)} = 2.5 \times 10^{-4}$ Wb. Now

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_c^{(2)} L_c + \Phi^{(2)} R_g = 200(8 \times 10^{-2}) + (2.5 \times 10^{-4})(7.1 \times 10^5) = 194$$

This is less than 200, meaning that the actual flux is slightly higher than 2.5×10^{-4} Wb. I will leave the answer at that, considering the lack of fine resolution in Fig. 9.11.

9.32. Determine the total energy stored in a spherical region 1cm in radius, centered at the origin in free space, in the uniform field:

a) $\mathbf{H}_1 = -600\mathbf{a}_y$ A/m: First we find the energy density:

$$w_{m1} = \frac{1}{2}\mathbf{B}_1 \cdot \mathbf{H}_1 = \frac{1}{2}\mu_0 H_1^2 = \frac{1}{2}(4\pi \times 10^{-7})(600)^2 = 0.226 \text{ J/m}^3$$

The energy within the sphere is then

$$W_{m1} = w_{m1}\left(\frac{4}{3}\pi a^3\right) = 0.226\left(\frac{4}{3}\pi \times 10^{-6}\right) = \underline{0.947\ \mu\text{J}}$$

b) $\mathbf{H}_2 = 600\mathbf{a}_x + 1200\mathbf{a}_y$ A/m: In this case the energy density is

$$w_{m2} = \frac{1}{2}\mu_0\left[(600)^2 + (1200)^2\right] = \frac{5}{2}\mu_0(600)^2$$

or five times the energy density that was found in part a. Therefore, the stored energy in this field is five times the amount in part a, or $W_{m2} = \underline{4.74\ \mu\text{J}}$.

c) $\mathbf{H}_3 = -600\mathbf{a}_x + 1200\mathbf{a}_y$. This field differs from $\mathbf{H}_2$ only by the negative x component, which is a non-issue since the component is squared when finding the energy density. Therefore, the stored energy will be the same as that in part b, or $W_{m3} = \underline{4.74\ \mu\text{J}}$.

d) $\mathbf{H}_4 = \mathbf{H}_2 + \mathbf{H}_3$, or $2400\mathbf{a}_y$ A/m: The energy density is now $w_{m4} = (1/2)\mu_0(2400)^2 = (1/2)\mu_0(16)(600)^2$ J/m^3, which is sixteen times the energy density in part a. The stored energy is therefore sixteen times that result, or $W_{m4} = 16(0.947) = \underline{15.2\ \mu\text{J}}$.

e) $1000\mathbf{a}_x$ A/m$+0.001\mathbf{a}_x$ T: The energy density is $w_{m5} = (1/2)\mu_0[1000 + .001/\mu_0]^2 = 2.03$ J/m^3. Then $W_{m5} = 2.03[(4/3)\pi \times 10^{-6}] = \underline{8.49\ \mu\text{J}}$.

9.33. A toroidal core has a square cross section, 2.5 cm $< \rho <$ 3.5 cm, -0.5 cm $< z <$ 0.5 cm. The upper half of the toroid, $0 < z <$ 0.5 cm, is constructed of a linear material for which $\mu_R = 10$, while the lower half, -0.5 cm $< z < 0$, has $\mu_R = 20$. An mmf of 150 A $\cdot$ t establishes a flux in the $\mathbf{a}_\phi$ direction. For $z > 0$, find:

a) $H_\phi(\rho)$: Ampere's circuital law gives:

$$2\pi\rho H_\phi = NI = 150 \;\Rightarrow\; H_\phi = \frac{150}{2\pi\rho} = \underline{23.9/\rho \text{ A/m}}$$

b) $B_\phi(\rho)$: We use $B_\phi = \mu_R\mu_0 H_\phi = (10)(4\pi \times 10^{-7})(23.9/\rho) = \underline{3.0 \times 10^{-4}/\rho \text{ Wb/m}^2}$.

c) $\Phi_{z>0}$: This will be

$$\Phi_{z>0} = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^{.005}\int_{.025}^{.035} \frac{3.0\times10^{-4}}{\rho} d\rho dz = (.005)(3.0\times10^{-4})\ln\left(\frac{.035}{.025}\right)$$
$$= \underline{5.0 \times 10^{-7} \text{ Wb}}$$

d) Repeat for $z < 0$: First, the magnetic field strength will be the same as in part a, since the calculation is material-independent. Thus $H_\phi = \underline{23.9/\rho \text{ A/m}}$. Next, B_ϕ is modified only by the new permeability, which is twice the value used in part a: Thus $B_\phi = \underline{6.0 \times 10^{-4}/\rho \text{ Wb/m}^2}$. Finally, since B_ϕ is twice that of part a, the flux will be increased by the same factor, since the area of integration for $z < 0$ is the same. Thus $\Phi_{z<0} = \underline{1.0 \times 10^{-6} \text{ Wb}}$.

e) Find Φ_{total}: This will be the sum of the values found for $z < 0$ and $z > 0$, or $\Phi_{\text{total}} = \underline{1.5 \times 10^{-6} \text{ Wb}}$.

9.34. Three planar current sheets are located in free space as follows: $-100\mathbf{a}_x$ A/m^2 at $z = -1$, $200\mathbf{a}_x$ A/m^2 at $z = 0$, $-100\mathbf{a}_x$ A/m^2 at $z = 1$. Let $w_H = (1/2)\mathbf{B}\cdot\mathbf{H}$ J/m^3, and find w_H for all z: Using the fact that the field on either side of a current sheet is given in magnitude by $H = K/2$, we find, in A/m:

$$\mathbf{H}(z > 1) = (1/2)(-200 + 100 + 100)\mathbf{a}_y = 0$$

$$\mathbf{H}(0 < z < 1) = (1/2)(-200 - 100 + 100)\mathbf{a}_y = -100\mathbf{a}_y$$

$$\mathbf{H}(-1 < z < 0) = (1/2)(200 - 100 + 100)\mathbf{a}_y = 100\mathbf{a}_y$$

and

$$\mathbf{H}(z < -1) = (1/2)(200 - 100 - 100)\mathbf{a}_y = 0$$

The energy densities are then

$$w_H(z > 1) = w_H(z < -1) = \underline{0}$$

$$w_H(0 < z < 1) = w_H(-1 < z < 0) = (1/2)\mu_0(100)^2 = \underline{6.28 \text{ mJ/m}^2}$$

9.35. The cones $\theta = 21^\circ$ and $\theta = 159^\circ$ are conducting surfaces and carry total currents of 40 A, as shown in Fig. 9.18. The currents return on a spherical conducting surface of 0.25 m radius.

a) Find $\mathbf{H}$ in the region $0 < r < 0.25$, $21^\circ < \theta < 159^\circ$, $0 < \phi < 2\pi$: We can apply Ampere's circuital law and take advantage of symmetry. We expect to see $\mathbf{H}$ in the $\mathbf{a}_\phi$ direction and it would be constant at a given distance from the z axis. We thus perform the line integral of $\mathbf{H}$ over a circle, centered on the z axis, and parallel to the xy plane:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \mathbf{a}_\phi \cdot r \sin\theta \mathbf{a}_\phi \, d\phi = I_{encl.} = 40 \text{ A}$$

Assuming that H_ϕ is constant over the integration path, we take it outside the integral and solve:

$$H_\phi = \frac{40}{2\pi r \sin\theta} \quad \Rightarrow \quad \mathbf{H} = \underline{\frac{20}{\pi r \sin\theta} \mathbf{a}_\phi \text{ A/m}}$$

b) How much energy is stored in this region? This will be

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 = \int_0^{2\pi} \int_{21^\circ}^{159^\circ} \int_0^{.25} \frac{200\mu_0}{\pi^2 r^2 \sin^2\theta} r^2 \sin\theta \, dr \, d\theta \, d\phi = \frac{100\mu_0}{\pi} \int_{21^\circ}^{159^\circ} \frac{d\theta}{\sin\theta}$$
$$= \frac{100\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \underline{1.35 \times 10^{-4} \text{ J}}$$

9.36. A filament carrying current I in the $\mathbf{a}_z$ direction lies on the z axis, and cylindrical current sheets of $5\mathbf{a}_z$ A/m and $-2\mathbf{a}_z$ A/m are located at $\rho = 3$ and $\rho = 10$, respectively.

a) Find I if $\mathbf{H} = 0$ for $\rho > 10$. Ampere's circuital law says, for $\rho > 10$:

$$2\pi\rho H = 2\pi(3)(5) - 2\pi(10)(2) + I = 0$$

from which $I = 2\pi(10)(3) - 2\pi(3)(5) = \underline{10\pi \text{ A}}$.

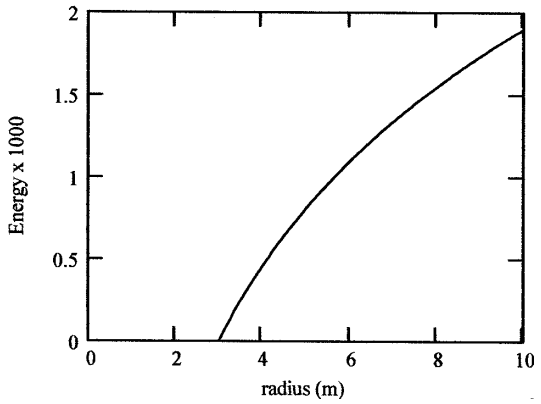
b) Using this value of I, calculate $\mathbf{H}$ for all ρ, $3 < \rho < 10$: Again, using Ampere's circuital law, we find

$$\mathbf{H}(3 < \rho < 10) = \frac{1}{2\pi\rho}[10\pi + 2\pi(3)(5)]\mathbf{a}_\phi = \underline{\frac{20}{\rho}\mathbf{a}_\phi \text{ A/m}}$$

c) Calculate and plot W_H versus ρ_0, where W_H is the total energy stored within the volume $0 < z < 1$, $0 < \phi < 2\pi$, $3 < \rho < \rho_0$: First the energy density will be $w_H = (1/2)\mu_0 H^2 = 200\mu_0/\rho^2 \text{ J/m}^3$. Then the energy is

$$W_H = \int_0^1 \int_0^{2\pi} \int_3^{\rho_0} \frac{200\mu_0}{\rho^2} \rho \, d\rho \, d\phi \, dz = 400\pi\mu_0 \ln\left(\frac{\rho_0}{3}\right) = \underline{(1.58 \times 10^{-3}) \ln\left(\frac{\rho_0}{3}\right) \text{ J}}$$

9.36c. (continued) A plot of the energy as a function of ρ_0 is shown below.



9.37. Find the inductance of the cone-sphere configuration described in Problem 9.35 and Fig. 9.18. The inductance is that offered at the origin between the vertices of the cone: From Problem 9.35, the magnetic flux density is $B_\phi = 20\mu_0/(\pi r \sin\theta)$. We integrate this over the crossectional area defined by $0 < r < 0.25$ and $21^\circ < \theta < 159^\circ$, to find the total flux:

$$\Phi = \int_{21^\circ}^{159^\circ} \int_0^{0.25} \frac{20\mu_0}{\pi r \sin\theta}\, r\, dr\, d\theta = \frac{5\mu_0}{\pi} \ln\left[\frac{\tan(159/2)}{\tan(21/2)}\right] = \frac{5\mu_0}{\pi}(3.37) = 6.74 \times 10^{-6} \text{ Wb}$$

Now $L = \Phi/I = 6.74 \times 10^{-6}/40 = \underline{0.17\ \mu\text{H}}$.
Second method: Use the energy computation of Problem 9.35, and write

$$L = \frac{2W_H}{I^2} = \frac{2(1.35 \times 10^{-4})}{(40)^2} = \underline{0.17\ \mu\text{H}}$$

9.38. A toroidal core has a rectangular cross section defined by the surfaces $\rho = 2$ cm, $\rho = 3$ cm, $z = 4$ cm, and $z = 4.5$ cm. The core material has a relative permeability of 80. If the core is wound with a coil containing 8000 turns of wire, find its inductance: First we apply Ampere's circuital law to a circular loop of radius ρ in the interior of the toroid, and in the $\mathbf{a}_\phi$ direction.

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = NI \quad \Rightarrow \quad H_\phi = \frac{NI}{2\pi\rho}$$

The flux in the toroid is then the integral over the cross section of **B**:

$$\Phi = \int\int \mathbf{B} \cdot d\mathbf{L} = \int_{.04}^{.045} \int_{.02}^{.03} \frac{\mu_R \mu_0 NI}{2\pi\rho}\, d\rho\, dz = (.005)\frac{\mu_R \mu_0 NI}{2\pi} \ln\left(\frac{.03}{.02}\right)$$

The flux linkage is then given by $N\Phi$, and the inductance is

$$L = \frac{N\Phi}{I} = \frac{(.005)(80)(4\pi \times 10^{-7})(8000)^2}{2\pi} \ln(1.5) = \underline{2.08 \text{ H}}$$

9.39. Conducting planes in air at $z = 0$ and $z = d$ carry surface currents of $\pm K_0 \mathbf{a}_x$ A/m.

a) Find the energy stored in the magnetic field per unit length ($0 < x < 1$) in a width w ($0 < y < w$): First, assuming current flows in the $+\mathbf{a}_x$ direction in the sheet at $z = d$, and in $-\mathbf{a}_x$ in the sheet at $z = 0$, we find that both currents together yield $\mathbf{H} = K_0 \mathbf{a}_y$ for $0 < z < d$ and zero elsewhere. The stored energy within the specified volume will be:

$$W_H = \int_v \frac{1}{2}\mu_0 H^2 dv = \int_0^d \int_0^w \int_0^1 \frac{1}{2}\mu_0 K_0^2 \, dx \, dy \, dz = \underline{\frac{1}{2} w d \mu_0 K_0^2 \text{ J/m}}$$

b) Calculate the inductance per unit length of this transmission line from $W_H = (1/2)LI^2$, where I is the total current in a width w in either conductor: We have $I = wK_0$, and so

$$L = \frac{2}{I^2}\frac{wd}{2}\mu_0 K_0^2 = \frac{2}{w^2 K_0^2}\frac{dw}{2}\mu_0 K_0^2 = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

c) Calculate the total flux passing through the rectangle $0 < x < 1$, $0 < z < d$, in the plane $y = 0$, and from this result again find the inductance per unit length:

$$\Phi = \int_0^d \int_0^1 \mu_0 H \mathbf{a}_y \cdot \mathbf{a}_y \, dx \, dz = \int_0^d \int_0^1 \mu_0 K_0 dx \, dy = \mu_0 d K_0$$

Then

$$L = \frac{\Phi}{I} = \frac{\mu_0 d K_0}{w K_0} = \underline{\frac{\mu_0 d}{w} \text{ H/m}}$$

9.40. A coaxial cable has conductor dimensions of 1 and 5 mm. The region between conductors is air for $0 < \phi < \pi/2$ and $\pi < \phi < 3\pi/2$, and a non-conducting material having $\mu_R = 8$ for $\pi/2 < \phi < \pi$ and $3\pi/2 < \phi < 2\pi$. Find the inductance per meter length: The interfaces between media all occur along radial lines, normal to the direction of **B** and **H** in the coax line. **B** is therefore continuous (and constant at constant radius) around a circular loop centered on the z axis. Ampere's circuital law can thus be written in this form:

$$\oint \mathbf{H} \cdot d\mathbf{L} = \frac{B}{\mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_R \mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_0}\left(\frac{\pi}{2}\rho\right) + \frac{B}{\mu_R \mu_0}\left(\frac{\pi}{2}\rho\right) = \frac{\pi \rho B}{\mu_R \mu_0}(\mu_R + 1) = I$$

and so

$$\mathbf{B} = \frac{\mu_R \mu_0 I}{\pi \rho (1 + \mu_R)} \mathbf{a}_\phi$$

The flux in the line per meter length in z is now

$$\Phi = \int_0^1 \int_{.001}^{.005} \frac{\mu_R \mu_0 I}{\pi \rho (1 + \mu_R)} \, d\rho \, dz = \frac{\mu_R \mu_0 I}{\pi (1 + \mu_R)} \ln(5)$$

And the inductance per unit length is:

$$L = \frac{\Phi}{I} = \frac{\mu_R \mu_0}{\pi (1 + \mu_R)} \ln(5) = \frac{8(4\pi \times 10^{-7})}{\pi (9)} \ln(5) = \underline{572 \text{ nH/m}}$$

9.41. A rectangular coil is composed of 150 turns of a filamentary conductor. Find the mutual inductance in free space between this coil and an infinite straight filament on the z axis if the four corners of the coil are located at

a) (0,1,0), (0,3,0), (0,3,1), and (0,1,1): In this case the coil lies in the yz plane. If we assume that the filament current is in the $+\mathbf{a}_z$ direction, then the $\mathbf{B}$ field from the filament penetrates the coil in the $-\mathbf{a}_x$ direction (normal to the loop plane). The flux through the loop will thus be

$$\Phi = \int_0^1 \int_1^3 \frac{-\mu_0 I}{2\pi y}\mathbf{a}_x \cdot (-\mathbf{a}_x)\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln 3$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = \frac{150\mu_0}{2\pi} \ln 3 = \underline{33\ \mu\text{H}}$$

b) (1,1,0), (1,3,0), (1,3,1), and (1,1,1): Now the coil lies in the $x = 1$ plane, and the field from the filament penetrates in a direction that is not normal to the plane of the coil. We write the $\mathbf{B}$ field from the filament at the coil location as

$$\mathbf{B} = \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}}$$

The flux through the coil is now

$$\Phi = \int_0^1 \int_1^3 \frac{\mu_0 I \mathbf{a}_\phi}{2\pi\sqrt{y^2+1}} \cdot (-\mathbf{a}_x)\, dy\, dz = \int_0^1 \int_1^3 \frac{\mu_0 I \sin\phi}{2\pi\sqrt{y^2+1}}\, dy\, dz$$
$$= \int_0^1 \int_1^3 \frac{\mu_0 I y}{2\pi(y^2+1)}\, dy\, dz = \frac{\mu_0 I}{2\pi} \ln(y^2+1)\Big|_1^3 = (1.6 \times 10^{-7})I$$

The mutual inductance is then

$$M = \frac{N\Phi}{I} = (150)(1.6 \times 10^{-7}) = \underline{24\ \mu\text{H}}$$

9.42. Find the mutual inductance of this conductor system in free space:

a) the solenoid of Fig. 8.11b and a square filamentary loop of side length b coaxially centered inside the solenoid, if $a > b/\sqrt{2}$; With the given side length, the loop lies entirely inside the solenoid, and so is linked over its entire cross section by the solenoid field. The latter is given by $\mathbf{B} = \mu_0 N I/d\ \mathbf{a}_z$ T. The flux through the loop area is now $\Phi = Bb^2$, and the mutual inductance is $M = \Phi/I = \underline{\mu_0 N b^2/d\ \text{H}}$.

b) a cylindrical conducting shell of a radius a, axis on the z axis, and a filament at $x = 0$, $y = d$, *and where* $d > a$ (omitted from problem statement); The $\mathbf{B}$ field from the cylinder is $\mathbf{B} = (\mu_0 I)/(2\pi\rho)\ \mathbf{a}_\phi$ for $\rho > a$, and so the flux per unit length between cylinder and wire is

$$\Phi = \int_0^1 \int_a^d \frac{\mu_0 I}{2\pi\rho}\, d\rho\, dz = \frac{\mu_0 I}{2\pi} \ln\left(\frac{d}{a}\right)\ \text{Wb}$$

Finally the mutual inductance is $M = \Phi/I = \underline{\mu_0/2\pi\ \ln(d/a)\ \text{H}}$.

9.43. a) Use energy relationships to show that the internal inductance of a nonmagnetic cylindrical wire of radius a carrying a uniformly-distributed current I is $\mu_0/(8\pi)$ H/m. We first find the magnetic field inside the conductor, then calculate the energy stored there. From Ampere's circuital law:

$$2\pi\rho H_\phi = \frac{\pi\rho^2}{\pi a^2} I \quad \Rightarrow \quad H_\phi = \frac{I\rho}{2\pi a^2} \text{ A/m}$$

Now

$$W_H = \int_v \frac{1}{2}\mu_0 H_\phi^2 \, dv = \int_0^1 \int_0^{2\pi} \int_0^a \frac{\mu_0 I^2 \rho^2}{8\pi^2 a^4} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{16\pi} \text{ J/m}$$

Now, with $W_H = (1/2)LI^2$, we find $L_{int} = \mu_0/(8\pi)$ as expected.

b) Find the internal inductance if the portion of the conductor for which $\rho < c < a$ is removed: The hollowed-out conductor still carries current I, so Ampere's circuital law now reads:

$$2\pi\rho H_\phi = \frac{\pi(\rho^2 - c^2)}{\pi(a^2 - c^2)} \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{\rho^2 - c^2}{a^2 - c^2}\right] \text{ A/m}$$

and the energy is now

$$W_H = \int_0^1 \int_0^{2\pi} \int_c^a \frac{\mu_0 I^2 (\rho^2 - c^2)^2}{8\pi^2 \rho^2 (a^2 - c^2)^2} \, \rho \, d\rho \, d\phi \, dz = \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2} \int_c^a \left[\rho^3 - 2c^2\rho + \frac{C^4}{\rho}\right] d\rho$$

$$= \frac{\mu_0 I^2}{4\pi(a^2 - c^2)^2}\left[\frac{1}{4}(a^4 - c^4) - c^2(a^2 - c^2) + c^4 \ln\left(\frac{a}{c}\right)\right] \text{ J/m}$$

The internal inductance is then

$$\underline{L_{int} = \frac{2W_H}{I^2} = \frac{\mu_0}{8\pi}\left[\frac{a^4 - 4a^2c^2 + 3c^4 + 4c^4 \ln(a/c)}{(a^2 - c^2)^2}\right] \text{ H/m}}$$

CHAPTER 10

10.1. In Fig. 10.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find:

a) $V_{ab}(t)$: Since B is constant over the loop area, the flux is $\Phi = \pi(0.15)^2 B = 1.41\times 10^{-2}\cos 120\pi t$ Wb. Now, $emf = V_{ba}(t) = -d\Phi/dt = (120\pi)(1.41\times 10^{-2})\sin 120\pi t$. Then $V_{ab}(t) = -V_{ba}(t) = \underline{-5.33\sin 120\pi t \text{ V}}$.

b) $I(t) = V_{ba}(t)/R = 5.33\sin(120\pi t)/250 = \underline{21.3\sin(120\pi t)\text{ mA}}$

10.2. Given the time-varying magnetic field, $\mathbf{B} = (0.5\mathbf{a}_x + 0.6\mathbf{a}_y - 0.3\mathbf{a}_z)\cos 5000t$ T, and a square filamentary loop with its corners at (2,3,0), (2,-3,0), (-2,3,0), and (-2,-3,0), find the time-varying current flowing in the general $\mathbf{a}_\phi$ direction if the total loop resistance is 400 kΩ: We write

$$emf = \oint \mathbf{E}\cdot d\mathbf{L} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int\int_{\text{loop area}} \mathbf{B}\cdot\mathbf{a}_z\, da = \frac{d}{dt}(0.3)(4)(6)\cos 5000t$$

where the loop normal is chosen as positive $\mathbf{a}_z$, so that the path integral for $\mathbf{E}$ is taken around the positive $\mathbf{a}_\phi$ direction. Taking the derivative, we find

$$emf = -7.2(5000)\sin 5000t \text{ so that } I = \frac{emf}{R} = \frac{-36000\sin 5000t}{400\times 10^3} = \underline{-90\sin 5000t \text{ mA}}$$

10.3. Given $\mathbf{H} = 300\,\mathbf{a}_z\cos(3\times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at

a) (0,0,0), (1,0,0), (1,1,0), and (0,1,0): The magnetic flux will be:

$$\Phi = \int_0^1\int_0^1 300\mu_0\cos(3\times 10^8 t - y)\,dx\,dy = 300\mu_0\sin(3\times 10^8 t - y)|_0^1$$
$$= 300\mu_0\left[\sin(3\times 10^8 t - 1) - \sin(3\times 10^8 t)\right] \text{ Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = -300(3\times 10^8)(4\pi\times 10^{-7})\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right]$$
$$= \underline{-1.13\times 10^5\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right] \text{ V}}$$

b) corners at (0,0,0), (2π,0,0), (2π,2π,0), (0,2π,0): In this case, the flux is

$$\Phi = 2\pi\times 300\mu_0\sin(3\times 10^8 t - y)|_0^{2\pi} = 0$$

The emf is therefore $\underline{0}$.

10.4. Conductor surfaces are located at $\rho = 1$cm and $\rho = 2$cm in free space. The volume $1\,\text{cm} < \rho < 2\,\text{cm}$ contains the fields $H_\phi = (2/\rho)\cos(6\times10^8\pi t - 2\pi z)$ A/m and $E_\rho = (240\pi/\rho)\cos(6\times10^8\pi t - 2\pi z)$ V/m.

a) Show that these two fields satisfy Eq. (6), Sec. 10.1: Have

$$\nabla\times\mathbf{E} = \frac{\partial E_\rho}{\partial z}\,\mathbf{a}_\phi = \frac{2\pi(240\pi)}{\rho}\sin(6\times10^8\pi t - 2\pi z)\,\mathbf{a}_\phi = \frac{480\pi^2}{\rho}\sin(6\times10^8\pi t - 2\pi z)\mathbf{a}_\phi$$

Then

$$\begin{aligned}-\frac{\partial\mathbf{B}}{\partial t} &= \frac{2\mu_0(6\times10^8)\pi}{\rho}\sin(6\times10^8\pi t - 2\pi z)\,\mathbf{a}_\phi \\ &= \frac{(8\pi\times10^{-7})(6\times10^8)\pi}{\rho}\sin(6\times10^8\pi t - 2\pi z) = \frac{480\pi^2}{\rho}\sin(6\times10^8\pi t - 2\pi z)\,\mathbf{a}_\phi\end{aligned}$$

b) Evaluate both integrals in Eq. (4) for the planar surface defined by $\phi = 0$, $1\text{cm} < \rho < 2\text{cm}$, $0 < z < 0.1$m (*note misprint in problem statement*), and its perimeter, and show that the same results are obtained: we take the normal to the surface as positive $\mathbf{a}_\phi$, so the the loop surrounding the surface (by the right hand rule) is in the negative $\mathbf{a}_\rho$ direction at $z = 0$, and is in the positive $\mathbf{a}_\rho$ direction at $z = 0.1$. Taking the left hand side first, we find

$$\begin{aligned}\oint\mathbf{E}\cdot d\mathbf{L} &= \int_{.02}^{.01}\frac{240\pi}{\rho}\cos(6\times10^8\pi t)\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,d\rho \\ &\quad+\int_{.01}^{.02}\frac{240\pi}{\rho}\cos(6\times10^8\pi t - 2\pi(0.1))\,\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,d\rho \\ &= 240\pi\cos(6\times10^8\pi t)\,\ln\left(\frac{1}{2}\right) + 240\pi\cos(6\times10^8\pi t - 0.2\pi)\,\ln\left(\frac{2}{1}\right) \\ &= 240(\ln 2)\left[\cos(6\times10^8\pi t - 0.2\pi) - \cos(6\times10^8\pi t)\right]\end{aligned}$$

Now for the right hand side. First,

$$\begin{aligned}\int\mathbf{B}\cdot d\mathbf{S} &= \int_0^{0.1}\int_{.01}^{.02}\frac{8\pi\times10^{-7}}{\rho}\cos(6\times10^8\pi t - 2\pi z)\,\mathbf{a}_\phi\cdot\mathbf{a}_\phi\,d\rho\,dz \\ &= \int_0^{0.1}(8\pi\times10^{-7})\,\ln 2\,\cos(6\times10^8\pi t - 2\pi z)\,dz \\ &= -4\times10^{-7}\,\ln 2\left[\sin(6\times10^8\pi t - 0.2\pi) - \sin(6\times10^8\pi t)\right]\end{aligned}$$

Then

$$-\frac{d}{dt}\int\mathbf{B}\cdot d\mathbf{S} = 240\pi(\ln 2)\left[\cos(6\times10^8\pi t - 0.2\pi) - \cos(6\times10^8\pi t)\right]\quad\text{(check)}$$

10.5. The location of the sliding bar in Fig. 10.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at:

a) $t = 0.4$ s: The flux through the loop will be

$$\Phi = \int_0^{0.2}\int_0^x 0.8(x')^2\,dx'\,dy = \frac{0.16}{3}x^3 = \frac{0.16}{3}(5t + 2t^3)^3\ \text{Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = \frac{0.16}{3}(3)(5t+2t^3)^2(5+6t^2) = -(0.16)[5(.4)+2(.4)^3]^2[5+6(.4)^2] = \underline{-4.32 \text{ V}}$$

b) $x = 0.6$ m: Have $0.6 = 5t + 2t^3$, from which we find $t = 0.1193$. Thus

$$\text{emf} = -(0.16)[5(.1193) + 2(.1193)^3]^2[5 + 6(.1193)^2] = \underline{-.293 \text{ V}}$$

10.6. A perfectly conducting filament containing a small 500-Ω resistor is formed into a square, as illustrated in Fig. 10.6. Find $I(t)$ if

a) $\mathbf{B} = 0.3\cos(120\pi t - 30^\circ)\,\mathbf{a}_z$ T: First the flux through the loop is evaluated, where the unit normal to the loop is $\mathbf{a}_z$. We find

$$\Phi = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.3)(0.5)^2 \cos(120\pi t - 30^\circ) \text{ Wb}$$

Then the current will be

$$I(t) = \frac{\text{emf}}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = \frac{(120\pi)(0.3)(0.25)}{500}\sin(120\pi t - 30^\circ) = \underline{57\sin(120\pi t - 30^\circ) \text{ mA}}$$

b) $\mathbf{B} = 0.4\cos[\pi(ct - y)]\,\mathbf{a}_z$ μT where $c = 3 \times 10^8$ m/s: Since the field varies with y, the flux is now

$$\Phi = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{S} = (0.5)(0.4)\int_0^{.5} \cos(\pi y - \pi ct)\,dy = \frac{0.2}{\pi}[\sin(\pi ct - \pi/2) - \sin(\pi ct)] \ \mu\text{Wb}$$

The current is then

$$\begin{aligned} I(t) &= \frac{\text{emf}}{R} = -\frac{1}{R}\frac{d\Phi}{dt} = \frac{-0.2c}{500}[\cos(\pi ct - \pi/2) - \cos(\pi ct)] \ \mu\text{A} \\ &= \frac{-0.2(3 \times 10^8)}{500}[\sin(\pi ct) - \cos(\pi ct)] \ \mu\text{A} = \underline{120\,[\cos(\pi ct) - \sin(\pi ct)] \text{ mA}} \end{aligned}$$

10.7. The rails in Fig. 10.7 each have a resistance of 2.2 Ω/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and

a) a 0.3 Ω resistor is present across the left end with the right end open-circuited: The flux in the left-hand closed loop is

$$\Phi_l = B \times \text{area} = (0.8)(0.2)(2 + 9t)$$

Then, $\text{emf}_l = -d\Phi_l/dt = -(0.16)(9) = -1.44$ V. With the bar in motion, the loop resistance is increasing with time, and is given by $R_l(t) = 0.3 + 2[2.2(2 + 9t)]$. The current is now

$$I_l(t) = \frac{\text{emf}_l}{R_l(t)} = \frac{-1.44}{\underline{9.1 + 39.6t}} \text{ A}$$

Note that the sign of the current indicates that it is flowing in the direction opposite that shown in the figure.

b) Repeat part a, but with a resistor of 0.3 Ω across each end: In this case, there will be a contribution to the current from the right loop, which is now closed. The flux in the right loop, whose area decreases with time, is

$$\Phi_r = (0.8)(0.2)[(16-2)-9t]$$

and $\text{emf}_r = -d\Phi_r/dt = (0.16)(9) = 1.44$ V. The resistance of the right loop is $R_r(t) = 0.3 + 2[2.2(14-9t)]$, and so the contribution to the current from the right loop will be

$$I_r(t) = \frac{-1.44}{61.9 - 39.6t} \text{ A}$$

The minus sign has been inserted because again the current must flow in the opposite direction as that indicated in the figure, with the flux decreasing with time. The total current is found by adding the part a result, or

$$I_T(t) = \underline{-1.44\left[\frac{1}{61.9-39.6t} + \frac{1}{9.1+39.6t}\right] \text{ A}}$$

10.8. Fig. 10.1 is modified to show that the rail separation is larger when y is larger. Specifically, let the separation $d = 0.2 + 0.02y$. Given a uniform velocity $v_y = 8$ m/s and a uniform magnetic flux density $B_z = 1.1$ T, find V_{12} as a function of time if the bar is located at $y = 0$ at $t = 0$: The flux through the loop as a function of y can be written as

$$\Phi = \int \mathbf{B}\cdot d\mathbf{S} = \int_0^y \int_0^{.2+.02y'} 1.1\, dx\, dy' = \int_0^y 1.1(.2+.02y')\, dy' = 0.22y(1+.05y)$$

Now, with $y = vt = 8t$, the above becomes $\Phi = 1.76t(1+.40t)$. Finally,

$$V_{12} = -\frac{d\Phi}{dt} = \underline{-1.76(1+.80t) \text{ V}}$$

10.9. A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at (0, 0, 0), (0.25, 0, 0), (0.25, 0.25, 0), and (0, 0.25, 0) at $t = 0$. The loop is moving with velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5\times10^8 t - 0.5x)\ \mu$T. Develop a function of time which expresses the ohmic power being delivered to the loop: First, since the field does not vary with y, the loop motion in the y direction does not produce any time-varying flux, and so this motion is immaterial. We can evaluate the flux at the original loop position to obtain:

$$\begin{aligned}\Phi(t) &= \int_0^{.25}\int_0^{.25} 8\times10^{-6}\cos(1.5\times10^8 t - 0.5x)\, dx\, dy \\ &= -(4\times10^{-6})\left[\sin(1.5\times10^8 t - 0.13x) - \sin(1.5\times10^8 t)\right] \text{ Wb}\end{aligned}$$

Now, $emf = V(t) = -d\Phi/dt = 6.0\times10^2\left[\cos(1.5\times10^8 t - 0.13x) - \cos(1.5\times10^8 t)\right]$, The total loop resistance is $R = 125(0.25+0.25+0.25+0.25) = 125\,\Omega$. Then the ohmic power is

$$P(t) = \frac{V^2(t)}{R} = \underline{2.9\times10^3\left[\cos(1.5\times10^8 t - 0.13x) - \cos(1.5\times10^8 t)\right] \text{ Watts}}$$

10.10a. Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$. First, $D = \epsilon E = \epsilon E_m \cos\omega t$. Then the displacement current density is $\partial D/\partial t = -\omega\epsilon E_m \sin\omega t$. Second, $J_c = \sigma E = \sigma E_m \cos\omega t$. Using these results we find $|J_c|/|J_d| = \sigma/\omega\epsilon$.

b. What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real? As before, find $D = \epsilon E = \epsilon E_m e^{-t/\tau}$, and so $J_d = \partial D/\partial t = -(\epsilon/\tau) E_m e^{-t/\tau}$. Also, $J_c = \sigma E_m e^{-t/\tau}$. Finally, $|J_c|/|J_d| = \underline{\sigma\tau/\epsilon}$.

10.11. Let the internal dimension of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho)\cos(10^5 t)\mathbf{a}_\rho$ V/m (note missing t in the argument of the cosine in the book), find:

a) **J**: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{\left(\frac{10}{\rho}\right)\cos(10^5 t)\mathbf{a}_\rho \text{ A/m}^2}$$

b) the total conduction current, I_c, through the capacitor: Have

$$I_c = \int\int \mathbf{J}\cdot d\mathbf{S} = 2\pi\rho l J = 20\pi l\cos(10^5 t) = \underline{8\pi\cos(10^5 t)\text{ A}}$$

c) the total displacement current, I_d, through the capacitor: First find

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon\mathbf{E}) = -\frac{(10^5)(10^{-11})(10^6)}{\rho}\sin(10^5 t)\mathbf{a}_\rho = -\frac{1}{\rho}\sin(10^5 t)\text{ A/m}$$

Now

$$I_d = 2\pi\rho l J_d = -2\pi l\sin(10^5 t) = \underline{-0.8\pi\sin(10^5 t)\text{ A}}$$

d) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor: This will be

$$\frac{|I_d|}{|I_c|} = \frac{0.8}{8} = \underline{0.1}$$

10.12. Given a coaxial transmission line with $b/a = e^{2.5}$, $\mu_R = \epsilon_R = 1$, and an electric field intensity $\mathbf{E} = (200/\rho)\cos(10^9 t - 3.336z)\,\mathbf{a}_\rho$ V/m, find:

a) V_{ab}, the voltage between the conductors, if it is known that electrostatic relationship $\mathbf{E} = -\nabla V$ is valid; We use

$$V_{ab} = -\int_b^a \frac{200}{\rho}\cos(10^9 t - 3.336z)\,d\rho = 200\ln\left(\frac{b}{a}\right)\cos(10^9 t - 3.336z)$$
$$= \underline{500\cos(10^9 t - 3.336z)\text{ V}}$$

b) the displacement current density;

$$\mathbf{J_d} = \frac{\partial\mathbf{D}}{\partial t} = \frac{-200\times 10^9\epsilon_0}{\rho}\sin(10^9 t - 3.336z)\mathbf{a}_\rho = \underline{-\frac{1.77}{\rho}\sin(10^9 t - 3.336z)\mathbf{a}_\rho \text{ A/m}^2}$$

10.13. Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_R = 5$, $\mu_R = 4$, and $\sigma = 0$. If $\mathbf{J}_d = 20\cos(1.5\times10^8 t - bx)\mathbf{a}_y\ \mu\text{A/m}^2$;

a) find $\mathbf{D}$ and $\mathbf{E}$: Since $\mathbf{J}_d = \partial\mathbf{D}/\partial t$, we write

$$\mathbf{D} = \int \mathbf{J}_d dt + C = \frac{20\times10^{-6}}{1.5\times10^8}\sin(1.5\times10^8 - bx)\mathbf{a}_y$$
$$= \underline{1.33\times10^{-13}\sin(1.5\times10^8 t - bx)\mathbf{a}_y\ \text{C/m}^2}$$

where the integration constant is set to zero (assuming no dc fields are present). Then

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1.33\times10^{-13}}{(5\times8.85\times10^{-12})}\sin(1.5\times10^8 t - bx)\mathbf{a}_y$$
$$= \underline{3.0\times10^{-3}\sin(1.5\times10^8 t - bx)\mathbf{a}_y\ \text{V/m}}$$

b) use the point form of Faraday's law and an integration with respect to time to find $\mathbf{B}$ and $\mathbf{H}$: In this case,

$$\nabla\times\mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = -b(3.0\times10^{-3})\cos(1.5\times10^8 t - bx)\mathbf{a}_z = -\frac{\partial\mathbf{B}}{\partial t}$$

Solve for $\mathbf{B}$ by integrating over time:

$$\mathbf{B} = \frac{b(3.0\times10^{-3})}{1.5\times10^8}\sin(1.5\times10^8 t - bx)\mathbf{a}_z = \underline{(2.0)b\times10^{-11}\sin(1.5\times10^8 t - bx)\mathbf{a}_z\ \text{T}}$$

Now

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{(2.0)b\times10^{-11}}{4\times4\pi\times10^{-7}}\sin(1.5\times10^8 t - bx)\mathbf{a}_z$$
$$= \underline{(4.0\times10^{-6})b\sin(1.5\times10^8 t - bx)\mathbf{a}_z\ \text{A/m}}$$

c) use $\nabla\times\mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$: Since $\sigma = 0$, there is no conduction current, so in this case

$$\nabla\times\mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{a}_y = \underline{4.0\times10^{-6}b^2\cos(1.5\times10^8 t - bx)\mathbf{a}_y\ \text{A/m}^2} = \mathbf{J}_d$$

d) What is the numerical value of b? We set the given expression for $\mathbf{J}_d$ equal to the result of part c to obtain:

$$20\times10^{-6} = 4.0\times10^{-6}b^2 \quad\Rightarrow\quad b = \underline{\sqrt{5.0}\ \text{m}^{-1}}$$

10.14. A voltage source, $V_0\sin\omega t$, is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_R\epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 5.10) and circuit analysis methods: First, solving Laplace's equation, we find the voltage between spheres (see Eq. 20, Chapter 7):

$$V(t) = \frac{(1/r) - (1/b)}{(1/a) - (1/b)}V_0\sin\omega t$$

10.14 (continued) Then

$$\mathbf{E} = -\nabla V = \frac{V_0 \sin \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r \quad \Rightarrow \quad \mathbf{D} = \frac{\epsilon_R \epsilon_0 V_0 \sin \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

Now

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon_R \epsilon_0 \omega V_0 \cos \omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

The displacement current is then

$$I_d = 4\pi r^2 J_d = \frac{4\pi \epsilon_R \epsilon_0 \omega V_0 \cos \omega t}{(1/a - 1/b)} = C\frac{dV}{dt}$$

where, from Eq. 47, Chapter 5,

$$C = \frac{4\pi \epsilon_R \epsilon_0}{(1/a - 1/b)}$$

The results are consistent.

10.15. Let $\mu = 3 \times 10^{-5}$ H/m, $\epsilon = 1.2 \times 10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10}t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for **B**, **D**, **E**, and β: First, $\mathbf{B} = \mu\mathbf{H} = \underline{6 \times 10^{-5}\cos(10^{10}t - \beta x)\mathbf{a}_z \text{ T}}$. Next we use

$$\nabla \times \mathbf{H} = -\frac{\partial \mathbf{H}}{\partial x}\mathbf{a}_y = 2\beta \sin(10^{10}t - \beta x)\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

from which

$$\mathbf{D} = \int 2\beta \sin(10^{10}t - \beta x)\, dt + C = \underline{-\frac{2\beta}{10^{10}}\cos(10^{10}t - \beta x)\mathbf{a}_y \text{ C/m}^2}$$

where the integration constant is set to zero, since no dc fields are presumed to exist. Next,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{2\beta}{(1.2 \times 10^{-10})(10^{10})}\cos(10^{10}t - \beta x)\mathbf{a}_y = \underline{-1.67\beta \cos(10^{10}t - \beta x)\mathbf{a}_y \text{ V/m}}$$

Now

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = 1.67\beta^2 \sin(10^{10}t - \beta x)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

So

$$\mathbf{B} = -\int 1.67\beta^2 \sin(10^{10}t - \beta x)\mathbf{a}_z dt = (1.67 \times 10^{-10})\beta^2 \cos(10^{10}t - \beta x)\mathbf{a}_z$$

We require this result to be consistent with the expression for **B** originally found. So

$$(1.67 \times 10^{-10})\beta^2 = 6 \times 10^{-5} \quad \Rightarrow \quad \beta = \underline{\pm 600 \text{ rad/m}}$$

10.16a. A certain material has $\sigma = 0$ and $\epsilon_R = 1$. If $\mathbf{H} = 4\sin(10^6 t - 0.01z)\mathbf{a}_y$ A/m, make use of Maxwell's equations to find μ_R: First find

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = 0.04\cos(10^6 t - 0.01z)\mathbf{a}_x = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So

$$\mathbf{E} = \int \frac{.04}{\epsilon_0}\cos(10^6 t - 0.01z)\mathbf{a}_x\, dt = \frac{.04}{10^6\epsilon_0}\sin(10^6 t - 0.01z)\mathbf{a}_x$$

where the integration constant is zero, since we assume no dc fields present. Next

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\frac{.04(.01)}{10^6\epsilon_0}\cos(10^6 t - 0.01z)\mathbf{a}_y = -\mu_R\mu_0\frac{\partial \mathbf{H}}{\partial t}$$

So

$$\mathbf{H} = \int \frac{.04(.01)}{10^6\epsilon_0\mu_0\mu_R}\cos(10^6 t - 0.01z)\mathbf{a}_y\, dt = \frac{.04(.01)}{10^{12}\epsilon_0\mu_0\mu_R}\sin(10^6 t - 0.01z)\mathbf{a}_y$$
$$= 4\sin(10^6 t - 0.01z)\mathbf{a}_y$$

where the last equality is required for consistency. Therefore

$$\frac{.04(.01)}{10^{12}\epsilon_0\mu_0\mu_R} = 4 \;\Rightarrow\; \mu_R = \frac{(.01)^2(9\times 10^{16})}{10^{12}} = \underline{9}$$

b) Find $\mathbf{E}(z,t)$: This we already found during the development in part a: We have

$$\mathbf{E}(z,t) = \frac{.04}{10^6\epsilon_0}\sin(10^6 t - 0.01z)\mathbf{a}_x \text{ V/m} = \underline{4.5\sin(10^6 t - 0.01z)\mathbf{a}_x \text{ kV/m}}$$

10.17. The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin(12y)\sin(az)\cos(2\times 10^{10}t)\,\mathbf{a}_x$ V/m. Beginning with the $\nabla \times \mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero: In this case we find

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\,\mathbf{a}_y - \frac{\partial E_z}{\partial y}\,\mathbf{a}_z$$
$$= C\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\cos(2\times 10^{10}t) = -\frac{\partial \mathbf{B}}{\partial t}$$

Then

$$\mathbf{H} = -\frac{1}{\mu_0}\int \nabla \times \mathbf{E}\, dt + C_1$$
$$= -\frac{C}{\mu_0(2\times 10^{10}}\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\sin(2\times 10^{10}t) \text{ A/m}$$

where the integration constant, $C_1 = 0$, since there are no initial conditions. Using this result, we now find

$$\nabla \times \mathbf{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\mathbf{a}_x = -\frac{C(144 + a^2)}{\mu_0(2\times 10^{10})}\sin(12y)\sin(az)\sin(2\times 10^{10}t)\,\mathbf{a}_x = \frac{\partial \mathbf{D}}{\partial t}$$

10.17. (continued) Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \int \frac{1}{\epsilon_0} \nabla \times \mathbf{H}\, dt + C_2 = \frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} \sin(12y) \sin(az) \cos(2 \times 10^{10} t)\, \mathbf{a}_x$$

where $C_2 = 0$. This field must be the same as the original field as stated, and so we require that

$$\frac{C(144 + a^2)}{\mu_0 \epsilon_0 (2 \times 10^{10})^2} = 1$$

Using $\mu_0 \epsilon_0 = (3 \times 10^8)^{-2}$, we find

$$a = \left[\frac{(2 \times 10^{10})^2}{(3 \times 10^8)^2} - 144 \right]^{1/2} = \underline{66 \text{ m}^{-1}}$$

10.18. The parallel plate transmission line shown in Fig. 10.8 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between plates is characterized by $\mu_R = 1$, $\epsilon_R = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5 \cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find:

a) β, if $\beta > 0$: Take

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = -5\beta \sin(10^9 t - \beta z)\mathbf{a}_x = 20\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So

$$\mathbf{E} = \int \frac{-5\beta}{20\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x\, dt = \frac{\beta}{(4 \times 10^9)\epsilon_0} \cos(10^9 t - \beta z)\mathbf{a}_x$$

Then

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = \frac{\beta^2}{(4 \times 10^9)\epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

So that

$$\mathbf{H} = \int \frac{-\beta^2}{(4 \times 10^9)\mu_0 \epsilon_0} \sin(10^9 t - \beta z)\mathbf{a}_x\, dt = \frac{\beta^2}{(4 \times 10^{18})\mu_0 \epsilon_0} \cos(10^9 t - \beta z)$$
$$= 5 \cos(10^9 t - \beta z)\mathbf{a}_y$$

where the last equality is required to maintain consistency. Therefore

$$\frac{\beta^2}{(4 \times 10^{18})\mu_0 \epsilon_0} = 5 \quad \Rightarrow \quad \beta = \underline{14.9 \text{ m}^{-1}}$$

b) the displacement current density at $z = 0$: Since $\sigma = 0$, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_d = -5\beta \sin(10^9 t - \beta z) = -74.5 \sin(10^9 t - 14.9z)\mathbf{a}_x$$
$$= \underline{-74.5 \sin(10^9 t)\mathbf{a}_x \text{ A/m}} \text{ at z} = 0$$

c) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, and $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction. We evaluate the flux integral of $\mathbf{J}_d$ over the given cross section:

$$I_d = -74.5b \int_0^{0.1} \sin(10^9 t - 14.9z)\, \mathbf{a}_x \cdot \mathbf{a}_x\, dz = \underline{0.20 \left[\cos(10^9 t - 1.49) - \cos(10^9 t) \right] \text{ A}}$$

10.19. In the first section of this chapter, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0\rho e^{kt}\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0 e^{kt}\mathbf{a}_z$ (note that the factor of ρ appearing in $\mathbf{E}$ was omitted from the original problem statement).

a) Show that these fields do not satisfy Maxwell's other curl equation: Note that $\mathbf{B}$ as stated is constant with position, and so will have zero curl. The electric field, however, varies with time, and so $\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ would have a zero left-hand side and a non-zero right-hand side. The equation is thus not valid with these fields.

b) If we let $B_0 = 1$ T and $k = 10^6\ s^{-1}$, we are establishing a fairly large magnetic flux density in 1 μs. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which B_z should (but does not) change with ρ is only about 5×10^{-6} T/m in free space at $t = 0$: Assuming that $\mathbf{B}$ varies with ρ, we write

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial \rho}\mathbf{a}_\phi = -\frac{1}{\mu_0}\frac{dB_0}{d\rho}e^{kt} = \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{2}\epsilon_0 k^2 B_0 \rho e^{kt}$$

Thus

$$\frac{dB_0}{d\rho} = \frac{1}{2}\mu_0\epsilon_0 k^2 \rho B_0 = \frac{10^{12}(1)\rho}{2(3\times 10^8)^2} = 5.6\times 10^{-6}\rho$$

which is near the stated value if ρ is on the order of 1m.

10.20. Point $C(-0.1, -0.2, 0.3)$ lies on the surface of a perfect conductor. The electric field intensity at C is $(500\mathbf{a}_x - 300\mathbf{a}_y + 600\mathbf{a}_z)\cos 10^7 t$ V/m, and the medium surrounding the conductor is characterized by $\mu_R = 5$, $\epsilon_R = 10$, and $\sigma = 0$.

a) Find a unit vector normal to the conductor surface at C, if the origin lies within the conductor: At $t = 0$, the field must be directed *out of* the surface, and will be normal to it, since we have a perfect conductor. Therefore

$$\mathbf{n} = \frac{+\mathbf{E}(t=0)}{|\mathbf{E}(t=0)|} = \frac{5\mathbf{a}_x - 3\mathbf{a}_y + 6\mathbf{a}_z}{\sqrt{25+9+36}} = \underline{0.60\mathbf{a}_x - 0.36\mathbf{a}_y + 0.72\mathbf{a}_z}$$

b) Find the surface charge density at C: Use

$$\begin{aligned}\rho_s = \mathbf{D}\cdot\mathbf{n}|_{surface} &= 10\epsilon_0\left[500\mathbf{a}_x - 300\mathbf{a}_y + 600\mathbf{a}_z\right]\cos(10^7 t)\cdot\left[.60\mathbf{a}_x - .36\mathbf{a}_y + .72\mathbf{a}_z\right] \\ &= 10\epsilon_0\left[300 + 108 + 432\right]\cos(10^7 t) = 7.4\times 10^{-8}\cos(10^7 t)\ \text{C/m}^2 = \underline{74\cos(10^7 t)\ \text{nC/m}^2}\end{aligned}$$

10.21. The surfaces $\rho = 3$ and 10 mm, and $z = 0$ and 25 cm are perfect conductors. The region enclosed by these surfaces has $\mu = 2.5\times 10^{-6}$ H/m, $\epsilon = 4\times 10^{-11}$ F/m, and $\sigma = 0$. Let $\mathbf{H} = (2/\rho)\cos(10\pi z)\cos(\omega t)\,\mathbf{a}_\phi$ A/m. Make use of Maxwell's equations to find

a) ω: We start with

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = \frac{20\pi}{\rho}\sin(10\pi z)\cos(\omega t)\,\mathbf{a}_\rho = \epsilon\frac{\partial \mathbf{E}}{\partial t}$$

We then find

$$\mathbf{E} = \int \frac{20\pi}{\rho\epsilon}\sin(10\pi z)\cos(\omega t)\,dt\,\mathbf{a}_\rho = \frac{20\pi}{\omega\rho\epsilon}\sin(10\pi z)\sin(\omega t)\,\mathbf{a}_\rho$$

10.21a. (continued) At this point, a flaw in the problem statement becomes apparent, since this field should vanish on the surface of the perfect conductor located at $z = 0.25$m. This does not happen with the $\sin(10\pi z)$ function. Nevertheless, we press on:

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\,\mathbf{a}_\phi = \frac{(20\pi)(10\pi)}{\omega\rho\epsilon}\cos(10\pi z)\sin(\omega t)\,\mathbf{a}_\phi = -\mu\frac{\partial \mathbf{H}}{\partial t}$$

So

$$\mathbf{H} = \int \frac{-200\pi^2}{\omega\rho\mu\epsilon}\cos(10\pi z)\sin(\omega t)\,\mathbf{a}_\phi\,dt = \frac{200\pi^2}{\omega^2\mu\epsilon\rho}\cos(10\pi z)\cos(\omega t)\,\mathbf{a}_\phi$$

This result must equal the given **H** field, so we require that

$$\frac{200\pi^2}{\omega^2\mu\epsilon\rho} = \frac{2}{\rho} \quad\Rightarrow\quad \omega = \frac{10\pi}{\sqrt{\mu\epsilon}} = \frac{10\pi}{\sqrt{(2.5\times10^{-6})(4\times10^{-11})}} = \underline{\pi\times10^9\ \text{sec}^{-1}}$$

b) **E**: We use the result of part *a*:

$$\mathbf{E} = \frac{20\pi}{\omega\rho\epsilon}\sin(10\pi z)\sin(\omega t)\,\mathbf{a}_\rho = \underline{\frac{500}{\rho}\sin(10\pi z)\sin(\omega t)\,\mathbf{a}_\rho\ \text{V/m}}$$

10.22. In free space, where $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 0$, $\mathbf{J} = 0$, and $\rho_v = 0$, assume a cartesian coordinate system in which **E** and **H** are both functions only of z and t.

a) If $\mathbf{E} = E_y\mathbf{a}_y$ and $\mathbf{H} = H_x\mathbf{a}_x$, begin with Maxwell's equations and determine the second order partial differential equation that E_y must satisfy: The procedure here is similar to the development that leads to Eq. 53. Begin by taking the curl of both sides of the Faraday law equation:

$$\nabla\times\nabla\times\mathbf{E} = \nabla\times\left(-\mu_0\frac{\partial\mathbf{H}}{\partial t}\right) = -\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{H})$$

where $\nabla\times\mathbf{H} = \epsilon_0\partial\mathbf{E}/\partial t$. Therefore

$$\nabla\times\nabla\times\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

where the first equality is found from Eq. 52. Noting that in free space, $\nabla\cdot\mathbf{D} = \epsilon_0\nabla\cdot\mathbf{E} = 0$, we obtain,

$$\nabla^2\mathbf{E} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \quad\Rightarrow\quad \underline{\frac{\partial^2 E_y}{\partial z^2} = \mu_0\epsilon_0\frac{\partial^2 E_y}{\partial t^2}}$$

since **E** varies only with z and t, and is y-directed.

b) Show that $E_y = 5(300t + bz)^2$ is a solution of that equation for a particular value of b, and find that value: Substituting, we find

$$\frac{\partial^2 E_y}{\partial z^2} = 10b^2 = \mu_0\epsilon_0\frac{\partial E_y}{\partial t^2} = 9\times10^5\mu_0\epsilon_0$$

Therefore

$$10b^2 = 9\times10^5\mu_0\epsilon_0 \quad\rightarrow\quad b = \underline{1.0\times10^{-6}\ \text{m}^{-1}}$$

10.23. In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z)\cos(10^9 t)$ V/m at $P_1(0, 0, 0^-)$.

a) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$: These will be

$$\mathbf{E}_{N1} = \underline{10\cos(10^9 t)\mathbf{a}_z \text{ V/m}} \quad \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

$$\mathbf{D}_{N1} = \epsilon_1 \mathbf{E}_{N1} = (2 \times 10^{-11})(10)\cos(10^9 t)\mathbf{a}_z \text{ C/m}^2 = \underline{200\cos(10^9 t)\mathbf{a}_z \text{ pC/m}^2}$$

$$\mathbf{D}_{t1} = \epsilon_1 \mathbf{E}_{t1} = (2 \times 10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(600\mathbf{a}_x + 400\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

b) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at P_1:

$$\mathbf{J}_{N1} = \sigma_1 \mathbf{E}_{N1} = (4 \times 10^{-3})(10\cos(10^9 t))\mathbf{a}_z = \underline{40\cos(10^9 t)\mathbf{a}_z \text{ mA/m}^2}$$

$$\mathbf{J}_{t1} = \sigma_1 \mathbf{E}_{t1} = (4 \times 10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(120\mathbf{a}_x + 80\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

c) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at P_1: By continuity of tangential $\mathbf{E}$,

$$\mathbf{E}_{t2} = \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

Then

$$\mathbf{D}_{t2} = \epsilon_2 \mathbf{E}_{t2} = (10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(300\mathbf{a}_x + 200\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

$$\mathbf{J}_{t2} = \sigma_2 \mathbf{E}_{t2} = (10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

d) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t - \partial D_{N1}/\partial t$ (note misprint in problem statement) and then determine $\mathbf{E}_{N2}$, $\mathbf{D}_{N2}$, and $\mathbf{J}_{N2}$: We assume the existence of a surface charge layer at the boundary having density ρ_s C/m^2. If we draw a cylindrical "pillbox" whose top and bottom surfaces (each of area Δa) are on either side of the interface, we may use the continuity condition to write

$$(J_{N2} - J_{N1})\Delta a = -\frac{\partial \rho_s}{\partial t}\Delta a$$

where $\rho_s = D_{N2} - D_{N1}$. Therefore,

$$J_{N1} - J_{N2} = \frac{\partial}{\partial t}(D_{N2} - D_{N1})$$

In terms of the normal electric field components, this becomes

$$\sigma_1 E_{N1} - \sigma_2 E_{N2} = \frac{\partial}{\partial t}(\epsilon_2 E_{N2} - \epsilon_1 E_{N1})$$

Now let $E_{N2} = A\cos(10^9 t) + B\sin(10^9 t)$, while from before, $E_{N1} = 10\cos(10^9 t)$.

10.23. (continued)

These, along with the permittivities and conductivities, are substituted to obtain

$$\begin{aligned}&(4\times10^{-3})(10)\cos(10^9t)-10^{-3}[A\cos(10^9t)+B\sin(10^9t)]\\&=\frac{\partial}{\partial t}\left[10^{-11}[A\cos(10^9t)+B\sin(10^9t)]-(2\times10^{-11})(10)\cos(10^9t)\right]\\&=-(10^{-2}A\sin(10^9t)+10^{-2}B\cos(10^9t)+(2\times10^{-1})\sin(10^9t)\end{aligned}$$

We now equate coefficients of the sin and cos terms to obtain two equations:

$$4\times10^{-2}-10^{-3}A=10^{-2}B$$

$$-10^{-3}B=-10^{-2}A+2\times10^{-1}$$

These are solved together to find $A=20.2$ and $B=2.0$. Thus

$$\mathbf{E}_{N2}=\left[20.2\cos(10^9t)+2.0\sin(10^9t)\right]\mathbf{a}_z=\underline{20.3\cos(10^9t+5.6^\circ)\mathbf{a}_z\ \text{V/m}}$$

Then

$$\mathbf{D}_{N2}=\epsilon_2\mathbf{E}_{N2}=\underline{203\cos(10^9t+5.6^\circ)\mathbf{a}_z\ \text{pC/m}^2}$$

and

$$\mathbf{J}_{N2}=\sigma_2\mathbf{E}_{N2}=\underline{20.3\cos(10^9t+5.6^\circ)\mathbf{a}_z\ \text{mA/m}^2}$$

10.24. Given the fields $V=80z\cos x\cos3\times10^8t$. kV and $\mathbf{A}=26.7z\sin x\sin3\times10^8t\,\mathbf{a}_x$ mWb/m in free space, find **E** and **H**: First, find **E** through

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}$$

where

$$-\nabla V=80\cos(3\times10^8t)[z\sin x\mathbf{a}_x-\cos x\mathbf{a}_z]\ \text{kV/m}$$

and

$$-\partial\mathbf{A}/\partial t=-(3\times10^8)(26.7)z\sin x\cos(3\times10^8t)\mathbf{a}_x\ \text{mV/m}$$

Finally,

$$\mathbf{E}=\underline{-\left[7.9\times10^6z\sin x\ \mathbf{a}_x+8.0\times10^4\cos x\ \mathbf{a}_z\right]\cos(3\times10^8t)\ \text{V/m}}$$

Now

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\partial A_x}{\partial z}\mathbf{a}_y=26.7\sin x\sin(3\times10^8t)\mathbf{a}_y\ \text{mWb/m}^2$$

Then

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}=\underline{2.12\times10^4\sin x\sin(3\times10^8t)\,\mathbf{a}_y\ \text{A/m}}$$

10.25. In a region where $\mu_R = \epsilon_R = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x[(z/c) - t]\mathbf{a}_z$ Wb/m, where $c = 1/\sqrt{\mu_0\epsilon_0}$.

a) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon(\partial V/\partial t)$:

First,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c} = x\sqrt{\mu_0\epsilon_0}$$

Second,

$$\frac{\partial V}{\partial t} = -cx = -\frac{x}{\sqrt{\mu_0\epsilon_0}}$$

so we observe that $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$ in free space, implying that the given statement would hold true in general media.

b) Find **B**, **H**, **E**, and **D**:

Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial x}\mathbf{a}_y = \underline{\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ T}}$$

Then

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \underline{\frac{1}{\mu_0}\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ A/m}}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -(z - ct)\mathbf{a}_x - x\mathbf{a}_z + x\mathbf{a}_z = \underline{(ct - z)\mathbf{a}_x \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0\mathbf{E} = \underline{\epsilon_0(ct - z)\mathbf{a}_x \text{ C/m}^2}$$

c) Show that these results satisfy Maxwell's equations if **J** and ρ_v are zero:

i. $\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_0(ct - z)\mathbf{a}_x = 0$

ii. $\nabla \cdot \mathbf{B} = \nabla \cdot (t - z/c)\mathbf{a}_y = 0$

iii.

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \frac{1}{\mu_0 c}\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

which we require to equal $\partial\mathbf{D}/\partial t$:

$$\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 c\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

iv.

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mathbf{a}_y$$

which we require to equal $-\partial\mathbf{B}/\partial t$:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{a}_y$$

So all four Maxwell equations are satisfied.

10.26. Let the current $I = 80t$ A be present in the $\mathbf{a}_z$ direction on the z axis in free space within the interval $-0.1 < z < 0.1$ m.

a) Find A_z at $P(0, 2, 0)$: The integral for the retarded vector potential will in this case assume the form

$$\mathbf{A} = \int_{-.1}^{.1} \frac{\mu_0 80(t - R/c)}{4\pi R} \mathbf{a}_z \, dz$$

where $R = \sqrt{z^2 + 4}$ and $c = 3 \times 10^8$ m/s. We obtain

$$A_z = \frac{80\mu_0}{4\pi} \left[\int_{-.1}^{.1} \frac{t}{\sqrt{z^2+4}} \, dz - \int_{-.1}^{.1} \frac{1}{c} \, dz \right] = 8 \times 10^{-6} t \ln(z + \sqrt{z^2+4}) \Big|_{-.1}^{.1} - \frac{8 \times 10^{-6}}{3 \times 10^8} z \Big|_{-.1}^{.1}$$

$$= 8 \times 10^{-6} \ln \left(\frac{.1 + \sqrt{4.01}}{-.1 + \sqrt{4.01}} \right) - 0.53 \times 10^{-14} = 8.0 \times 10^{-7} t - 0.53 \times 10^{-14}$$

So finally, $\mathbf{A} = \underline{\left[8.0 \times 10^{-7} t - 5.3 \times 10^{-15}\right] \mathbf{a}_z \text{ Wb/m}}$.

b) Sketch A_z versus t over the time interval $-0.1 < t < 0.1$ μs: The sketch is linearly increasing with time, beginning with $A_z = -8.53 \times 10^{-14}$ Wb/m at $t = -0.1$ μs, crossing the time axis and going positive at $t = 6.6$ ns, and reaching a maximum value of 7.46×10^{-14} Wb/m at $t = 0.1$ μs.

CHAPTER 11

11.1. Show that $E_{xs} = Ae^{jk_0z+\phi}$ is a solution to the vector Helmholtz equation, Sec. 11.1, Eq. (16), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any ϕ and A: We take

$$\frac{d^2}{dz^2}\,Ae^{jk_0z+\phi} = (jk_0)^2 Ae^{jk_0z+\phi} = -k_0^2 E_{xs}$$

11.2. Let $\mathbf{E}(z,t) = 200\sin 0.2z\cos 10^8 t\,\mathbf{a}_x + 500\cos(0.2z+50^\circ)\sin 10^8 t\,\mathbf{a}_y$ V/m. Find:

a) $\mathbf{E}$ at $P(0,2,0.6)$ at $t = 25$ ns: Obtain

$$\begin{aligned} E_P(t=25) &= 200\sin[(0.2)(0.6)]\cos(2.5)\mathbf{a}_x + 500\cos[(0.2)(0.6)+50(2\pi)/360]\sin(2.5)\mathbf{a}_y \\ &= \underline{-19.2\mathbf{a}_x + 164\mathbf{a}_y\ \text{V/m}} \end{aligned}$$

b) $|\mathbf{E}|$ at P at $t = 20$ ns:

$$\begin{aligned} E_P(t=20) &= 200\sin[(0.2)(0.6)]\cos(2.0)\mathbf{a}_x + 500\cos[(0.2)(0.6)+50(2\pi)/360]\sin(2.0)\mathbf{a}_y \\ &= -9.96\mathbf{a}_x + 248\mathbf{a}_y\ \text{V/m} \end{aligned}$$

Thus $|\mathbf{E}_P| = \sqrt{(9.96)^2+(248)^2} = \underline{249\ \text{V/m}}$.

c) E_s at P: $E_s = 200\sin 0.2z\mathbf{a}_x - j500\cos(0.2z+50^\circ)\mathbf{a}_y$. Thus

$$\begin{aligned} E_{sP} &= 200\sin[(0.2)(0.6)]\,\mathbf{a}_x - j500\cos[(0.2)(0.6)+2\pi(50)/360]\,\mathbf{a}_y \\ &= \underline{23.9\mathbf{a}_x - j273\mathbf{a}_y\ \text{V/m}} \end{aligned}$$

11.3. An $\mathbf{H}$ field in free space is given as $\mathbf{H}(x,t) = 10\cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find

a) β: Since we have a uniform plane wave, $\beta = \omega/c$, where we identify $\omega = 10^8$ sec^{-1}. Thus $\beta = 10^8/(3\times 10^8) = \underline{0.33\ \text{rad/m}}$.

b) λ: We know $\lambda = 2\pi/\beta = \underline{18.9\ \text{m}}$.

c) $\mathbf{E}(x,t)$ at $P(0.1,0.2,0.3)$ at $t = 1$ ns: Use $E(x,t) = -\eta_0 H(x,t) = -(377)(10)\cos(10^8 t - \beta x) = -3.77\times 10^3\cos(10^8 t - \beta x)$. The vector direction of $\mathbf{E}$ will be $-\mathbf{a}_z$, since we require that $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, where $\mathbf{S}$ is x-directed. At the given point, the relevant coordinate is $x = 0.1$. Using this, along with $t = 10^{-9}$ sec, we finally obtain

$$\begin{aligned} \mathbf{E}(x,t) &= -3.77\times 10^3\cos[(10^8)(10^{-9}) - (0.33)(0.1)]\mathbf{a}_z = -3.77\times 10^3\cos(6.7\times 10^{-2})\mathbf{a}_z \\ &= \underline{-3.76\times 10^3\mathbf{a}_z\ \text{V/m}} \end{aligned}$$

11.4. In phasor form, the electric field intensity of a uniform plane wave in free space is expressed as $\mathbf{E}_s = (40 - j30)e^{-j20z}\mathbf{a}_x$ V/m. Find:

a) ω: From the given expression, we identify $\beta = 20$ rad/m. Then $\omega = c\beta = (3\times 10^8)(20) = \underline{6.0\times 10^9\ \text{rad/s}}$.

b) $\beta = \underline{20\ \text{rad/m}}$ from part a.

11.4. (continued)

c) $f = \omega/2\pi = \underline{956\text{ MHz}}$.

d) $\lambda = 2\pi/\beta = 2\pi/20 = \underline{0.314\text{ m}}$.

e) $\mathbf{H}_s$: In free space, we find $\mathbf{H}_s$ by dividing $\mathbf{E}_s$ by η_0, and assigning vector components such that $\mathbf{E}_s \times \mathbf{H}_s$ gives the required direction of wave travel: We find

$$\mathbf{H}_s = \frac{40 - j30}{377} e^{-j20z} \mathbf{a}_y = \underline{(0.11 - j0.08) e^{-j20z}\, \mathbf{a}_y \text{ A/m}}$$

f) $\mathbf{H}(z, t)$ at $P(6, -1, 0.07)$, $t = 71$ ps:

$$\mathbf{H}(z, t) = \text{Re}\left[\mathbf{H}_s e^{j\omega t}\right] = \left[0.11\cos(6.0 \times 10^9 t - 20z) + 0.08\sin(6.0 \times 10^9 t - 20z)\right] \mathbf{a}_y$$

Then

$$\begin{aligned}\mathbf{H}(.07, t = 71\text{ps}) &= \left[0.11\cos\left[(6.0 \times 10^9)(7.1 \times 10^{-11}) - 20(.07)\right]\right. \\ &\quad \left. + .08\sin\left[(6.0 \times 10^9)(7.1 \times 10^{-11}) - 20(.07)\right]\right] \mathbf{a}_y \\ &= [0.11(0.562) - 0.08(0.827)]\mathbf{a}_y = \underline{-6.2 \times 10^{-3} \mathbf{a}_y \text{ A/m}}\end{aligned}$$

11.5. A 150-MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m.

a) Find numerical values for ω, λ, and β: First, $\omega = 2\pi \times 150 \times 10^6 = \underline{3\pi \times 10^8 \text{ sec}^{-1}}$. Second, for a uniform plane wave in free space, $\lambda = 2\pi c/\omega = c/f = (3 \times 10^8)/(1.5 \times 10^8) = \underline{2\text{ m}}$. Third, $\beta = 2\pi/\lambda = \underline{\pi \text{ rad/m}}$.

b) Find $\mathbf{H}(z, t)$ at $t = 1.5$ ns, $z = 20$ cm: Use

$$\begin{aligned}\mathbf{H}(z, t) &= \text{Re}\{\mathbf{H}_s e^{j\omega t}\} = \text{Re}\{(4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\} \\ &= [8\cos(\omega t - \beta z) - 20\sin(\omega t - \beta z)]\,\mathbf{a}_x - [10\cos(\omega t - \beta z) + 4\sin(\omega t - \beta z)]\,\mathbf{a}_y\end{aligned}$$

. Now at the given position and time, $\omega t - \beta z = (3\pi \times 10^8)(1.5 \times 10^{-9}) - \pi(0.20) = \pi/4$. And $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. So finally,

$$\mathbf{H}(z = 20\text{cm}, t = 1.5\text{ns}) = -\frac{1}{\sqrt{2}}\left(12\mathbf{a}_x + 14\mathbf{a}_y\right) = \underline{-8.5\mathbf{a}_x - 9.9\mathbf{a}_y \text{ A/m}}$$

c) What is $|E|_{max}$? Have $|E|_{max} = \eta_0 |H|_{max}$, where

$$|H|_{max} = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = [4(4 + j10)(4 - j10) + (j)(-j)(4 + j10)(4 - j10)]^{1/2} = 24.1\text{ A/m}$$

Then $|E|_{max} = 377(24.1) = \underline{9.08\text{ kV/m}}$.

11.6. Let $\mu_R = \epsilon_R = 1$ for the field $\mathbf{E}(z,t) = (25\mathbf{a}_x - 30\mathbf{a}_y)\cos(\omega t - 50z)$ V/m.

a) Find ω: $\omega = c\beta = (3\times10^8)(50) = \underline{15.0\times10^9 \text{ s}^{-1}}$.

b) Determine the displacement current density, $\mathbf{J}_d(z,t)$:

$$\mathbf{J}_d(z,t) = \frac{\partial \mathbf{D}}{\partial t} = -\epsilon_0\omega(25\mathbf{a}_x - 30\mathbf{a}_y)\sin(\omega t - 50z)$$
$$= \underline{(-3.32\mathbf{a}_x + 3.98\mathbf{a}_y)\sin(1.5\times10^{10}t - 50z) \text{ A/m}^2}$$

c) Find the total magnetic flux Φ passing through the rectangle defined by $0 < x < 1$, $y = 0$, $0 < z < 1$, at $t = 0$: In free space, the magnetic field of the uniform plane wave can be easily found using the intrinsic impedance:

$$\mathbf{H}(z,t) = \left(\frac{25}{\eta_0}\mathbf{a}_y + \frac{30}{\eta_0}\mathbf{a}_x\right)\cos(\omega t - 50z) \text{ A/m}$$

Then $\mathbf{B}(z,t) = \mu_0\mathbf{H}(z,t) = (1/c)(25\mathbf{a}_y + 30\mathbf{a}_x)\cos(\omega t - 50z)$ Wb/m^2, where $\mu_0/\eta_0 = \sqrt{\mu_0\epsilon_0} = 1/c$. The flux at $t = 0$ is now

$$\Phi = \int_0^1\int_0^1 \mathbf{B}\cdot\mathbf{a}_y\,dx\,dz = \int_0^1 \frac{25}{c}\cos(50z)\,dz = \frac{25}{50(3\times10^8)}\sin(50) = \underline{-0.44 \text{ nWb}}$$

11.7. The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find β, η, λ, v_p, ϵ_R, μ_R, and $\mathbf{H}(x,y,z,t)$: First, from the phasor expression, we identify $\beta = \underline{25 \text{ m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H}\cdot\mathbf{H}^*} = \sqrt{2^2+5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25$ m $= \underline{25 \text{ cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi\times400\times10^6}{25} = \underline{1.01\times10^8 \text{ m/s}}$$

Now we note that

$$\eta = 278.5 = 377\sqrt{\frac{\mu_R}{\epsilon_R}} \;\Rightarrow\; \frac{\mu_R}{\epsilon_R} = 0.546$$

And

$$v_p = 1.01\times10^8 = \frac{c}{\sqrt{\mu_R\epsilon_R}} \;\Rightarrow\; \mu_R\epsilon_R = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_R = \underline{4.01}$ and $\mu_R = \underline{2.19}$. Finally,

$$\mathbf{H}(x,y,z,t) = Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}e^{j\omega t}\right\}$$
$$= 2\cos(2\pi\times400\times10^6t - 25x)\mathbf{a}_y + 5\sin(2\pi\times400\times10^6t - 25x)\mathbf{a}_z$$
$$= \underline{2\cos(8\pi\times10^8t - 25x)\mathbf{a}_y + 5\sin(8\pi\times10^8t - 25x)\mathbf{a}_z \text{ A/m}}$$

11.8. Let the fields, $\mathbf{E}(z,t) = 1800\cos(10^7\pi t - \beta z)\mathbf{a}_x$ V/m and $\mathbf{H}(z,t) = 3.8\cos(10^7\pi t - \beta z)\mathbf{a}_y$ A/m, represent a uniform plane wave propagating at a velocity of 1.4×10^8 m/s in a perfect dielectric. Find:

a) $\beta = \omega/v = (10^7\pi)/(1.4 \times 10^8) = \underline{0.224\,\text{m}^{-1}}$.

b) $\lambda = 2\pi/\beta = 2\pi/.224 = \underline{28.0\,\text{m}}$.

c) $\eta = |\mathbf{E}|/|\mathbf{H}| = 1800/3.8 = \underline{474\,\Omega}$.

d) μ_R: Have two equations in the two unknowns, μ_R and ϵ_R: $\eta = \eta_0\sqrt{\mu_R/\epsilon_R}$ and $\beta = \omega\sqrt{\mu_R\epsilon_R}/c$. Eliminate ϵ_R to find

$$\mu_R = \left[\frac{\beta c\eta}{\omega\eta_0}\right]^2 = \left[\frac{(.224)(3\times 10^8)(474)}{(10^7\pi)(377)}\right]^2 = \underline{2.69}$$

e) $\epsilon_R = \mu_R(\eta_0/\eta)^2 = (2.69)(377/474)^2 = \underline{1.70}$.

11.9. A certain lossless material has $\mu_R = 4$ and $\epsilon_R = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns.

a) Find β, λ, v_p, and η: For a uniform plane wave,

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_R\epsilon_R} = \frac{2\pi\times 10^7}{3\times 10^8}\sqrt{(4)(9)} = \underline{0.4\pi\ \text{rad/m}}$$

Then $\lambda = (2\pi)/\beta = (2\pi)/(0.4\pi) = \underline{5\,\text{m}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi\times 10^7}{4\pi\times 10^{-1}} = \underline{5\times 10^7\ \text{m/s}}$$

Finally,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_R}{\epsilon_R}} = 377\sqrt{\frac{4}{9}} = \underline{251\ \Omega}$$

b) Find $E(t)$ (at P): We are given the amplitude at $t = 60$ ns and at $y = 0.6$ m. Let the maximum amplitude be E_{max}, so that in general, $E_x = E_{max}\cos(\omega t - \beta y)$. At the given position and time,

$$E_x = 400 = E_{max}\cos[(2\pi\times 10^7)(60\times 10^{-9}) - (4\pi\times 10^{-1})(0.6)] = E_{max}\cos(0.96\pi)$$
$$= -0.99E_{max}$$

So $E_{max} = (400)/(-0.99) = -403$ V/m. Thus at P, $E(t) = \underline{-403\cos(2\pi\times 10^7 t)\ \text{V/m}}$.

c) Find $H(t)$: First, we note that if E at a given instant points in the negative x direction, while the wave propagates in the forward y direction, then H at that same position and time must point in the positive z direction. Since we have a lossless homogeneous medium, η is real, and we are allowed to write $H(t) = E(t)/\eta$, where η is treated as negative and real. Thus

$$H(t) = H_z(t) = \frac{E_x(t)}{\eta} = \frac{-403}{-251}\cos(2\pi\times 10^{-7}t) = \underline{1.61\cos(2\pi\times 10^{-7}t)\ \text{A/m}}$$

11.10. Given a 20MHz uniform plane wave with $\mathbf{H}_s = (6\mathbf{a}_x - j2\mathbf{a}_y)e^{-jz}$ A/m, assume propagation in a lossless medium characterized by $\epsilon_R = 5$ and an unknown μ_R.

a) Find λ, v_p, μ_R, and η: First, $\beta = 1$, so $\lambda = 2\pi/\beta = \underline{2\pi \text{ m}}$. Next, $v_p = \omega/\beta = 2\pi \times 20 \times 10^6 = \underline{4\pi \times 10^7 \text{ m/s}}$. Then, $\mu_R = (\beta^2 c^2)/(\omega^2 \epsilon_R) = (3 \times 10^8)^2/(4\pi \times 10^7)^2(5) = \underline{1.14}$. Finally, $\eta = \eta_0\sqrt{\mu_R/\epsilon_R} = 377\sqrt{1.14/5} = \underline{180}$.

b) Determine $\mathbf{E}$ at the origin at $t = 20$ns: We use the relation $|\mathbf{E}| = \eta|\mathbf{H}|$ and note that for positive z propagation, a positive x component of $\mathbf{H}$ is coupled to a negative y component of $\mathbf{E}$, and a negative y component of $\mathbf{H}$ is coupled to a negative x component of $\mathbf{E}$. We obtain $\mathbf{E}_s = -\eta(6\mathbf{a}_y + j2\mathbf{a}_x)e^{-jz}$. Then $\mathbf{E}(z,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = -6\eta\cos(\omega t - z)\mathbf{a}_y + 2\eta\sin(\omega t - z)\mathbf{a}_x = 360\sin(\omega t - z)\mathbf{a}_x - 1080\cos(\omega t - z)\mathbf{a}_y$. With $\omega = 4\pi \times 10^7$ sec^{-1}, $t = 2 \times 10^{-8}$ s, and $z = 0$, $\mathbf{E}$ evaluates as $\mathbf{E}(0, 20\text{ns}) = 360(0.588)\mathbf{a}_x - 1080(-0.809)\mathbf{a}_y = \underline{212\mathbf{a}_x + 874\mathbf{a}_y \text{ V/m}}$.

11.11. A 2-GHz uniform plane wave has an amplitude of $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\,\mu$H/m. Find:

a) E_y at $P(0, 0, 1.8\text{cm})$ at 0.2 ns: To begin, we have the ratio, $\epsilon''/\epsilon' = 1.6/3.0 = 0.533$. So

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$

$$= (2\pi \times 2 \times 10^9)\sqrt{\frac{(2.5 \times 10^{-6})(3.0 \times 10^{-11})}{2}}\left[\sqrt{1 + (.533)^2} - 1\right]^{1/2} = 28.1\,\text{Np/m}$$

Then

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = 112\,\text{rad/m}$$

Thus in general,

$$E_y(z,t) = 1.4e^{-28.1z}\cos(4\pi \times 10^9 t - 112z)\text{ kV/m}$$

Evaluating this at $t = 0.2$ ns and $z = 1.8$ cm, find

$$E_y(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{0.74\,\text{kV/m}}$$

b) H_x at P at 0.2 ns: We use the phasor relation, $H_{xs} = -E_{ys}/\eta$ where

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.5 \times 10^{-6}}{3.0 \times 10^{-11}}}\frac{1}{\sqrt{1 - j(.533)}} = 263 + j65.7 = 271\angle 14^\circ\ \Omega$$

So now

$$H_{xs} = -\frac{E_{ys}}{\eta} = -\frac{(1.4 \times 10^3)e^{-28.1z}e^{-j112z}}{271e^{j14^\circ}} = -5.16e^{-28.1z}e^{-j112z}e^{-j14^\circ}\text{ A/m}$$

Then

$$H_x(z,t) = -5.16e^{-28.1z}\cos(4\pi \times 10^{-9}t - 112z - 14^\circ)$$

This, when evaluated at $t = 0.2$ ns and $z = 1.8$ cm, yields

$$H_x(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{-3.0\,\text{A/m}}$$

11.12. The plane wave $\mathbf{E}_s = 300e^{-jkx}\mathbf{a}_y$ V/m is propagating in a material for which $\mu = 2.25\ \mu$H/m, $\epsilon' = 9$ pF/m, and $\epsilon'' = 7.8$ pF/m. If $\omega = 64$ Mrad/s, find:

a) α: We use the general formula, Eq. (35):

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}-1\right]^{1/2}$$

$$= (64\times10^6)\sqrt{\frac{(2.25\times10^{-6})(9\times10^{-12})}{2}}\left[\sqrt{1+(.867)^2}-1\right]^{1/2} = \underline{0.116\ \text{Np/m}}$$

b) β: Using (36), we write

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2}+1\right]^{1/2} = \underline{.311\ \text{rad/m}}$$

c) $v_p = \omega/\beta = (64\times10^6)/(.311) = \underline{2.06\times10^8\ \text{m/s}}$.

d) $\lambda = 2\pi/\beta = 2\pi/(.311) = \underline{20.2\ \text{m}}$.

e) η: Using (39):

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1-j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.25\times10^{-6}}{9\times10^{-12}}}\frac{1}{\sqrt{1-j(.867)}} = 407+j152 = \underline{434.5e^{j.36}\ \Omega}$$

f) $\mathbf{H}_s$: With $\mathbf{E}_s$ in the positive y direction (at a given time) and propagating in the positive x direction, we would have a positive z component of $\mathbf{H}_s$, at the same time. We write (with $jk = \alpha + j\beta$):

$$\mathbf{H}_s = \frac{E_s}{\eta}\mathbf{a}_z = \frac{300}{434.5e^{j.36}}e^{-jkx}\mathbf{a}_z = 0.69e^{-\alpha x}e^{-j\beta x}e^{-j.36}\mathbf{a}_z$$

$$= \underline{0.69e^{-.116x}e^{-j.311x}e^{-j.36}\mathbf{a}_z\ \text{A/m}}$$

g) $\mathbf{E}(3, 2, 4, 10\text{ns})$: The real instantaneous form of $\mathbf{E}$ will be

$$\mathbf{E}(x, y, z, t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = 300e^{-\alpha x}\cos(\omega t - \beta x)\mathbf{a}_y$$

Therefore

$$\mathbf{E}(3, 2, 4, 10\text{ns}) = 300e^{-.116(3)}\cos[(64\times10^6)(10^{-8}) - .311(3)]\mathbf{a}_y = \underline{203\ \text{V/m}}$$

11.13. Let $jk = 0.2 + j1.5\,\text{m}^{-1}$ and $\eta = 450 + j60\ \Omega$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find μ, ϵ', and ϵ'': We begin with

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1-j(\epsilon''/\epsilon')}} = 450 + j60$$

and

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1-j(\epsilon''/\epsilon')} = 0.2 + j1.5$$

11.13. (continued) Then

$$\eta\eta^* = \frac{\mu}{\epsilon'}\frac{1}{\sqrt{1+(\epsilon''/\epsilon')^2}} = (450+j60)(450-j60) = 2.06\times 10^5 \tag{1}$$

and

$$(jk)(jk)^* = \omega^2\mu\epsilon'\sqrt{1+(\epsilon''/\epsilon')^2} = (0.2+j1.5)(0.2-j1.5) = 2.29 \tag{2}$$

Taking the ratio of (2) to (1),

$$\frac{(jk)(jk)^*}{\eta\eta^*} = \omega^2(\epsilon')^2\left(1+(\epsilon''/\epsilon')^2\right) = \frac{2.29}{2.06\times 10^5} = 1.11\times 10^{-5}$$

Then with $\omega = 3\times 10^8$,

$$(\epsilon')^2 = \frac{1.11\times 10^{-5}}{(3\times 10^8)^2\left(1+(\epsilon''/\epsilon')^2\right)} = \frac{1.23\times 10^{-22}}{\left(1+(\epsilon''/\epsilon')^2\right)} \tag{3}$$

Now, we use Eqs. (35) and (36). Squaring these and taking their ratio gives

$$\frac{\alpha^2}{\beta^2} = \frac{\sqrt{1+(\epsilon''/\epsilon')^2}}{\sqrt{1+(\epsilon''/\epsilon')^2}} = \frac{(0.2)^2}{(1.5)^2}$$

We solve this to find $\epsilon''/\epsilon' = 0.271$. Substituting this result into (3) gives $\epsilon' = 1.07\times 10^{-11}$ F/m. Since $\epsilon''/\epsilon' = 0.271$, we then find $\epsilon'' = 2.90\times 10^{-12}$ F/m. Finally, using these results in either (1) or (2) we find $\mu = 2.28\times 10^{-6}$ H/m. Summary: $\mu = \underline{2.28\times 10^{-6}\,\text{H/m}}$,
$\epsilon' = \underline{1.07\times 10^{-11}\,\text{F/m}}$, and $\epsilon'' = \underline{2.90\times 10^{-12}\,\text{F/m}}$.

11.14. A certain nonmagnetic material has the material constants $\epsilon'_R = 2$ and $\epsilon''/\epsilon' = 4\times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before:

a) it is attenuated by 1 Np: First, $\epsilon'' = (4\times 10^4)(2)(8.854\times 10^{-12}) = 7.1\times 10^{-15}$ F/m. Then, since $\epsilon''/\epsilon' << 1$, we use the approximate form for α, given by Eq. (51) (written in terms of ϵ''):

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{(1.5\times 10^9)(7.1\times 10^{-15})}{2}\frac{377}{\sqrt{2}} = 1.42\times 10^{-3}\ \text{Np/m}$$

The required distance is now $z_1 = (1.42\times 10^{-3})^{-1} = \underline{706\ \text{m}}$

b) the power level is reduced by one-half: The governing relation is $e^{-2\alpha z_{1/2}} = 1/2$, or $z_{1/2} = \ln 2/2\alpha = \ln 2/2(1.42\times 10^-3) = \underline{244\ \text{m}}$.

c) the phase shifts 360°: This distance is defined as one wavelength, where $\lambda = 2\pi/\beta$
$= (2\pi c)/(\omega\sqrt{\epsilon'_R}) = [2\pi(3\times 10^8)]/[(1.5\times 10^9)\sqrt{2}] = \underline{0.89\ \text{m}}$.

11.15. A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon'_R = 1$ and $\epsilon''_R = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_R}{2}}\left[\sqrt{1+\left(\frac{\epsilon''_R}{\epsilon'_R}\right)^2}-1\right]^{1/2}$$

11.15. (continued) and

$$\beta = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon'_R}{2}}\left[\sqrt{1+\left(\frac{\epsilon''_R}{\epsilon'_R}\right)^2}+1\right]^{1/2}$$

With the given values of ϵ'_R and ϵ''_R, it is clear that $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$, and so $\lambda = 2\pi/\beta = 2\pi c/\omega = 3\times10^{10}/10^{10} = \underline{3\text{ cm}}$. It is also clear that $\alpha = \underline{0}$.

b) $\epsilon'_R = 1.04$ and $\epsilon''_R = 9.00\times10^{-4}$: In this case $\epsilon''_R/\epsilon'_R << 1$, and so $\beta \doteq \omega\sqrt{\epsilon'_R}/c = 2.13\text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95\text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon''_R}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon'_R}} = \frac{\omega}{2c}\frac{\epsilon''_R}{\sqrt{\epsilon'_R}} = \frac{2\pi\times10^{10}}{2\times3\times10^8}\frac{(9.00\times10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24\times10^{-2}\text{ Np/m}}$$

c) $\epsilon'_R = 2.5$ and $\epsilon''_R = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi\times10^{10}\sqrt{2.5}}{(3\times10^{10})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}+1\right]^{1/2} = 4.71\text{ cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33\text{ cm}}$. Then

$$\alpha = \frac{2\pi\times10^{10}\sqrt{2.5}}{(3\times10^{8})\sqrt{2}}\left[\sqrt{1+\left(\frac{7.2}{2.5}\right)^2}-1\right]^{1/2} = \underline{335\text{ Np/m}}$$

11.16. The power factor of a capacitor is defined as the cosine of the impedance phase angle, and its Q is ωCR, where R is the parallel resistance. Assume an idealized parallel plate capacitor having a dielecric characterized by σ, ϵ', and μ_R. Find both the power factor and Q in terms of the loss tangent: First, the impedance will be:

$$Z = \frac{R\left(\frac{1}{j\omega C}\right)}{R+\left(\frac{1}{j\omega C}\right)} = R\frac{1-jR\omega C}{1+(R\omega C)^2} = R\frac{1-jQ}{1+Q^2}$$

Now $R = d/(\sigma A)$ and $C = \epsilon' A/d$, and so $Q = \omega\epsilon'/\sigma = \underline{1/l.t.}$ Then the power factor is P.F $=$ $\cos[\tan^{-1}(-Q)] = \underline{1/\sqrt{1+Q^2}}$.

11.17. Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\,\text{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find:

a) $\mathbf{P}_{z,av}$ at $z = 0$ and $z = 60$ cm: Assume x-polarization for the electric field. Then

$$\begin{aligned}\mathbf{P}_{z,av} &= \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{400e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x \times \frac{400}{\eta^*}e^{-\alpha z}e^{j\beta z}\mathbf{a}_y\right\} \\ &= \frac{1}{2}(400)^2 e^{-2\alpha z}\text{Re}\left\{\frac{1}{\eta^*}\right\}\mathbf{a}_z = 8.0\times 10^4 e^{-2(0.2)z}\text{Re}\left\{\frac{1}{250-j30}\right\}\mathbf{a}_z \\ &= 315\,e^{-2(0.2)z}\,\mathbf{a}_z\ \text{W/m}^2\end{aligned}$$

Evaluating at $z = 0$, obtain $\mathbf{P}_{z,av}(z=0) = \underline{315\,\mathbf{a}_z\ \text{W/m}^2}$,
and at $z = 60$ cm, $\mathbf{P}_{z,av}(z=0.6) = 315e^{-2(0.2)(0.6)}\mathbf{a}_z = \underline{248\,\mathbf{a}_z\ \text{W/m}^2}$.

b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm: At this point a flaw becomes evident in the problem statement, since solving this part in two different ways gives results that are not the same. I will demonstrate: In the first method, we use Poynting's theorem in point form (first equation at the top of p. 366), which we modify for the case of time-average fields to read:

$$-\nabla\cdot\mathbf{P}_{z,av} = <\mathbf{J}\cdot\mathbf{E}>$$

where the right hand side is the average power dissipation per volume. Note that the additional right-hand-side terms in Poynting's theorem that describe changes in energy stored in the fields will both be zero in steady state. We apply our equation to the result of part a:

$$<\mathbf{J}\cdot\mathbf{E}> = -\nabla\cdot\mathbf{P}_{z,av} = -\frac{d}{dz}315\,e^{-2(0.2)z} = (0.4)(315)e^{-2(0.2)z} = 126e^{-0.4z}\ \text{W/m}^3$$

At $z = 60$ cm, this becomes $<\mathbf{J}\cdot\mathbf{E}> = 99.1\ \text{W/m}^3$. In the second method, we solve for the conductivity and evaluate $<\mathbf{J}\cdot\mathbf{E}> = \sigma < E^2 >$. We use

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1-j(\epsilon''/\epsilon')}}$$

We take the ratio,

$$\frac{jk}{\eta} = j\omega\epsilon'\left[1 - j\left(\frac{\epsilon''}{\epsilon'}\right)\right] = j\omega\epsilon' + \omega\epsilon''$$

Identifying $\sigma = \omega\epsilon''$, we find

$$\sigma = \text{Re}\left\{\frac{jk}{\eta}\right\} = \text{Re}\left\{\frac{0.2+j2}{250+j30}\right\} = 1.74\times 10^{-3}\ \text{S/m}$$

Now we find the dissipated power per volume:

$$\sigma < E^2 > = 1.74\times 10^{-3}\left(\frac{1}{2}\right)\left(400e^{-0.2z}\right)^2$$

11.17b. (continued) At $z = 60$ cm, this evaluates as 109 W/m^3. One can show that consistency between the two methods requires that

$$\text{Re}\left\{\frac{1}{\eta^*}\right\} = \frac{\sigma}{2\alpha}$$

This relation does not hold using the numbers as given in the problem statement and the value of σ found above. Note that in Problem 11.13, where all values are worked out, the relation does hold and consistent results are obtained using both methods.

11.18a. Find $P(\mathbf{r}, t)$ if $\mathbf{E}_s = 400e^{-j2x}\mathbf{a}_y$ V/m in free space: A positive y component of $\mathbf{E}$ requires a positive z component of $\mathbf{H}$ for propagation in the forward x direction. Thus $\mathbf{H}_s = (400/\eta_0)e^{-j2x}\mathbf{a}_z = 1.06e^{-j2x}\mathbf{a}_z$ A/m. In real form, the field are $\mathbf{E}(x,t) = 400\cos(\omega t - 2x)\mathbf{a}_y$ and $\mathbf{H}(x,t) = 1.06\cos(\omega t - 2x)\mathbf{a}_z$. Now $P(\mathbf{r}, t) = P(x,t) = \mathbf{E}(x,t) \times \mathbf{H}(x,t) = \underline{424.4\cos^2(\omega t - 2x)\mathbf{a}_x \text{ W/m}^2}$.

b) Find P at $t = 0$ for $\mathbf{r} = (a, 5, 10)$, where a = 0,1,2, and 3: At $t = 0$, we find from part a, $P(a, 0) = 424.4\cos^2(2a)$, which leads to the values (in W/m^2): $\underline{424.4 \text{ at } a = 0}$, $\underline{73.5 \text{ at } a = 1}$, $\underline{181.3 \text{ at } a = 2}$, and $\underline{391.3 \text{ at } a = 3}$.

c) Find P at the origin for $T = 0$, $0.2T$, $0.4T$, and $0.6T$, where T is the oscillation period. At the origin, we have $P(0,t) = 424.4\cos^2(\omega t) = 424.4\cos^2(2\pi t/T)$. Using this, we obtain the following values (in W/m^2): $\underline{424.4 \text{ at } t = 0}$, $\underline{42.4 \text{ at } t = 0.2T}$, $\underline{277.8 \text{ at } t = 0.4T}$, and $\underline{277.8 \text{ at } t = 0.6T}$.

11.19. Perfectly-conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_R = 1$. If $\mathbf{E}$ in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find:

a) ω, with the help of Maxwell's equations in cylindrical coordinates: We use the two curl equations, beginning with $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$, where in this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2000}{\rho}\sin(\omega t - 4z)\mathbf{a}_\phi = -\frac{\partial B_\phi}{\partial t}\mathbf{a}_\phi$$

So

$$B_\phi = \int \frac{2000}{\rho}\sin(\omega t - 4z)dt = \frac{2000}{\omega\rho}\cos(\omega t - 4z) \text{ T}$$

Then

$$H_\phi = \frac{B_\phi}{\mu_0} = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z) \text{ A/m}$$

We next use $\nabla \times \mathbf{H} = \partial\mathbf{D}/\partial t$, where in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z$$

where the second term on the right hand side becomes zero when substituting our H_ϕ. So

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)\mathbf{a}_\rho = \frac{\partial D_\rho}{\partial t}\mathbf{a}_\rho$$

And

$$D_\rho = \int -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)dt = \frac{8000}{(4\pi \times 10^{-7})\omega^2\rho}\cos(\omega t - 4z) \text{ C/m}^2$$

11.19a. (continued) Finally, using the given ϵ,

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{8000}{(10^{-16})\omega^2\rho}\cos(\omega t - 4z)\ \text{V/m}$$

This must be the same as the given field, so we require

$$\frac{8000}{(10^{-16})\omega^2\rho} = \frac{500}{\rho} \quad \Rightarrow \quad \omega = \underline{4\times 10^8\ \text{rad/s}}$$

b) $\mathbf{H}(\rho, z, t)$: From part *a*, we have

$$\mathbf{H}(\rho, z, t) = \frac{2000}{(4\pi\times 10^{-7})\omega\rho}\cos(\omega t - 4z)\mathbf{a}_\phi = \underline{\frac{4.0}{\rho}\cos(4\times 10^8 t - 4z)\mathbf{a}_\phi\ \text{A/m}}$$

c) $\mathbf{P}(\rho, \phi, z)$: This will be

$$\begin{aligned}\mathbf{P}(\rho,\phi,z) = \mathbf{E}\times\mathbf{H} &= \frac{500}{\rho}\cos(4\times 10^8 t - 4z)\mathbf{a}_\rho \times \frac{4.0}{\rho}\cos(4\times 10^8 t - 4z)\mathbf{a}_\phi \\ &= \underline{\frac{2.0\times 10^{-3}}{\rho^2}\cos^2(4\times 10^8 t - 4z)\mathbf{a}_z\ \text{W/m}^2}\end{aligned}$$

d) the average power passing through every cross-section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$. Using the result of part *c*, we find $\mathbf{P}_{avg} = (1.0\times 10^3)/\rho^2\mathbf{a}_z$ W/m^2. The power through the given cross-section is now

$$\text{P} = \int_0^{2\pi}\int_{.008}^{.020}\frac{1.0\times 10^3}{\rho^2}\,\rho\,d\rho\,d\phi = 2\pi\times 10^3\ln\left(\frac{20}{8}\right) = \underline{5.7\,\text{kW}}$$

11.20. If $\mathbf{E}_s = (60/r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\theta$ V/m, and $\mathbf{H}_s = (1/4\pi r)\sin\theta\, e^{-j2r}\,\mathbf{a}_\phi$ A/m in free space, find the average power passing outward through the surface $r = 10^6$, $0 < \theta < \pi/3$, and $0 < \phi < 2\pi$.

$$P_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s\times\mathbf{H}_s^*\right\} = \frac{15\sin^2\theta}{2\pi r^2}\mathbf{a}_r\ \text{W/m}^2$$

Then, the requested power will be

$$\begin{aligned}\Phi &= \int_0^{2\pi}\int_0^{\pi/3}\frac{15\sin^2\theta}{2\pi r^2}\mathbf{a}_r\cdot\mathbf{a}_r\,r^2\sin\theta d\theta d\phi = 15\int_0^{\pi/3}\sin^3\theta\,d\theta \\ &= 15\left(-\frac{1}{3}\cos\theta(\sin^2\theta+2)\right)\Big|_0^{\pi/3} = \frac{25}{8} = \underline{3.13\ \text{W}}\end{aligned}$$

Note that the radial distance at the surface, $r = 10^6$ m, makes no difference, since the power density dimishes as $1/r^2$.

11.21. The cylindrical shell, 1 cm < ρ < 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are non-conducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm.

a) Find **H** everywhere: Use Ampere's circuital law, which states:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho(2000) = 2\pi(1.2 \times 10^{-2})(2000) = 48\pi \text{ A} = I_{encl}$$

Then in this case

$$\mathbf{J} = \frac{I}{Area}\mathbf{a}_z = \frac{48}{(1.44 - 1.00) \times 10^{-4}}\mathbf{a}_z = 1.09 \times 10^6\, \mathbf{a}_z \text{ A/m}^2$$

With this result we again use Ampere's circuital law to find **H** everywhere within the shell as a function of ρ (in meters):

$$H_{\phi 1}(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi}\int_{.01}^{\rho} 1.09 \times 10^6\, \rho\, d\rho\, d\phi = \underline{\frac{54.5}{\rho}(10^4\rho^2 - 1) \text{ A/m } (.01 < \rho < .012)}$$

Outside the shell, we would have

$$H_{\phi 2}(\rho) = \frac{48\pi}{2\pi\rho} = \underline{24/\rho \text{ A/m } (\rho > .012)}$$

Inside the shell ($\rho < .01$ m), $H_\phi = 0$ since there is no enclosed current.

b) Find **E** everywhere: We use

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1.09 \times 10^6}{10^6}\mathbf{a}_z = \underline{1.09\, \mathbf{a}_z \text{ V/m}}$$

which is valid, presumeably, outside as well as inside the shell.

c) Find **P** everywhere: Use

$$\begin{aligned}\mathbf{P} = \mathbf{E} \times \mathbf{H} &= 1.09\, \mathbf{a}_z \times \frac{54.5}{\rho}(10^4\rho^2 - 1)\, \mathbf{a}_\phi \\ &= \underline{-\frac{59.4}{\rho}(10^4\rho^2 - 1)\, \mathbf{a}_\rho \text{ W/m}^2 \ (.01 < \rho < .012 \text{ m})}\end{aligned}$$

Outside the shell,

$$\mathbf{P} = 1.09\, \mathbf{a}_z \times \frac{24}{\rho}\, \mathbf{a}_\phi = \underline{-\frac{26}{\rho}\, \mathbf{a}_\rho \text{ W/m}^2 \ (\rho > .012 \text{ m})}$$

11.22. The inner and outer dimensions of a copper coaxial transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than δ. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the:

a) inner conductor: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(4\times 10^8)(4\pi\times 10^{-7})(5.8\times 10^7)}} = 3.3\times 10^{-6}\text{m} = 3.3\mu\text{m}$$

Now, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a\sigma\delta} = \frac{1}{2\pi(2\times 10^{-3})(5.8\times 10^7)(3.3\times 10^{-6})} = \underline{0.42 \text{ ohms/m}}$$

b) outer conductor: Again, (70) applies but with a different conductor radius. Thus

$$R_{out} = \frac{a}{b}R_{in} = \frac{2}{7}(0.42) = \underline{0.12 \text{ ohms/m}}$$

c) transmission line: Since the two resistances found above are in series, the line resistance is their sum, or $R = R_{in} + R_{out} = \underline{0.54 \text{ ohms/m}}$.

11.23. A hollow tubular conductor is constructed from a type of brass having a conductivity of 1.2×10^7 S/m. The inner and outer radii are 9 mm and 10 mm respectively. Calculate the resistance per meter length at a frequency of

a) dc: In this case the current density is uniform over the entire tube cross-section. We write:

$$R(\text{dc}) = \frac{L}{\sigma A} = \frac{1}{(1.2\times 10^7)\pi(.01^2 - .009^2)} = \underline{1.4\times 10^{-3}\ \Omega/\text{m}}$$

b) 20 MHz: Now the skin effect will limit the effective cross-section. At 20 MHz, the skin depth is

$$\delta(20\text{MHz}) = [\pi f \mu_0 \sigma]^{-1/2} = [\pi(20\times 10^6)(4\pi\times 10^{-7})(1.2\times 10^7)]^{-1/2} = 3.25\times 10^{-5}\text{ m}$$

This is much less than the outer radius of the tube. Therefore we can approximate the resistance using the formula:

$$R(20\text{MHz}) = \frac{L}{\sigma A} = \frac{1}{2\pi b\delta} = \frac{1}{(1.2\times 10^7)(2\pi(.01))(3.25\times 10^{-5})} = \underline{4.1\times 10^{-2}\ \Omega/\text{m}}$$

c) 2 GHz: Using the same formula as in part *b*, we find the skin depth at 2 GHz to be $\delta = 3.25\times 10^{-6}$ m. The resistance (using the other formula) is $R(2\text{GHz}) = \underline{4.1\times 10^{-1}\ \Omega/\text{m}}$.

11.24a. Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_R = 500$ for the stainless steel interior, and find the depth of penetration:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.45 \times 10^9)(4\pi \times 10^{-7})(1.2 \times 10^6)}} = 9.28 \times 10^{-6}\text{m} = 9.28\mu\text{m}$$

b) Let $E_s = 50\angle 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of E_s vs. the angle of E_s as the field propagates into the stainless steel: Since the conductivity is high, we use (62) to write $\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma} = 1/\delta$. So, assuming that the direction into the conductor is z, the depth-dependent field is written as

$$E_s(z) = 50e^{-\alpha z}e^{-j\beta z} = 50e^{-z/\delta}e^{-jz/\delta} = \underbrace{50\exp(-z/9.28)}_{\text{amplitude}}\exp(-j\underbrace{z/9.28}_{\text{angle}})$$

where z is in microns. Therefore, the plot of amplitude versus angle is simply a plot of e^{-x} versus x, where $x = z/9.28$; the starting amplitude is 50 and the $1/e$ amplitude (at $z = 9.28$ μm) is 18.4.

11.25. A good conductor is planar in form and carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of 3×10^5 m/s. Assuming the conductor is non-magnetic, determine the frequency and the conductivity: First, we use

$$f = \frac{v}{\lambda} = \frac{3 \times 10^5}{3 \times 10^{-4}} = 10^9 \text{ Hz} = \underline{1 \text{ GHz}}$$

Next, for a good conductor,

$$\delta = \frac{\lambda}{2\pi} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \Rightarrow \quad \sigma = \frac{4\pi}{\lambda^2 f \mu} = \frac{4\pi}{(9 \times 10^{-8})(10^9)(4\pi \times 10^{-7})} = \underline{1.1 \times 10^5 \text{ S/m}}$$

11.26. The dimensions of a certain coaxial transmission line are $a = 0.8$mm and $b = 4$mm. The outer conductor thickness is 0.6mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m.

a) Find R, the resistance per unit length, at an operating frequency of 2.4 GHz: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.4 \times 10^8)(4\pi \times 10^{-7})(1.6 \times 10^7)}} = 2.57 \times 10^{-6}\text{m} = 2.57\mu\text{m}$$

Then, using (70) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(0.8 \times 10^{-3})(1.6 \times 10^7)(2.57 \times 10^{-6})} = 4.84 \text{ ohms/m}$$

The outer conductor resistance is then found from the inner through

$$R_{out} = \frac{a}{b}R_{in} = \frac{0.8}{4}(4.84) = 0.97 \text{ ohms/m}$$

The net resistance per length is then the sum, $R = R_{in} + R_{out} = \underline{5.81 \text{ ohms/m}}$.

11.26b. Use information from Secs. 5.10 and 9.10 to find C and L, the capacitance and inductance per unit length, respectively. The coax is air-filled. From those sections, we find (in free space)

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.854\times10^{-12})}{\ln(4/.8)} = \underline{3.46\times10^{-11}\ \text{F/m}}$$

$$L = \frac{\mu_0}{2\pi}\ln(b/a) = \frac{4\pi\times10^{-7}}{2\pi}\ln(4/.8) = \underline{3.22\times10^{-7}\ \text{H/m}}$$

c) Find α and β if $\alpha + j\beta = \sqrt{j\omega C(R+j\omega L)}$: Taking real and imaginary parts of the given expression, we find

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R+j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{R}{\omega L}\right)^2}-1\right]^{1/2}$$

and

$$\beta = \text{Im}\left\{\sqrt{j\omega C(R+j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}}\left[\sqrt{1+\left(\frac{R}{\omega L}\right)^2}+1\right]^{1/2}$$

These can be found by writing out $\alpha = \text{Re}\left\{\sqrt{j\omega C(R+j\omega L)}\right\} = (1/2)\sqrt{j\omega C(R+j\omega L)}+c.c.$, where $c.c$ denotes the complex conjugate. The result is squared, terms collected, and the square root taken. Now, using the values of R, C, and L found in parts a and b, we find $\alpha = \underline{3.0\times10^{-2}\ \text{Np/m}}$ and $\beta = \underline{50.3\ \text{rad/m}}$.

11.27. The planar surface at $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4\times10^{10}$ rad/s:

a) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$: From the appendix we find $\epsilon''/\epsilon' = .0003$ for Teflon, making the material a good dielectric. Also, for Teflon, $\epsilon'_R = 2.1$. For brass, we find $\sigma = 1.5\times10^7$ S/m, making brass a good conductor at the stated frequency. For a good dielectric (Teflon) we use the approximations:

$$\alpha \doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} = \left(\frac{\epsilon''}{\epsilon'}\right)\left(\frac{1}{2}\right)\omega\sqrt{\mu\epsilon'} = \frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\omega}{c}\sqrt{\epsilon'_R}$$

$$\beta \doteq \omega\sqrt{\mu\epsilon'}\left[1+\frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)\right] \doteq \omega\sqrt{\mu\epsilon\prime} = \frac{\omega}{c}\sqrt{\epsilon'_R}$$

For brass (good conductor) we have

$$\alpha \doteq \beta \doteq \sqrt{\pi f\mu\sigma_{\text{brass}}} = \sqrt{\pi\left(\frac{1}{2\pi}\right)(4\times10^{10})(4\pi\times10^{-7})(1.5\times10^7)} = 6.14\times10^5\ \text{m}^{-1}$$

Now

$$\frac{\alpha_{\text{Tef}}}{\alpha_{\text{brass}}} = \frac{1/2\left(\epsilon''/\epsilon'\right)(\omega/c)\sqrt{\epsilon'_R}}{\sqrt{\pi f\mu\sigma_{\text{brass}}}} = \frac{(1/2)(.0003)(4\times10^{10}/3\times10^8)\sqrt{2.1}}{6.14\times10^5} = \underline{4.7\times10^{-8}}$$

b)

$$\frac{\lambda_{\text{Tef}}}{\lambda_{\text{brass}}} = \frac{(2\pi/\beta_{\text{Tef}})}{(2\pi/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \frac{c\sqrt{\pi f\mu\sigma_{\text{brass}}}}{\omega\sqrt{\epsilon'_{R\,\text{Tef}}}} = \frac{(3\times10^8)(6.14\times10^5)}{(4\times10^{10})\sqrt{2.1}} = \underline{3.2\times10^3}$$

11.27. (continued)

c)

$$\frac{v_{\text{Tef}}}{v_{\text{brass}}} = \frac{(\omega/\beta_{\text{Tef}})}{(\omega/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \underline{3.2 \times 10^3} \text{ as before}$$

11.28. A uniform plane wave in free space has electric field given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m.

a) Describe the wave polarization: Since the two components have a fixed phase difference (in this case zero) with respect to time and position, the wave has <u>linear polarization</u>, with the field vector in the yz plane at angle $\phi = \tan^{-1}(10/15) = 33.7^\circ$ to the y axis.

b) Find $\mathbf{H}_s$: With propagation in forward x, we would have

$$\mathbf{H}_s = \frac{-10}{377}e^{-j\beta x}\mathbf{a}_y + \frac{15}{377}e^{-j\beta x}\mathbf{a}_z \text{ A/m} = \underline{-26.5e^{-j\beta x}\mathbf{a}_y + 39.8e^{-j\beta x}\mathbf{a}_z \text{ mA/m}}$$

c) determine the average power density in the wave in W/m^2: Use

$$\mathbf{P}_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left[\frac{(10)^2}{377}\mathbf{a}_x + \frac{(15)^2}{377}\mathbf{a}_x\right] = 0.43\mathbf{a}_x \text{ W/m}^2 \text{ or } P_{avg} = \underline{0.43 \text{ W/m}^2}$$

11.29. Consider a left-circularly polarized wave in free space that propagates in the forward z direction. The electric field is given by the appropriate form of Eq. (80).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (80), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the x component of $\mathbf{E}_s$ is associated with a y component of $\mathbf{H}_s$, and the y component of $\mathbf{E}_s$ is associated with a negative x component of $\mathbf{H}_s$. The result is

$$\underline{\mathbf{H}_s = \frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right)e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m^2 by direct application of Eq. (57): We have

$$\mathbf{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \text{ (assuming } E_0 \text{ is real)}$$

11.30. The electric field of a uniform plane wave in free space is given by $\mathbf{E}_s = 10(\mathbf{a}_y + j\mathbf{a}_z)e^{-j25x}$.

a) Determine the frequency, f: Use

$$f = \frac{\beta c}{2\pi} = \frac{(25)(3\times 10^8)}{2\pi} = \underline{1.2\text{ GHz}}$$

b) Find the magnetic field phasor, $\mathbf{H}_s$: With the Poynting vector in the positive x direction, a positive y component for $\mathbf{E}$ requires a positive z component for $\mathbf{H}$. Similarly, a positive z component for $\mathbf{E}$ requires a negative y component for $\mathbf{H}$. Therefore,

$$\mathbf{H}_s = \underline{\frac{10}{\eta_0}\left[\mathbf{a}_z - j\mathbf{a}_y\right]e^{-j25x}}$$

c) Describe the polarization of the wave: This is most clearly seen by first converting the given field to real instantaneous form:

$$\mathbf{E}(x,t) = \text{Re}\left\{\mathbf{E}_s e^{j\omega t}\right\} = 10\left[\cos(\omega t - 25x)\mathbf{a}_y - \sin(\omega t - 25x)\mathbf{a}_z\right]$$

At $x = 0$, this becomes,

$$\mathbf{E}(0,t) = 10\left[\cos(\omega t)\mathbf{a}_y - \sin(\omega t)\mathbf{a}_z\right]$$

With the wave traveling in the forward x direction, we recognize the polarization as <u>left circular</u>.

11.31. A linearly-polarized uniform plane wave, propagating in the forward z direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along y (ϵ_{Ry}) differs from that seen by waves polarized along x (ϵ_{Rx}). Suppose $\epsilon_{Rx} = 2.15$, $\epsilon_{Ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive x and y axes. *Assume free space wavelength* λ.

a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized: With the input field at 45°, the x and y components are of equal magnitude, and circular polarization will result if the phase difference between the components is $\pi/2$. Our requirement over length L is thus $\beta_x L - \beta_y L = \pi/2$, or

$$L = \frac{\pi}{2(\beta_x - \beta_y)} = \frac{\pi c}{2\omega(\sqrt{\epsilon_{Rx}} - \sqrt{\epsilon_{Ry}})}$$

With the given values, we find,

$$L = \frac{(58.3)\pi c}{2\omega} = 58.3\,\frac{\lambda}{4} = \underline{14.6\,\lambda}$$

b) Will the output wave be right- or left-circularly-polarized? With the dielectric constant greater for x-polarized waves, the x component will lag the y component in time at the output. The field can thus be written as $\mathbf{E} = E_0(\mathbf{a}_y - j\mathbf{a}_x)$, which is <u>left circular polarization</u>.

11.32. Suppose that the length of the medium of Problem 11.31 is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case: With the length doubled, a phase shift of π radians develops between the two components. At the input, we can write the field as $\mathbf{E}_s(0) = E_0(\mathbf{a}_x + \mathbf{a}_y)$. After propagating through length L, we would have,

$$\mathbf{E}_s(L) = E_0[e^{-j\beta_x L}\mathbf{a}_x + e^{-j\beta_y L}\mathbf{a}_y] = E_0 e^{-j\beta_x L}[\mathbf{a}_x + e^{-j(\beta_y-\beta_x)L}\mathbf{a}_y]$$

where $(\beta_y - \beta_x)L = -\pi$ (since $\beta_x > \beta_y$), and so $\mathbf{E}_s(L) = E_0 e^{-j\beta_x L}[\mathbf{a}_x - \mathbf{a}_y]$. With the reversal of the y component, the wave polarization is rotated by 90°, but is still linear polarization.

11.33. Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m, propagating in a medium characterized by complex intrinsic impedance, η.

a) Find $\mathbf{H}_s$: With the wave propagating in the forward z direction, we find:

$$\mathbf{H}_s = \frac{1}{\eta}\left[-18e^{j\phi}\mathbf{a}_x + 15\mathbf{a}_y\right]e^{-j\beta z}\ \text{A/m}$$

b) Determine the average power density in W/m^2: We find

$$P_{z,avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\text{Re}\left\{\frac{(15)^2}{\eta^*} + \frac{(18)^2}{\eta^*}\right\} = 275\,\text{Re}\left\{\frac{1}{\eta^*}\right\}\ \text{W/m}^2$$

11.34. Given the general elliptically-polarized wave as per Eq. (73):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

a) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where δ is a constant: Adding the two fields gives

$$\begin{aligned}\mathbf{E}_{s,tot} &= \left[E_{x0}\left(1 + e^{j\delta}\right)\mathbf{a}_x + E_{y0}\left(e^{j\phi} + e^{-j\phi}e^{j\delta}\right)\mathbf{a}_y\right]e^{-j\beta z} \\ &= \left[E_{x0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2} + e^{j\delta/2}\right)}_{2\cos(\delta/2)}\mathbf{a}_x + E_{y0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}e^{j\phi} + e^{-j\phi}e^{j\delta/2}\right)}_{2\cos(\phi-\delta/2)}\mathbf{a}_y\right]e^{-j\beta z}\end{aligned}$$

This simplifies to $\mathbf{E}_{s,tot} = 2\left[E_{x0}\cos(\delta/2)\mathbf{a}_x + E_{y0}\cos(\phi - \delta/2)\mathbf{a}_y\right]e^{j\delta/2}e^{-j\beta z}$, which is linearly polarized.

b) Find δ in terms of ϕ such that the resultant wave is polarized along x: By inspecting the part a result, we achieve a zero y component when $2\phi - \delta = \pi$ (or odd multiples of π).

CHAPTER 12

12.1. A uniform plane wave in air, $E_{x1}^{+} = E_{x10}^{+}\cos(10^{10}t - \beta z)$ V/m, is normally-incident on a copper surface at $z = 0$. What percentage of the incident power density is transmitted into the copper? We need to find the reflection coefficient. The intrinsic impedance of copper (a good conductor) is

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\sqrt{\frac{10^{10}(4\pi \times 10^{7})}{2(5.8 \times 10^{7})}} = (1+j)(.0104)$$

Note that the accuracy here is questionable, since we know the conductivity to only two significant figures. We nevertheless proceed: Using $\eta_0 = 376.7288$ ohms, we write

$$\Gamma = \frac{\eta_c - \eta_0}{\eta_c + \eta_0} = \frac{.0104 - 376.7288 + j.0104}{.0104 + 376.7288 + j.0104} = -.9999 + j.0001$$

Now $|\Gamma|^2 = .9999$, and so the transmitted power fraction is $1 - |\Gamma|^2 = .0001$, or about $\underline{0.01\%}$ is transmitted.

12.2. The plane $y = 0$ defines the boundary between two different dielectrics. For $y < 0$, $\epsilon'_{R1} = 1$, $\mu_1 = \mu_0$, and $\epsilon''_{R1} = 0$; and for $y > 0$, $\epsilon'_{R2} = 5$, $\mu_2 = \mu_0$, and $\epsilon''_{R2} = 0$. Let $E_{z1}^{+} = 150\cos(\omega t - 8y)$ V/m, and find

a) ω: Have $\beta = 8 = \omega/c \Rightarrow \omega = 8c = \underline{2.4 \times 10^{9}\ \text{sec}^{-1}}$.

b) $\mathbf{H}_1^{+}$: With E in the z direction, and propagation in the forward y direction, H will lie in the positive x direction, and its amplitude will be $H_x = E_y/\eta_0$ in region 1.
Thus $\mathbf{H}_1^{+} = (150/\eta_0)\cos(\omega t - 8y)\mathbf{a}_x = \underline{0.40\cos(2.4 \times 10^{9}t - 8y)\mathbf{a}_x\ \text{A/m}}$.

c) $\mathbf{H}_1^{-}$: First,

$$E_{z1}^{-} = \Gamma E_{z1}^{+} = \frac{\eta_0/\sqrt{5} - \eta_0/1}{\eta_0/\sqrt{5} + \eta_0/1} = \frac{1-\sqrt{5}}{1+\sqrt{5}}E_{z1}^{+} = -0.38E_{z1}^{+}$$

Then

$$H_{x1}^{-} = +(0.38/\eta_0)E_{z1}^{+} = \frac{0.38(150)}{377}\cos(\omega t + 8y)$$

So finally, $\mathbf{H}_{x1}^{-} = \underline{0.15\cos(2.4 \times 10^{9}t + 8y)\mathbf{a}_x\ \text{A/m}}$.

12.3. A uniform plane wave in region 1 is normally-incident on the planar boundary separating regions 1 and 2. If $\epsilon''_1 = \epsilon''_2 = 0$, while $\epsilon'_{R1} = \mu_{R1}^3$ and $\epsilon'_{R2} = \mu_{R2}^3$, find the ratio $\epsilon'_{R2}/\epsilon'_{R1}$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers. First, since $|\Gamma|^2 = .20$, and since both permittivities and permeabilities are real, $\Gamma = \pm 0.447$. we then set up

$$\Gamma = \pm 0.447 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{(\mu_{R2}/\epsilon'_{R2})} - \eta_0\sqrt{(\mu_{R1}/\epsilon'_{R1})}}{\eta_0\sqrt{(\mu_{R2}/\epsilon'_{R2})} + \eta_0\sqrt{(\mu_{R1}/\epsilon'_{R1})}}$$

$$= \frac{\sqrt{(\mu_{R2}/\mu_{R2}^3)} - \sqrt{(\mu_{R1}/\mu_{R1}^3)}}{\sqrt{(\mu_{R2}/\mu_{R2}^3)} + \sqrt{(\mu_{R1}/\mu_{R1}^3)}} = \frac{\mu_{R1} - \mu_{R2}}{\mu_{R1} + \mu_{R2}}$$

12.3. (continued) Therefore

$$\frac{\mu_{R2}}{\mu_{R1}} = \frac{1 \mp 0.447}{1 \pm 0.447} = (0.382, 2.62) \quad \Rightarrow \quad \frac{\epsilon'_{R2}}{\epsilon'_{R1}} = \left(\frac{\mu_{R2}}{\mu_{R1}}\right)^3 = \underline{(0.056, 17.9)}$$

12.4. The magnetic field intensity in a region where $\epsilon'' = 0$ is given as $\mathbf{H} = 5\cos\omega t \cos\beta z\, \mathbf{a}_y$ A/m, where $\omega = 5$ Grad/s and $\beta = 30$ rad/m. If the amplitude of the associated electric field intensity is 2kV/m, find

a) μ and ϵ' for the medium: In phasor form, the magnetic field is $H_{ys} = H_0 e^{-j\beta z} + H_0 e^{+\beta z} = 5\cos\beta z \;\Rightarrow\; H_0 = 2.5$. The electric field will be x directed, and is $E_{xs} = \eta(2.5)e^{-j\beta z} - \eta(2.5)e^{+j\beta z} = (2j)\eta(2.5)\sin\beta z$. Given the electric field amplitude of 2 kV/m, we write $2\times10^3 = 5\eta$, or $\eta = 400\,\Omega$. Now $\eta = 400 = \eta_0\sqrt{\mu_r/\epsilon'_R}$ and we also have $\beta = 30 = (\omega/c)\sqrt{\mu_R\epsilon'_R}$. We solve these two equations simultaneously for μ_R and ϵ'_R to find $\mu_R = 1.91$ and $\epsilon'_R = 1.70$. Therefore $\mu = 1.91 \times 4\pi \times 10^{-7} = \underline{2.40\,\mu\text{H/m}}$ and $\epsilon' = 1.70 \times 8.854 \times 10^{-12} = \underline{15.1\text{ pF/m}}$.

b) $\mathbf{E}$: From part a, electric field in phasor form is $E_{xs} = j2\sin\beta z$ kV/m, and so, in real form: $\mathbf{E}(z,t) = \text{Re}(E_{xs}e^{j\omega t})\mathbf{a}_x = \underline{2\sin\beta z\sin\omega t\, \mathbf{a}_x\text{ kV/m}}$ with ω and β as given.

12.5. The region $z < 0$ is characterized by $\epsilon'_R = \mu_R = 1$ and $\epsilon''_R = 0$. The total $\mathbf{E}$ field here is given as the sum of the two uniform plane waves, $\mathbf{E}_s = 150e^{-j10z}\,\mathbf{a}_x + (50\angle 20^\circ)e^{j10z}\,\mathbf{a}_x$ V/m.

a) What is the operating frequency? In free space, $\beta = k_0 = 10 = \omega/c = \omega/3\times10^8$. Thus, $\omega = 3\times10^9$ s^{-1}, or $f = \omega/2\pi = \underline{4.7\times10^8\text{ Hz}}$.

b) Specify the intrinsic impedance of the region $z > 0$ that would provide the appropriate reflected wave: Use

$$\Gamma = \frac{E_r}{E_{inc}} = \frac{50e^{j20^\circ}}{150} = \frac{1}{3}e^{j20^\circ} = 0.31 + j0.11 = \frac{\eta - \eta_0}{\eta + \eta_0}$$

Now

$$\eta = \eta_0\left(\frac{1+\Gamma}{1-\Gamma}\right) = 377\left(\frac{1+0.31+j0.11}{1-0.31-j0.31}\right) = \underline{691 + j177\,\Omega}$$

c) At what value of z ($-10\text{ cm} < z < 0$) is the total electric field intensity a maximum amplitude? We found the phase of the reflection coefficient to be $\phi = 20^\circ = .349$rad, and we use

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-.349}{20} = -0.017\text{ m} = \underline{-1.7\text{ cm}}$$

12.6. Region 1, $z < 0$, and region 2, $z > 0$, are described by the following parameters: $\epsilon'_1 = 100$ pF/m, $\mu_1 = 25\,\mu$H/m, $\epsilon''_1 = 0$, $\epsilon'_2 = 200$ pF/m, $\mu_2 = 50\,\mu$H/m, and $\epsilon''_2/\epsilon'_2 = 0.5$.
If $\mathbf{E}_1^+ = 600e^{-\alpha_1 z}\cos(5\times10^{10}t - \beta_1 z)\mathbf{a}_x$ V/m, find:

a) α_1: From Eq. (35), Chapter 11, we note that since $\epsilon''_1 = 0$, it follows that $\alpha_1 = \underline{0}$.

b) β_1: $\beta_1 = \omega\sqrt{\mu_1\epsilon'_1} = (5\times10^{10})\sqrt{(25\times10^{-6})(100\times10^{-12})} = \underline{2.50\times10^3\text{ rad/m}}$.

c) $\mathbf{E}_{s1}^+ = \underline{600e^{-j2.50\times10^3 z}\mathbf{a}_x\text{ V/m}}$.

d) $\mathbf{E}_{s1}^-$: To find this, we need to evaluate the reflection coefficient, which means that we first need the two intrinsic impedances. First, $\eta_1 = \sqrt{\mu_1/\epsilon'_1} = \sqrt{(25\times10^{-6})/(100\times10^{-12})} = 500$.

12.6d) (continued) Next, using Eq. (39), Chapter 11,

$$\eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2'}}\frac{1}{\sqrt{1-j(\epsilon_2''/\epsilon_2')}} = \sqrt{\frac{50\times10^{-6}}{2\times10^{-10}}}\frac{1}{\sqrt{1-j0.5}} = 460 + j109$$

Then

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{460+j109-500}{460+j109+500} = -2.83\times10^{-2} + j1.16\times10^{-1} = 0.120e^{j104^\circ}$$

Now we multiply $\mathbf{E}_{s1}^+$ by Γ and reverse the propagation direction to obtain

$$\mathbf{E}_{s1}^- = \underline{71.8e^{j104^\circ}e^{j2.5\times10^3 z}\ \text{V/m}}$$

e) $\mathbf{E}_{s2}^+$: This wave will experience loss in region 2, along with a different phase constant. We need to evaluate α_2 and β_2. First, using Eq. (35), Chapter 11,

$$\alpha_2 = \omega\sqrt{\frac{\mu_2\epsilon_2'}{2}}\left[\sqrt{1+\left(\frac{\epsilon_2''}{\epsilon_2'}\right)^2}-1\right]^{1/2}$$

$$= (5\times10^{10})\sqrt{\frac{(50\times10^6)(200\times10^{-12})}{2}}\left[\sqrt{1+(0.5)^2}-1\right]^{1/2} = 1.21\times10^3\ \text{Np/m}$$

Then, using Eq. (36), Chapter 11,

$$\beta_2 = \omega\sqrt{\frac{\mu_2\epsilon_2'}{2}}\left[\sqrt{1+\left(\frac{\epsilon_2''}{\epsilon_2'}\right)^2}+1\right]^{1/2} = 5.15\times10^3\ \text{rad/m}$$

Then, the transmission coefficient will be

$$\tau = 1+\Gamma = 1-2.83\times10^{-2}+j1.16\times10^{-1} = 0.972e^{j7^\circ}$$

The complex amplitude of $\mathbf{E}_{s2}^+$ is then found by multiplying the amplitude of $\mathbf{E}_{s1}^+$ by τ. The field in region 2 is then constructed by using the resulting amplitude, along with the attenuation and phase constants that are appropriate for region 2. The result is

$$\mathbf{E}_{s2}^+ = \underline{587e^{-1.21\times10^3 z}e^{j7^\circ}e^{-j5.15\times10^3 z}\ \text{V/m}}$$

12.7. The semi-infinite regions $z<0$ and $z>1$ m are free space. For $0<z<1$ m, $\epsilon_R'=4$, $\mu_R=1$, and $\epsilon_R''=0$. A uniform plane wave with $\omega = 4\times10^8$ rad/s is travelling in the $\mathbf{a}_z$ direction toward the interface at $z=0$.

a) Find the standing wave ratio in each of the three regions: First we find the phase constant in the middle region,

$$\beta_2 = \frac{\omega\sqrt{\epsilon_R'}}{c} = \frac{2(4\times10^8)}{3\times10^8} = 2.67\,\text{rad/m}$$

12.7a. (continued) Then, with the middle layer thickness of 1 m, $\beta_2 d = 2.67$ rad. Also, the intrinsic impedance of the middle layer is $\eta_2 = \eta_0/\sqrt{\epsilon'_R} = \eta_0/2$. We now find the input impedance:

$$\eta_{in} = \eta_2 \left[\frac{\eta_0 \cos(\beta_2 d) + j\eta_2 \sin(\beta_2 d)}{\eta_2 \cos(\beta_2 d) + j\eta_0 \sin(\beta_2 d)} \right] = \frac{377}{2} \left[\frac{2\cos(2.67) + j\sin(2.67)}{\cos(2.67) + j2\sin(2.67)} \right] = 231 + j141$$

Now, at the first interface,

$$\Gamma_{12} = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{231 + j141 - 377}{231 + j141 + 377} = -.176 + j.273 = .325\angle 123^\circ$$

The standing wave ratio measured in region 1 is thus

$$s_1 = \frac{1 + |\Gamma_{12}|}{1 - |\Gamma_{12}|} = \frac{1 + 0.325}{1 - 0.325} = \underline{1.96}$$

In region 2 the standing wave ratio is found by considering the reflection coefficient for waves incident from region 2 on the second interface:

$$\Gamma_{23} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1 - 1/2}{1 + 1/2} = \frac{1}{3}$$

Then

$$s_2 = \frac{1 + 1/3}{1 - 1/3} = \underline{2}$$

Finally, $s_3 = \underline{1}$, since no reflected waves exist in region 3.

b) Find the location of the maximum $|\mathbf{E}|$ for $z < 0$ that is nearest to $z = 0$. We note that the phase of Γ_{12} is $\phi = 123^\circ = 2.15$ rad. Thus

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-2.15}{2(4/3)} = \underline{-.81 \text{ m}}$$

12.8. A wave starts at point a, propagates 100m through a lossy dielectric for which $\alpha = 0.5$ Np/m, reflects at normal incidence at a boundary at which $\Gamma = 0.3 + j0.4$, and then returns to point a. Calculate the ratio of the final power to the incident power after this round trip: Final power, P_f, and incident power, P_i, are related through

$$P_f = P_i e^{-2\alpha L} |\Gamma|^2 e^{-2\alpha L} \quad \Rightarrow \quad \frac{P_f}{P_i} = |0.3 + j0.4|^2 e^{-2(0.5)100} = \underline{3.5 \times 10^{-88}} (!)$$

Try measuring that.

12.9. Region 1, $z < 0$, and region 2, $z > 0$, are both perfect dielectrics ($\mu = \mu_0$, $\epsilon'' = 0$). A uniform plane wave traveling in the $\mathbf{a}_z$ direction has a radian frequency of 3×10^{10} rad/s. Its wavelengths in the two regions are $\lambda_1 = 5$ cm and $\lambda_2 = 3$ cm. What percentage of the energy incident on the boundary is

a) reflected; We first note that

$$\epsilon'_{R1} = \left(\frac{2\pi c}{\lambda_1 \omega} \right)^2 \text{ and } \epsilon'_{R2} = \left(\frac{2\pi c}{\lambda_2 \omega} \right)^2$$

12.9a. (continued) Therefore $\epsilon'_{R1}/\epsilon'_{R2} = (\lambda_2/\lambda_1)^2$. Then with $\mu = \mu_0$ in both regions, we find

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{1/\epsilon'_{R2}} - \eta_0\sqrt{1/\epsilon'_{R1}}}{\eta_0\sqrt{1/\epsilon'_{R2}} + \eta_0\sqrt{1/\epsilon'_{R1}}} = \frac{\sqrt{\epsilon'_{R1}/\epsilon'_{R2}} - 1}{\sqrt{\epsilon'_{R1}/\epsilon'_{R2}} + 1} = \frac{(\lambda_2/\lambda_1) - 1}{(\lambda_2/\lambda_1) + 1}$$
$$= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3-5}{3+5} = -\frac{1}{4}$$

The fraction of the incident energy that is reflected is then $|\Gamma|^2 = 1/16 = \underline{6.25 \times 10^{-2}}$.

b) transmitted? We use part *a* and find the transmitted fraction to be $1 - |\Gamma|^2 = 15/16 = \underline{0.938}$.

c) What is the standing wave ratio in region 1? Use

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/4}{1 - 1/4} = \frac{5}{3} = \underline{1.67}$$

12.10. In Fig. 12.1, let region 2 be free space, while $\mu_{R1} = 1$, $\epsilon''_{R1} = 0$, and ϵ'_{R1} is unknown. Find $\epsilon'_{R!}$ if

a) the amplitude of $\mathbf{E}_1^-$ is one-half that of $\mathbf{E}_1^+$: Since region 2 is free space, the reflection coefficient is

$$\Gamma = \frac{|\mathbf{E}_1^-|}{|\mathbf{E}_1^+|} = \frac{\eta_0 - \eta_1}{\eta_0 + \eta_1} = \frac{\eta_0 - \eta_0/\sqrt{\epsilon'_{R1}}}{\eta_0 + \eta_0/\sqrt{\epsilon'_{R1}}} = \frac{\sqrt{\epsilon'_{R1}} - 1}{\sqrt{\epsilon'_{R1}} + 1} = \frac{1}{2} \;\Rightarrow\; \epsilon'_{R1} = \underline{9}$$

.

b) $P^-_{1,avg}$ is one-half of $P^+_{1,avg}$: This time

$$|\Gamma|^2 = \left|\frac{\sqrt{\epsilon'_{R1}} - 1}{\sqrt{\epsilon'_{R1}} + 1}\right|^2 = \frac{1}{2} \;\Rightarrow\; \epsilon'_{R1} = \underline{34}$$

c) $|\mathbf{E}_1|_{min}$ is one-half $|\mathbf{E}_1|_{max}$: Use

$$\frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2 \;\Rightarrow\; |\Gamma| = \Gamma = \frac{1}{3} = \frac{\sqrt{\epsilon'_{R1}} - 1}{\sqrt{\epsilon'_{R1}} + 1} \;\Rightarrow\; \epsilon'_{R1} = \underline{4}$$

12.11. A 150 MHz uniform plane wave in normally-incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material: First, the field minimum is used to find the phase of the reflection coefficient, where

$$z_{min} = -\frac{1}{2\beta}(\phi + \pi) = -0.3\lambda \;\Rightarrow\; \phi = 0.2\pi$$

where $\beta = 2\pi/\lambda$ has been used. Next,

$$|\Gamma| = \frac{s-1}{s+1} = \frac{3-1}{3+1} = \frac{1}{2}$$

12.11. (continued) So we now have

$$\Gamma = 0.5e^{j0.2\pi} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

We solve for η_u to find

$$\eta_u = \eta_0(1.70 + j1.33) = \underline{641 + j501\ \Omega}$$

12.12. A 50MHz uniform plane wave is normally incident from air onto the surface of a calm ocean. For seawater, $\sigma = 4$ S/m, and $\epsilon'_R = 78$.

a) Determine the fractions of the incident power that are reflected and transmitted: First we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50 \times 10^6)(78)(8.854 \times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 65, Chapter 11), the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1 + j)$, and the reflection coefficient becomes

$$\Gamma = \frac{\sqrt{\pi f \mu/\sigma}\ (1 + j) - \eta_0}{\sqrt{\pi f \mu/\sigma}\ (1 + j) + \eta_0}$$

where $\sqrt{\pi f \mu/\sigma} = \sqrt{\pi(50 \times 10^6)(4\pi \times 10^{-7})/4} = 7.0$. The fraction of the power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = \frac{[\sqrt{\pi f \mu/\sigma} - \eta_0]^2 + \pi f \mu/\sigma}{[\sqrt{\pi f \mu/\sigma} + \eta_0]^2 + \pi f \mu/\sigma} = \frac{[7.0 - 377]^2 + 49.0}{[7.0 + 377]^2 + 49.0} = \underline{0.93}$$

The transmitted fraction is then

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.93 = \underline{0.07}$$

b) Qualitatively, how will these answers change (if at all) as the frequency is increased? Within the limits of our good conductor approximation (loss tangent greater than about ten), the reflected power fraction, using the formula derived in part a, is found to <u>decrease</u> with increasing frequency. The transmitted power fraction thus increases.

12.13. A right-circularly-polarized plane wave is normally incident from air onto a semi-infinite slab of plexiglas ($\epsilon'_R = 3.45$, $\epsilon''_R = 0$). Calculate the fractions of the incident power that are reflected and transmitted. Also, describe the polarizations of the reflected and transmitted waves. First, the impedance of the plexiglas will be $\eta = \eta_0/\sqrt{3.45} = 203\,\Omega$. Then

$$\Gamma = \frac{203 - 377}{203 + 377} = -0.30$$

The reflected power fraction is thus $|\Gamma|^2 = \underline{0.09}$. The total electric field in the plane of the interface must rotate in the same direction as the incident field, in order to continually satisfy the boundary condition of tangential electric field continuity across the interface. Therefore, the reflected wave will have to be <u>left circularly polarized</u> in order to make this happen. The transmitted power fraction is now $1 - |\Gamma|^2 = \underline{0.91}$. The transmitted field will be <u>right circularly polarized</u> (as the incident field) for the same reasons.

12.14. A left-circularly-polarized plane wave is normally-incident onto the surface of a perfect conductor.

a) Construct the superposition of the incident and reflected waves in phasor form: Assume positive z travel for the incident electric field. Then, with reflection coefficient, $\Gamma = -1$, the incident and reflected fields will add to give the total field:

$$\mathbf{E}_{tot} = \mathbf{E}_i + \mathbf{E}_r = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} - E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{+j\beta z}$$

$$= E_0\left[\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)}\mathbf{a}_x + j\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)}\mathbf{a}_y\right] = \underline{2E_0\sin(\beta z)\left[\mathbf{a}_y - j\mathbf{a}_x\right]}$$

b) Determine the real instantaneous form of the result of part a:

$$\mathbf{E}(z,t) = \text{Re}\left\{\mathbf{E}_{tot}e^{j\omega t}\right\} = \underline{2E_0\sin(\beta z)\left[\cos(\omega t)\mathbf{a}_y + \sin(\omega t)\mathbf{a}_x\right]}$$

c) Describe the wave that is formed: This is a standing wave exhibiting circular polarization in time. At each location along the z axis, the field vector rotates clockwise in the xy plane, and has amplitude (constant with time) given by $2E_0\sin(\beta z)$.

12.15. Consider these regions in which $\epsilon'' = 0$: region 1, $z < 0$, $\mu_1 = 4\,\mu$H/m and $\epsilon_1' = 10$ pF/m; region 2, $0 < z < 6$ cm, $\mu_2 = 2\,\mu H/m$, $\epsilon_2' = 25$ pF/m; region 3, $z > 6$ cm, $\mu_3 = \mu_1$ and $\epsilon_3' = \epsilon_1'$.

a) What is the lowest frequency at which a uniform plane wave incident from region 1 onto the boundary at $z = 0$ will have no reflection? This frequency gives the condition $\beta_2 d = \pi$, where $d = 6$ cm, and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2'}$ Therefore

$$\beta_2 d = \pi \;\Rightarrow\; \omega = \frac{\pi}{(.06)\sqrt{\mu_2\epsilon_2'}} \;\Rightarrow\; f = \frac{1}{0.12\sqrt{(2\times10^{-6})(25\times10^{-12})}} = \underline{1.2\text{ GHz}}$$

b) If $f = 50$ MHz, what will the standing wave ratio be in region 1? At the given frequency, $\beta_2 = (2\pi \times 5\times10^7)\sqrt{(2\times10^{-6})(25\times10^{-12})} = 2.22$ rad/m. Thus $\beta_2 d = 2.22(.06) = 0.133$. The intrinsic impedance of regions 1 and 3 is $\eta_1 = \eta_3 = \sqrt{(4\times10^{-6})/(10^{-11})} = 632\,\Omega$. The input impedance at the first interface is now

$$\eta_{in} = 283\left[\frac{632\cos(.133) + j283\sin(.133)}{283\cos(.133) + j632\sin(.133)}\right] = 589 - j138 = 605\angle -.23$$

The reflection coefficient is now

$$\Gamma = \frac{\eta_{in} - \eta_1}{\eta_{in} + \eta_1} = \frac{589 - j138 - 632}{589 - j138 + 632} = .12\angle -1.7$$

The standing wave ratio is now

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .12}{1 - .12} = \underline{1.27}$$

12.16. A uniform plane wave in air is normally-incident onto a lossless dielectric plate of thickness $\lambda/8$, and of intrinsic impedance $\eta = 260\ \Omega$. Determine the standing wave ratio in front of the plate. Also find the fraction of the incident power that is transmitted to the other side of the plate: With the a thickness of $\lambda/8$, we have $\beta d = \pi/4$, and so $\cos(\beta d) = \sin(\beta d) = 1\sqrt{2}$. The input impedance thus becomes

$$\eta_{in} = 260\left[\frac{377 + j260}{260 + j377}\right] = 243 - j92\ \Omega$$

12.16. (continued)

The reflection coefficient is then

$$\Gamma = \frac{(243 - j92) - 377}{(243 - j92) + 377} = -0.19 - j0.18 = 0.26\angle -2.4\text{rad}$$

Therefore

$$s = \frac{1 + .26}{1 - .26} = \underline{1.7} \text{ and } 1 - |\Gamma|^2 = 1 - (.26)^2 = \underline{0.93}$$

12.17. Repeat Problem 12.16 for the cases in which the frequency is

a) doubled: If this is true, then $d = \lambda/4$, and thus $\eta_{in} = (260)^2/377 = 179$. The reflection coefficient becomes

$$\Gamma = \frac{179 - 377}{179 + 377} = -0.36 \;\Rightarrow\; s = \frac{1 + .36}{1 - .36} = \underline{2.13}$$

Then $1 - |\Gamma|^2 = 1 - (.36)^2 = \underline{0.87}$.

b) quadrupled: Now, $d = \lambda/2$, and so we have a half-wave section surrounded by air. Transmission will be total, and so $s = \underline{1}$ and $1 - |\Gamma|^2 = \underline{1}$.

12.18. In Fig. 12.6, let $\eta_1 = \eta_3 = 377\Omega$, and $\eta_2 = 0.4\eta_1$. A uniform plane wave is normally incident from the left, as shown. Plot a curve of the standing wave ratio, s, in the region to the left:

a) as a function of l if $f = 2.5$GHz: With $\eta_1 = \eta_3 = \eta_0$ and with $\eta_2 = 0.4\eta_0$, Eq. (41) becomes

$$\eta_{in} = 0.4\eta_0 \left[\frac{\cos(\beta l) + j0.4\sin(\beta l)}{0.4\cos(\beta l) + j\sin(\beta l)}\right] \times \left[\frac{0.4\cos(\beta l) - j\sin(\beta l)}{0.4\cos(\beta l) - j\sin(\beta l)}\right]$$
$$= \eta_0 \left[\frac{1 - j1.05\sin(2\beta l)}{\cos^2(\beta l) + 6.25\sin^2(\beta l)}\right]$$

Then $\Gamma = (\eta_{in} - \eta_0)/(\eta_{in} + \eta_0)$, from which we find

$$|\Gamma| = \sqrt{\Gamma\Gamma^*} = \left[\frac{\left[1 - \cos^2(\beta l) - 6.25\sin^2(\beta l)\right]^2 + (1.05)^2\sin^2(2\beta l)}{\left[1 + \cos^2(\beta l) + 6.25\sin^2(\beta l)\right]^2 + (1.05)^2\sin^2(2\beta l)}\right]^{1/2}$$

Then $s = (1 + |\Gamma|)/(1 - |\Gamma|)$. Now for a uniform plane wave, $\beta = \omega\sqrt{\mu\epsilon} = n\omega/c$. Given that $\eta_2 = 0.4\eta_0 = \eta_0/n$, we find $n = 2.5$ (assuming $\mu = \mu_0$). Thus, at 2.5 GHz,

$$\beta l = \frac{n\omega}{c} l = \frac{(2.5)(2\pi)(2.5 \times 10^9)}{3 \times 10^8} l = 12.95\, l \text{ (l in m)} = 0.1295\, l \text{ (l in cm)}$$

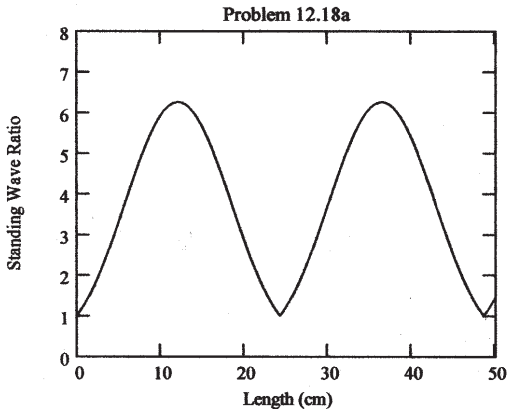
Using this in the expression for $|\Gamma|$, and calculating s as a function of l in cm leads to the first plot shown on the next page.

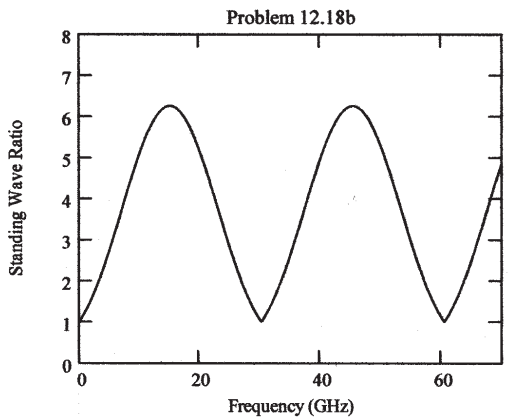
b) as a function of frequency if $l = 2$cm. In this case we use

$$\beta l = \frac{(2.5)(2\pi)(0.02)}{3 \times 10^8} f = 1.04 \times 10^{-10}\, f \text{ (f in Hz)} = 0.104\, f \text{ (f in GHz)}$$

Using this in the expression for $|\Gamma|$, and calculating s as a function of f in GHz leads to the second plot shown on the next page. MathCad was used in both cases.

12.18 (continued) Plots for parts *a* and *b*





12.19. You are given four slabs of lossless dielectric, all with the same intrinsic impedance, η, known to be different from that of free space. The thickness of each slab is $\lambda/4$, where λ is the wavelength as measured in the slab material. The slabs are to be positioned parallel to one another, and the combination lies in the path of a uniform plane wave, normally-incident. The slabs are to be arranged such that the air spaces between them are either zero, one-quarter wavelength, or one-half wavelength in thickness. Specify an arrangement of slabs and air spaces such that

a) the wave is totally transmitted through the stack: In this case, we look for a combination of half-wave sections. Let the inter-slab distances be d_1, d_2, and d_3 (from left to right). Two possibilities are i.) $\underline{d_1 = d_2 = d_3 = 0}$, thus creating a single section of thickness λ, or ii.) $\underline{d_1 = d_3 = 0,\ d_2 = \lambda/2}$, thus yielding two half-wave sections separated by a half-wavelength.

b) the stack presents the highest reflectivity to the incident wave: The best choice here is to make $\underline{d_1 = d_2 = d_3 = \lambda/4}$. Thus every thickness is one-quarter wavelength. The impedances transform as follows: First, the input impedance at the front surface of the last slab (slab 4) is $\eta_{in,1} = \eta^2/\eta_0$. We transform this back to the back surface of slab 3, moving through a distance of $\lambda/4$ in free space: $\eta_{in,2} = \eta_0^2/\eta_{in,1} = \eta_0^3/\eta^2$. We next transform this impedance to the front surface of slab 3, producing $\eta_{in,3} = \eta^2/\eta_{in,2} = \eta^4/\eta_0^3$. We continue in this manner until reaching the front surface of slab 1, where we find $\eta_{in,7} = \eta^8/\eta_0^7$. Assuming $\eta < \eta_0$, the ratio η^n/η_0^{n-1} becomes smaller as n increases (as the number of slabs increases). The reflection coefficient for waves incident on the front slab thus gets close to unity, and approaches 1 as the number of slabs approaches infinity.

12.20. The 50MHz plane wave of Problem 12.12 is incident onto the ocean surface at an angle to the normal of 60°. Determine the fractions of the incident power that are reflected and transmitted for

a) s polarization: To review Problem 12, we first we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50\times10^6)(78)(8.854\times10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 65, Chapter 11), and with $\mu = \mu_0$, the intrinsic impedance is $\eta_s = \sqrt{\pi f\mu/\sigma}(1+j) = 7.0(1+j)$.

12.20a. (continued)

Next we need the angle of refraction, which means that we need to know the refractive index of seawater at 50MHz. For a uniform plane wave in a good conductor, the phase constant is

$$\beta = \frac{n_{sea}\,\omega}{c} \doteq \sqrt{\pi f \mu \sigma} \;\;\Rightarrow\;\; n_{sea} \doteq c\sqrt{\frac{\mu\sigma}{4\pi f}} = 26.8$$

Then, using Snell's law, the angle of refraction is found:

$$\sin\theta_2 = \frac{n_{sea}}{n_1}\sin\theta_1 = 26.8\sin(60^\circ) \;\;\Rightarrow\;\; \theta_2 = 1.9^\circ$$

This angle is small enough so that $\cos\theta_2 \doteq 1$. Therefore, for s polarization,

$$\Gamma_s \doteq \frac{\eta_{s2}-\eta_{s1}}{\eta_{s2}+\eta_{s1}} = \frac{7.0(1+j)-377/\cos 60^\circ}{7.0(1+j)+377/\cos 60^\circ} = -0.98 + j0.018 = 0.98\angle 179^\circ$$

The fraction of the power reflected is now $|\Gamma_s|^2 = \underline{0.96}$. The fraction transmitted is then $\underline{0.04}$.

b) p polarization: Again, with the refracted angle close to zero, the relection coefficient for p polarization is

$$\Gamma_p \doteq \frac{\eta_{p2}-\eta_{p1}}{\eta_{p2}+\eta_{p1}} = \frac{7.0(1+j)-377\cos 60^\circ}{7.0(1+j)+377\cos 60^\circ} = -0.93 + j0.069 = 0.93\angle 176^\circ$$

The fraction of the power reflected is now $|\Gamma_p|^2 = \underline{0.86}$. The fraction transmitted is then $\underline{0.14}$.

12.21. A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_R = 3.45$, $\epsilon''_R = 0$, $\mu = \mu_0$).

a) Determine the fractions of the incident power that are reflected and transmitted: In plexiglas, Brewster's angle is $\theta_B = \theta_1 = \tan^{-1}(\epsilon'_{R2}/\epsilon'_{R1}) = \tan^{-1}(\sqrt{3.45}) = 61.7^\circ$. Then the angle of refraction is $\theta_2 = 90^\circ - \theta_B$ (see Example 12.9), or $\theta_2 = 28.3^\circ$. With incidence at Brewster's angle, all p-polarized power will be transmitted — only s-polarized power will be reflected. This is found through

$$\Gamma_s = \frac{\eta_{2s}-\eta_{1s}}{\eta_{2s}+\eta_{1s}} = \frac{.614\eta_0 - 2.11\eta_0}{.614\eta_0 + 2.11\eta_0} = -0.549$$

where $\eta_{1s} = \eta_1\sec\theta_1 = \eta_0\sec(61.7^\circ) = 2.11\eta_0$,
and $\eta_{2s} = \eta_2\sec\theta_2 = (\eta_0/\sqrt{3.45})\sec(28.3^\circ) = 0.614\eta_0$. Now, the reflected power fraction is $|\Gamma|^2 = (-.549)^2 = .302$. Since the wave is circularly-polarized, the s-polarized component represents one-half the total incident wave power, and so the fraction of the *total* power that is reflected is $.302/2 = 0.15$, or $\underline{15\%}$. The fraction of the incident power that is transmitted is then the remainder, or $\underline{85\%}$.

b) Describe the polarizations of the reflected and transmitted waves: Since all the p-polarized component is transmitted, the reflected wave will be entirely s-polarized (linear). The transmitted wave, while having all the incident p-polarized power, will have a reduced s-component, and so this wave will be right-elliptically polarized.

12.22. A dielectric waveguide is shown in Fig. 12.18 with refractive indices as labeled. Incident light enters the guide at angle ϕ from the front surface normal as shown. Once inside, the light totally reflects at the upper $n_1 - n_2$ interface, where $n_1 > n_2$. All subsequent reflections from the upper an lower boundaries will be total as well, and so the light is confined to the guide. Express, in terms of n_1 and n_2, the maximum value of ϕ such that total confinement will occur, with $n_0 = 1$. The quantity $\sin\phi$ is known as the *numerical aperture* of the guide.

From the illustration we see that ϕ_1 maximizes when θ_1 is at its minimum value. This minimum will be the critical angle for the $n_1 - n_2$ interface, where $\sin\theta_c = \sin\theta_1 = n_2/n_1$. Let the refracted angle to the right of the vertical interface (not shown) be ϕ_2, where $n_0 \sin\phi_1 = n_1 \sin\phi_2$. Then we see that $\phi_2 + \theta_1 = 90^\circ$, and so $\sin\theta_1 = \cos\phi_2$. Now, the numerical aperture becomes

$$\sin\phi_{1max} = \frac{n_1}{n_0}\sin\phi_2 = n_1\cos\theta_1 = n_1\sqrt{1-\sin^2\theta_1} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

Finally, $\phi_{1max} = \underline{\sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)}$ is the numerical aperture angle.

12.23. Suppose that ϕ_1 in Fig. 12.18 is Brewster's angle, and that θ_1 is the critical angle. Find n_0 in terms of n_1 and n_2: With the incoming ray at Brewster's angle, the refracted angle of this ray (measured from the inside normal to the front surface) will be $90^\circ - \phi_1$. Therefore, $\phi_1 = \theta_1$, and thus $\sin\phi_1 = \sin\theta_1$. Thus

$$\sin\phi_1 = \frac{n_1}{\sqrt{n_0^2 + n_1^2}} = \sin\theta_1 = \frac{n_2}{n_1} \quad \Rightarrow \quad n_0 = \underline{(n_1/n_2)\sqrt{n_1^2 - n_2^2}}$$

Alternatively, we could have used the result of Problem 12.22, in which it was found that $\sin\phi_1 = (1/n_0)\sqrt{n_1^2 - n_2^2}$, which we then set equal to $\sin\theta_1 = n_2/n_1$ to get the same result.

12.24. A *Brewster prism* is designed to pass p-polarized light without any reflective loss. The prism of Fig. 12.19 is made of glass ($n = 1.45$), and is in air. Considering the light path shown, determine the apex angle, α: With entrance and exit rays at Brewster's angle (to eliminate reflective loss), the interior ray must be horizontal, or parallel to the bottom surface of the prism. From the geometry, the angle between the interior ray and the normal to the prism surfaces that it intersects is $\alpha/2$. Since this angle is also Brewster's angle, we may write:

$$\alpha = 2\sin^{-1}\left(\frac{1}{\sqrt{1+n^2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{1+(1.45)^2}}\right) = 1.21\,\text{rad} = \underline{69.2^\circ}$$

12.25. In the Brewster prism of Fig. 12.19, determine for s-polarized light the fraction of the incident power that is transmitted through the prism: We use $\Gamma_s = (\eta_{s2} - \eta_{s1})/(\eta_{s2} + \eta_{s1})$, where

$$\eta_{s2} = \frac{\eta_2}{\cos(\theta_{B2})} = \frac{\eta_2}{n/\sqrt{1+n^2}} = \frac{\eta_0}{n^2}\sqrt{1+n^2}$$

and

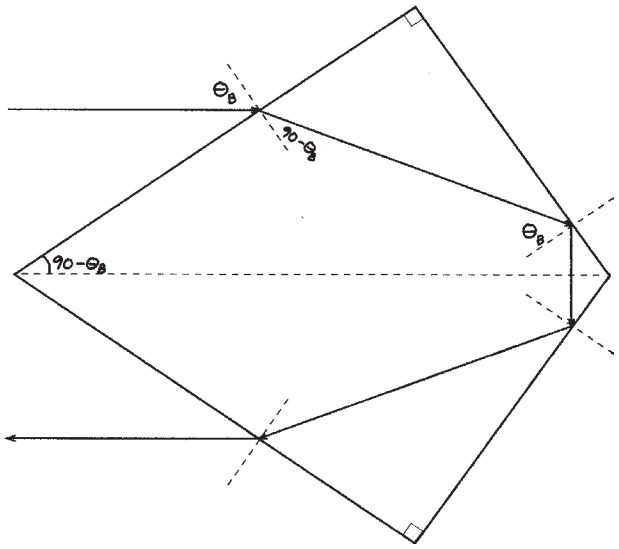
$$\eta_{s1} = \frac{\eta_1}{\cos(\theta_{B1})} = \frac{\eta_1}{1/\sqrt{1+n^2}} = \eta_0\sqrt{1+n^2}$$

12.25. (continued) Thus, at the first interface, $\Gamma = (1-n^2)/(1+n^2)$. At the second interface, Γ will be equal but of opposite sign to the above value. The power transmission coefficient through each interface is $1-|\Gamma|^2$, so that for both interfaces, we have, with $n = 1.45$:

$$\frac{P_{tr}}{P_{inc}} = \left(1-|\Gamma|^2\right)^2 = \left[1-\left(\frac{n^2-1}{n^2+1}\right)^2\right]^2 = \underline{0.76}$$

12.26. Show how a single block of glass can be used to turn a p-polarized beam of iight through 180°, with the light suffering, in principle, zero reflective loss. The light is incident from air, and the returning beam (also in air) may be displaced sideways from the incident beam. Specify all pertinent angles and use $n = 1.45$ for glass. More than one design is possible here.

The prism below is designed such that light enters at Brewster's angle, and once inside, is turned around using total reflection. Using the result of Example 12.9, we find that with glass, $\theta_B = 55.4°$, which, by the geometry, is also the incident angle for total reflection at the back of the prism. For this to work, the Brewster angle must be greater than or equal to the critical angle. This is in fact the case, since $\theta_c = \sin^{-1}(n_2/n_1) = \sin^{-1}(1/1.45) = 43.6°$.



12.27. Using Eq. (59) in Chapter 11 as a starting point, determine the ratio of the group and phase velocities of an electromagnetic wave in a good conductor. Assume conductivity does not vary with frequency: In a good conductor:

$$\beta = \sqrt{\pi f \mu\sigma} = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad \rightarrow \qquad \frac{d\beta}{d\omega} = \frac{1}{2}\left[\frac{\omega\mu\sigma}{2}\right]^{-1/2}\frac{\mu\sigma}{2}$$

Thus

$$\frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1} = 2\sqrt{\frac{2\omega}{\mu\sigma}} = v_g \qquad \text{and} \qquad v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega\mu\sigma/2}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Therefore $v_g/v_p = \underline{2}$.

12.28. Over a certain frequency range, the refractive index of a certain material varies approximately linearly with frequency: $n(\omega) \doteq n_a + n_b(\omega - \omega_a)$, where n_a, n_b, and ω_a are constants. Using $\beta = n\omega/c$:

a) determine the group velocity as a function (or perhaps not a function) of frequency: $v_g = (d\beta/d\omega)^{-1}$, where

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left[\frac{n_a\omega}{c} + \frac{n_b(\omega-\omega_a)\omega}{c}\right] = \frac{1}{c}\left[n_a + n_b(2\omega - \omega_a)\right]$$

so that

$$v_g(\omega) = \underline{c\left[n_a + n_b(2\omega - \omega_a)\right]^{-1}}$$

b) determine the group dispersion parameter, β_2:

$$\beta_2 = \frac{d^2\beta}{d\omega^2}\Big|_{\omega_0} = \frac{d}{d\omega}\frac{1}{c}\left[n_a + n_b(2\omega - \omega_a)\right]\Big|_{\omega_0} = \underline{2n_b/c}$$

c) Discuss the implications of these results, if any, on pulse broadening: The point of this problem was to show that higher order terms (involving $d^3\beta/d\omega^3$ and higher) in the Taylor series expansion, Eq. (89), do not exist if the refractive index varies linearly with ω. These higher order terms would be necessary in cases involving pulses of exremely large bandwidth, or in media exhibiting complicated variations in their ω-β curves over relatively small frequency ranges. With $d^2\beta/d\omega^2$ constant, the three-term Taylor expansion of Eq. (89) describes the phase constant of this medium exactly. The pulse will broaden and will acquire a frequency sweep (chirp) that is precisely linear with time. Additionally, a pulse of a given bandwidth will broaden by the same amount, regardless of what carrier frequency is used.

12.29. A $T = 5$ ps transform-limited pulse propagates in a dispersive channel for which $\beta_2 = 10$ ps^2/km. Over what distance will the pulse spread to twice its initial width? After propagation, the width is $T' = \sqrt{T^2 + (\Delta\tau)^2} = 2T$. Thus $\Delta\tau = \sqrt{3}T$, where $\Delta\tau = \beta_2 z/T$. Therefore

$$\frac{\beta_2 z}{T} = \sqrt{3}T \text{ or } z = \frac{\sqrt{3}T^2}{\beta_2} = \frac{\sqrt{3}(5\text{ ps})^2}{10\text{ ps}^2/\text{km}} = \underline{4.3\text{ km}}$$

12.30. A $T = 20$ ps transform-limited pulse propagates through 10 km of a dispersive channel for which $\beta_2 = 12$ ps^2/km. The pulse then propagates through a second 10 km channel for which $\beta_2 = -12$ ps^2/km. Describe the pulse at the output of the second channel and give a physical explanation for what happened.

Our theory of pulse spreading will allow for changes in β_2 down the length of the channel. In fact, we may write in general:

$$\Delta\tau = \frac{1}{T}\int_0^L \beta_2(z)\,dz$$

Having β_2 change sign at the midpoint, yields a zero $\Delta\tau$, and so the pulse emerges from the output unchanged! Physically, the pulse acquires a positive linear chirp (frequency increases with time over the pulse envelope) during the first half of the channel. When β_2 switches sign, the pulse begins to acquire a negative chirp in the second half, which, over an equal distance, will completely eliminate the chirp acquired during the first half. The pulse, if originally transform-limited at input, will emerge, again transform-limited, at its original width. More generally, complete *dispersion compensation* is achieved using a two-segment channel when $\beta_2 L = -\beta_2' L'$, assuming dispersion terms of higher order than β_2 do not exist.

CHAPTER 13

13.1. The parameters of a certain transmission line operating at 6×10^8 rad/s are $L = 0.4\ \mu$H/m, $C = 40$ pF/m, $G = 80$ mS/m, and $R = 20\ \Omega/m$.

a) Find γ, α, β, λ, and Z_0: We use

$$\begin{aligned}\gamma &= \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{[20 + j(6 \times 10^8)(0.4 \times 10^{-6})][80 \times 10^{-3} + j(6 \times 10^8)(40 \times 10^{-12})]} \\ &= \underline{2.8 + j3.5\ \text{m}^{-1}} = \alpha + j\beta\end{aligned}$$

Therefore, $\alpha = \underline{2.8\ \text{Np/m}}$, $\beta = \underline{3.5\ \text{rad/m}}$, and $\lambda = 2\pi/\beta = \underline{1.8\ \text{m}}$. Finally,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{20 + j2.4 \times 10^2}{80 \times 10^{-3} + j2.4 \times 10^{-2}}} = \underline{44 + j30\ \Omega}$$

b) If a voltage wave travels 20 m down the line, what percentage of the original amplitude remains, and by how many degrees is it phase shifted? First,

$$\frac{V_{20}}{V_0} = e^{-\alpha L} = e^{-(2.8)(20)} = 4.8 \times 10^{-25} \text{ or } \underline{4.8 \times 10^{-23}\ \text{percent!}}$$

Then the phase shift is given by βL, which in degrees becomes

$$\phi = \beta L \left(\frac{360}{2\pi}\right) = (3.5)(20)\left(\frac{360}{2\pi}\right) = \underline{4.0 \times 10^3\ \text{degrees}}$$

13.2. A lossless transmission line with $Z_0 = 60\ \Omega$ is being operated at 60 MHz. The velocity on the line is 3×10^8 m/s. If the line is short-circuited at $z = 0$, find Z_{in} at:

a) $z = -1$m: We use the expression for input impedance (Eq. 12), under the conditions $Z_2 = 60$ and $Z_3 = 0$:

$$Z_{in} = Z_2\left[\frac{Z_3\cos(\beta l) + jZ_2\sin(\beta l)}{Z_2\cos(\beta l) + jZ_3\sin(\beta l)}\right] = j60\tan(\beta l)$$

where $l = -z$, and where the phase constant is $\beta = 2\pi c/f = 2\pi(3 \times 10^8)/(6 \times 10^7) = (2/5)\pi$ rad/m. Now, with $z = -1$ ($l = 1$), we find $Z_{in} = j60\tan(2\pi/5) = \underline{j184.6\ \Omega}$.

b) $z = -2$ m: $Z_{in} = j60\tan(4\pi/5) = \underline{-j43.6\ \Omega}$

c) $z = -2.5$ m: $Z_{in} = j60\tan(5\pi/5) = \underline{0}$

d) $z = -1.25$ m: $Z_{in} = j60\tan(\pi/2) = \underline{j\infty\ \Omega}$ (open circuit)

13.3. The characteristic impedance of a certain lossless transmission line is 72 Ω. If $L = 0.5\,\mu$H/m, find:

a) C: Use $Z_0 = \sqrt{L/C}$, or

$$C = \frac{L}{Z_0^2} = \frac{5 \times 10^{-7}}{(72)^2} = 9.6 \times 10^{-11}\ \text{F/m} = \underline{96\ \text{pF/m}}$$

13.3b) v_p:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5\times10^{-7})(9.6\times10^{-11})}} = \underline{1.44\times10^8 \text{ m/s}}$$

c) β if $f = 80$ MHz:

$$\beta = \omega\sqrt{LC} = \frac{2\pi \times 80\times10^6}{1.44\times10^8} = \underline{3.5 \text{ rad/m}}$$

d) The line is terminated with a load of 60 Ω. Find Γ and s:

$$\Gamma = \frac{60-72}{60+72} = \underline{-0.09} \qquad s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+.09}{1-.09} = \underline{1.2}$$

13.4. A lossless transmission line having $Z_0 = 120\Omega$ is operating at $\omega = 5\times10^8$ rad/s. If the velocity on the line is 2.4×10^8 m/s, find:

a) L: With $Z_0 = \sqrt{L/C}$ and $v = 1/\sqrt{LC}$, we find $L = Z_0/v = 120/2.4\times10^8 = \underline{0.50\ \mu\text{H/m}}$.

b) C: Use $Z_0 v = \sqrt{L/C}/\sqrt{LC} \Rightarrow C = 1/(Z_0 v) = [120(2.4\times10^8)]^{-1} = \underline{35 \text{ pF/m}}$.

c) Let Z_L be represented by an inductance of 0.6 μH in series with a 100-Ω resistance. Find Γ and s: The inductive impedance is $j\omega L = j(5\times10^8)(0.6\times10^{-6}) = j300$. So the load impedance is $Z_L = 100 + j300$ Ω. Now

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100+j300-120}{100+j300+120} = 0.62 + j0.52 = \underline{0.808\angle 40^\circ}$$

Then

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.808}{1-0.808} = \underline{9.4}$$

13.5. Two characteristics of a certain lossless transmission line are $Z_0 = 50$ Ω and $\gamma = 0 + j0.2\pi$ m^{-1} at $f = 60$ MHz.

a) Find L and C for the line: We have $\beta = 0.2\pi = \omega\sqrt{LC}$ and $Z_0 = 50 = \sqrt{L/C}$. Thus

$$\frac{\beta}{Z_0} = \omega C \;\Rightarrow\; C = \frac{\beta}{\omega Z_0} = \frac{0.2\pi}{(2\pi\times60\times10^6)(50)} = \frac{1}{3}\times10^{10} = \underline{33.3 \text{ pF/m}}$$

Then $L = CZ_0^2 = (33.3\times10^{-12})(50)^2 = 8.33\times10^{-8}$ H/m $= \underline{83.3 \text{ nH/m}}$.

b) A load, $Z_L = 60 + j80$ Ω is located at $z = 0$. What is the shortest distance from the load to a point at which $Z_{in} = R_{in} + j0$? I will do this using two different methods:

The Hard Way: We use the general expression

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}\right]$$

We can then normalize the impedances with respect to Z_0 and write

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{(Z_L/Z_0) + j\tan(\beta l)}{1 + j(Z_L/Z_0)\tan(\beta l)}\right] = \left[\frac{z_L + j\tan(\beta l)}{1 + jz_L\tan(\beta l)}\right]$$

where $z_L = (60 + j80)/50 = 1.2 + j1.6$.

13.5b. (continued) Using this, and defining $x = \tan(\beta l)$, we find

$$z_{in} = \left[\frac{1.2 + j(1.6 + x)}{(1 - 1.6x) + j1.2x}\right]\left[\frac{(1 - 1.6x) - j1.2x}{(1 - 1.6x) - j1.2x}\right]$$

The second bracketed term is a factor of one, composed of the complex conjugate of the denominator of the first term, divided by itself. Carrying out this product, we find

$$z_{in} = \left[\frac{1.2(1 - 1.6x) + 1.2x(1.6 + x) - j[(1.2)^2 x - (1.6 + x)(1 - 1.6x)]}{(1 - 1.6x)^2 + (1.2)^2 x^2}\right]$$

We require the imaginary part to be zero. Thus

$$(1.2)^2 x - (1.6 + x)(1 - 1.6x) = 0 \;\Rightarrow\; 1.6x^2 + 3x - 1.6 = 0$$

So

$$x = \tan(\beta l) = \frac{-3 \pm \sqrt{9 + 4(1.6)^2}}{2(1.6)} = (.433, -2.31)$$

We take the positive root, and find

$$\beta l = \tan^{-1}(.433) = 0.409 \;\Rightarrow\; l = \frac{0.409}{0.2\pi} = 0.65 \text{ m} = \underline{65 \text{ cm}}$$

The Easy Way: We find

$$\Gamma = \frac{60 + j80 - 50}{60 + j80 + 50} = 0.405 + j0.432 = 0.59\angle 0.818$$

Thus $\phi = 0.818$ rad, and we use the fact that the input impedance will be purely real at the location of a voltage minimum or maximum. The first voltage maximum will occur at a distance in front of the load given by

$$z_{max} = \frac{\phi}{2\beta} = \frac{0.818}{2(0.2\pi)} = 0.65 \text{ m}$$

13.6. The propagation constant of a lossy transmission line is $1 + j2$ m^{-1}, and its characteristic impedance is $20 + j0\,\Omega$ at $\omega = 1$ Mrad/s. Find L, C, R, and G for the line: Begin with

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega L}} = 20 \;\Rightarrow\; R + j\omega L = 400(G + j\omega C) \tag{1}$$

Then

$$\gamma^2 = (R + j\omega L)(G + j\omega C) = (1 + j2)^2 \;\Rightarrow\; 400(G + j\omega C)^2 = (1 + j2)^2 \tag{2}$$

where (1) has been used. Eq. 2 now becomes $G + j\omega C = (1 + j2)/20$. Equating real and imaginary parts leads to $G = \underline{.05 \text{ S/m}}$ and $C = 1/(10\omega) = 10^{-7} = \underline{0.1\,\mu\text{F/m}}$.

13.6. (continued) Now, (1) becomes

$$20 = \sqrt{\frac{R + j\omega L}{1 + j2}}\sqrt{20} \;\Rightarrow\; 20 = \frac{R + j\omega L}{1 + j2} \;\Rightarrow\; 20 + j40 = R + j\omega L$$

Again, equating real and imaginary parts leads to $R = \underline{20\,\Omega/\text{m}}$ and $L = 40/\omega = \underline{40\,\mu\text{H/m}}$.

13.7. The dimensions of the outer conductor of a coaxial cable are b and c, $c > b$. Assume $\sigma = \sigma_c$ and let $\mu = \mu_0$. Find the magnetic energy stored per unit length in the region $b < r < c$ for a uniformly distributed total current I flowing in opposite directions in the inner and outer conductors: First, from the inner conductor, the magnetic field will be

$$\mathbf{H}_1 = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

The contribution from the outer conductor to the magnetic field within that conductor is found from Ampere's circuital law to be:

$$\mathbf{H}_2 = -\frac{I}{2\pi\rho}\,\frac{\rho^2 - b^2}{c^2 - b^2}\,\mathbf{a}_\phi$$

The total magnetic field within the outer conductor will be the sum of the two fields, or

$$\mathbf{H}_\text{T} = \mathbf{H}_1 + \mathbf{H}_2 = \frac{I}{2\pi\rho}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]\mathbf{a}_\phi$$

The energy density is

$$w_m = \frac{1}{2}\mu_0 H_T^2 = \frac{\mu_0 I^2}{8\pi^2}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]^2 \text{ J/m}^3$$

The stored energy per unit length in the outer conductor is now

$$W_m = \int_0^1\int_0^{2\pi}\int_b^c \frac{\mu_0 I^2}{8\pi^2}\left[\frac{c^2 - \rho^2}{c^2 - b^2}\right]^2 \rho\, d\rho\, d\phi\, dz = \frac{\mu_0 I^2}{4\pi(c^2 - b^2)^2}\int_b^c \left[\frac{c^4}{\rho} - 2c^2\rho + \rho^3\right] d\rho$$
$$= \underline{\frac{\mu_0 I^2}{4\pi}\left[\frac{c^4}{(c^2 - b^2)^2}\,\ln\left(\frac{c}{b}\right) + \frac{b^2 - (3/4)c^2}{(c^2 - b^2)}\right]\text{ J}}$$

13.8. The conductors of a coaxial transmission line are copper ($\sigma_c = 5.8 \times 10^{-7}$ S/m) and the dielectric is polyethylene ($\epsilon'_R = 2.26$, $\sigma/\omega\epsilon' = 0.0002$). If the inner radius of the outer conductor is 4 mm, find the radius of the inner conductor so that (assuming a lossless line):

a) $Z_0 = 50\,\Omega$: Use

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = 50 \;\Rightarrow\; \ln\left(\frac{b}{a}\right) = \frac{2\pi\sqrt{\epsilon'_R}(50)}{377} = 1.25$$

Thus $b/a = e^{1.25} = 3.50$, or $a = 4/3.50 = \underline{1.142\,\text{mm}}$

13.8b. $C = 100$ pF/m: Begin with

$$C = \frac{2\pi\epsilon'}{\ln(b/a)} = 10^{-10} \;\Rightarrow\; \ln\left(\frac{b}{a}\right) = 2\pi(2.26)(8.854\times10^{-2}) = 1.257$$

So $b/a = e^{1.257} = 3.51$, or $a = 4/3.51 = \underline{1.138\,\text{mm}}$.

c) $L = 0.2\,\mu$H/m: Use

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) = 0.2\times10^{-6} \;\Rightarrow\; \ln\left(\frac{b}{a}\right) = \frac{2\pi(0.2\times10^{-6})}{4\pi\times10^{-7}} = 1$$

Thus $b/a = e^1 = 2.718$, or $a = b/2.718 = \underline{1.472\,\text{mm}}$.

13.9. Two aluminum-clad steel conductors are used to construct a two-wire transmission line. Let $\sigma_{Al} = 3.8\times10^7$ S/m, $\sigma_{St} = 5\times10^6$ S/m, and $\mu_{St} = 100\,\mu$H/m. The radius of the steel wire is 0.5 in., and the aluminum coating is 0.05 in. thick. The dielectric is air, and the center-to-center wire separation is 4 in. Find C, L, G, and R for the line at 10 MHz: The first question is whether we are in the high frequency or low frequency regime. Calculation of the skin depth, δ, will tell us. We have, for aluminum,

$$\delta = \frac{1}{\sqrt{\pi f\mu_0\sigma_{Al}}} = \frac{1}{\sqrt{\pi(10^7)(4\pi\times10^{-7})(3.8\times10^7)}} = 2.58\times10^{-5}\text{ m}$$

so we are clearly in the high frequency regime, where uniform current distributions cannot be assumed. Furthermore, the skin depth is considerably less than the aluminum layer thickness, so the bulk of the current resides in the aluminum, and we may neglect the steel. Assuming solid aluminum wires of radius $a = 0.5 + 0.05 = 0.55$ in $= 0.014$ m, the resistance of the two-wire line is now

$$R = \frac{1}{\pi a\delta\sigma_{Al}} = \frac{1}{\pi(.014)(2.58\times10^{-5})(3.8\times10^7)} = \underline{0.023\ \Omega/\text{m}}$$

Next, since the dielectric is air, no leakage will occur from wire to wire, and so $G = \underline{0\text{ mho/m}}$. Now the capacitance will be

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\times8.85\times10^{-12}}{\cosh^{-1}(4/(2\times0.55))} = 1.42\times10^{-11}\text{ F/m} = \underline{14.2\text{ pF/m}}$$

Finally, the inductance per unit length will be

$$L = \frac{\mu_0}{\pi}\cosh(d/2a) = \frac{4\pi\times10^{-7}}{\pi}\cosh(4/(2\times0.55)) = 7.86\times10^{-7}\text{ H/m} = \underline{0.786\,\mu\text{H/m}}$$

13.10. Each conductor of a two-wire transmission line has a radius of 0.5mm; their center-to-center distance is 0.8cm. Let $f = 150MHz$ and assume $\sigma = 0$ and $\sigma_c \to \infty$ (note error in problem statement). Find the dielectric constant of the insulating medium if

a) $Z_0 = 300\,\Omega$: Use

$$300 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\epsilon'_R\epsilon_0}}\cosh^{-1}\left(\frac{d}{2a}\right) \;\Rightarrow\; \sqrt{\epsilon'_R} = \frac{120\pi}{300\pi}\cosh^{-1}\left(\frac{8}{2(.5)}\right) = 1.107 \;\Rightarrow\; \epsilon'_R = \underline{1.23}$$

b) $C = 20$ pF/m: Use

$$20\times10^{-12} = \frac{\pi\epsilon'}{\cosh^{-1}(d/2a)} \;\Rightarrow\; \epsilon'_R = \frac{20\times10^{-12}}{\pi\epsilon_0}\cosh^{-1}(8) = \underline{1.99}$$

c) $v_p = 2.6\times10^8$ m/s:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0\epsilon_0\epsilon'_R}} = \frac{c}{\sqrt{\epsilon'_R}} \;\Rightarrow\; \epsilon'_R = \left(\frac{3.0\times10^8}{2.6\times10^8}\right)^2 = \underline{1.33}$$

13.11. Pertinent dimensions for the transmission line shown in Fig. 13.4 are $b = 3$ mm, and $d = 0.2$ mm. The conductors and the dielectric are non-magnetic.

a) If the characteristic impedance of the line is 15 Ω, find ϵ'_R: We use

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\left(\frac{d}{b}\right) = 15 \;\Rightarrow\; \epsilon'_R = \left(\frac{377}{15}\right)^2\frac{.04}{9} = \underline{2.8}$$

b) Assume copper conductors and operation at 2×10^8 rad/s. If $RC = GL$, determine the loss tangent of the dielectric: For copper, $\sigma_c = 5.8\times10^7$ S/m, and the skin depth is

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma_c}} = \sqrt{\frac{2}{(2\times10^8)(4\pi\times10^{-7})(5.8\times10^7)}} = 1.2\times10^{-5}\text{ m}$$

Then

$$R = \frac{2}{\sigma_c\delta b} = \frac{2}{(5.8\times10^7)(1.2\times10^{-5})(.003)} = 0.98\,\Omega/\text{m}$$

Now

$$C = \frac{\epsilon' b}{d} = \frac{(2.8)(8.85\times10^{-12})(3)}{0.2} = 3.7\times10^{-10}\text{ F/m}$$

and

$$L = \frac{\mu_0 d}{b} = \frac{(4\pi\times10^{-7})(0.2)}{3} = 8.4\times10^{-8}\text{ H/m}$$

Then, with $RC = GL$,

$$G = \frac{RC}{L} = \frac{(.98)(3.7\times10^{-10})}{(8.4\times10^{-8})} = 4.4\times10^{-3}\text{ mho/m} = \frac{\sigma_d b}{d}$$

Thus $\sigma_d = (4.4\times10^{-3})(0.2/3) = 2.9\times10^{-4}$ S/m. The loss tangent is

$$l.t. = \frac{\sigma_d}{\omega\epsilon'} = \frac{2.9\times10^{-4}}{(2\times10^8)(2.8)(8.85\times10^{-12})} = \underline{5.85\times10^{-2}}$$

13.12. A transmission line constructed from perfect conductors and an air dielectric is to have a maximum dimension of 8mm for its cross-section. The line is to be used at high frequencies. Specify its dimensions if it is:

a) a two-wire line with $Z_0 = 300\,\Omega$: With the maximum dimension of 8mm, we have, using (27):

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon'}}\cosh^{-1}\left(\frac{8-2a}{2a}\right) = 300 \;\Rightarrow\; \frac{8-2a}{2a} = \cosh\left(\frac{300\pi}{120\pi}\right) = 6.13$$

Solve for a to find $a = \underline{0.56\,\text{mm}}$. Then $d = 8 - 2a = \underline{6.88\,\text{mm}}$.

b) a planar line with $Z_0 = 15\,\Omega$: In this case our maximum dimension dictates that $\sqrt{d^2+b^2} = 8$. So, using (34), we write

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\,\frac{\sqrt{64-b^2}}{b} = 15 \;\Rightarrow\; \sqrt{64-b^2} = \frac{15}{377}b$$

Solving, we find $b = \underline{7.99\,\text{mm}}$ and $d = \underline{0.32\,\text{mm}}$.

c) a 72 Ω coax having a zero-thickness outer conductor: With a zero-thickness outer conductor, we note that the outer radius is $b = 8/2 = 4$mm. Using (18), we write

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\,\ln\left(\frac{b}{a}\right) = 72 \;\Rightarrow\; \ln\left(\frac{b}{a}\right) = \frac{2\pi(72)}{120\pi} = 1.20 \;\Rightarrow\; a = be^{-1.20} = 4e^{-1.20} = 1.2$$

Summarizing, $a = \underline{1.2\,\text{mm}}$ and $b = \underline{4\,\text{mm}}$.

13.13. The incident voltage wave on a certain lossless transmission line for which $Z_0 = 50\,\Omega$ and $v_p = 2\times10^8$ m/s is $V^+(z,t) = 200\cos(\omega t - \pi z)$ V.

a) Find ω: We know $\beta = \pi = \omega/v_p$, so $\omega = \pi(2\times10^8) = \underline{6.28\times10^8\ \text{rad/s}}$.

b) Find $I^+(z,t)$: Since Z_0 is real, we may write

$$I^+(z,t) = \frac{V^+(z,t)}{Z_0} = \underline{4\cos(\omega t - \pi z)\ \text{A}}$$

The section of line for which $z > 0$ is replaced by a load $Z_L = 50 + j30\ \Omega$ at $z = 0$. Find

c) Γ_L: This will be

$$\Gamma_L = \frac{50+j30-50}{50+j30+50} = .0825 + j0.275 = \underline{0.287\angle 1.28\ \text{rad}}$$

d) $V_s^-(z) = \Gamma_L V_s^+(z)e^{j2\beta z} = 0.287(200)e^{j\pi z}e^{j1.28} = \underline{57.5e^{j(\pi z+1.28)}}$

e) V_s at $z = -2.2$ m:

$$V_s(-2.2) = V_s^+(-2.2) + V_s^-(-2.2) = 200e^{j2.2\pi} + 57.5e^{-j(2.2\pi-1.28)} = 257.5e^{j0.63}$$
$$= \underline{257.5\angle 36^\circ}$$

13.14. Coaxial lines 1 and 2 have the following parameters: $\mu_1 = \mu_2 = \mu_0$, $\sigma_1 = \sigma_2 = 0$, $\epsilon'_{R1} = 2.25$, $\epsilon'_{R2} = 4$, $a_1 = a_2 = 0.8$mm, $b_1 = 6$mm, $b_2 = 3$mm, $Z_{L2} = Z_{02}$, and Z_{L1} is Z_{in2}.

a) Find Z_{01} and Z_{02}. For either line, we have

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = \frac{377}{2\pi\sqrt{\epsilon'_R}}\ln\left(\frac{b}{a}\right) \qquad \text{leading to}$$

$$Z_{01} = \frac{377}{2\pi\sqrt{2.25}}\ln\left(\frac{6}{.8}\right) = \underline{80.6\ \Omega} \qquad \text{and} \qquad Z_{02} = \frac{377}{2\pi\sqrt{4}}\ln\left(\frac{3}{.8}\right) = \underline{39.7\ \Omega}$$

b) Find s on line 1: Line 1's load is line 2's input impedance (they are connected end-to-end). Also, since line 2 is matched, its input impedance is just it's characteristic impedance. Therefore, $Z_{L1} = Z_{in2} = Z_{02}$. The reflection coefficient encountered by waves incident on Z_{L1} from line 1 can now be found, along with the standing wave ratio:

$$\Gamma_{12} = \frac{39.7 - 80.6}{39.7 + 80.6} = -0.34 \ \Rightarrow\ s = \frac{1 + .34}{1 - .34} = \underline{2.03}$$

c) If a 20cm length of line 1 is inserted immediately in front of Z_{L2} and $f = 300$MHz, find s on line 2: The line 1 length now has a load impedance of 39.7 Ω and it is 20cm long. We need to find its input impedance. At 300 MHz, the free space wavelength is 1m. In line 1, having a dielectric constant of 2.25, the wavelength is $\lambda = 1\text{m}/\sqrt{2.25} = 0.67$m. Therefore $\beta l = 2\pi l/\lambda = 2\pi(20)/(67) = 1.87$. We now find the input impedance for this situation through

$$Z_{in} = Z_{01}\left[\frac{Z_{L2}\cos(\beta l) + jZ_{01}\sin(\beta l)}{Z_{01}\cos(\beta l) + jZ_{L2}\sin(\beta l)}\right] = 80.6\left[\frac{39.7\cos(1.87) + j80.6\sin(1.87)}{80.6\cos(1.87) + j39.7\sin(1.87)}\right]$$
$$= 128.7 - j55.8 = 140.3\angle -23.4^\circ$$

Now for waves incident at the line 1 - line 2 junction from line 2, the reflection coefficient will be

$$\Gamma_{21} = \frac{Z_{in} - Z_{02}}{Z_{in} + Z_{02}} = \frac{128.7 - 39.7 - j55.8}{128.7 + 39.7 - j55.8} = 0.58 - j0.14 = 0.59\angle -13.7^\circ$$

The standing wave ratio is now

$$s = \frac{1 + .59}{1 - .59} = \underline{3.9}$$

13.15. For the transmission line represented in Fig. 13.26, find $V_{s,out}$ if $f =$:

a) 60 Hz: At this frequency,

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \times 60}{(2/3)(3\times 10^8)} = 1.9\times 10^{-6}\ \text{rad/m}\ \ \text{So}\ \beta l = (1.9\times 10^{-6})(80) = 1.5\times 10^{-4} << 1$$

The line is thus essentially a lumped circuit, where $Z_{in} \doteq Z_L = 80\ \Omega$. Therefore

$$V_{s,out} = 120\left[\frac{80}{12+80}\right] = \underline{104\ \text{V}}$$

b) 500 kHz: In this case

$$\beta = \frac{2\pi \times 5\times 10^5}{2\times 10^8} = 1.57\times 10^{-2}\ \text{rad/s}\ \ \text{So}\ \beta l = 1.57\times 10^{-2}(80) = 1.26\ \text{rad}$$

Now

$$Z_{in} = 50\left[\frac{80\cos(1.26) + j50\sin(1.26)}{50\cos(1.26) + j80\sin(1.26)}\right] = 33.17 - j9.57 = 34.5\angle -.28$$

The equivalent circuit is now the voltage source driving the series combination of Z_{in} and the 12 ohm resistor. The voltage across Z_{in} is thus

$$V_{in} = 120\left[\frac{Z_{in}}{12+Z_{in}}\right] = 120\left[\frac{33.17 - j9.57}{12+33.17-j9.57}\right] = 89.5 - j6.46 = 89.7\angle -.071$$

The voltage at the line input is now the sum of the forward and backward-propagating waves just to the right of the input. We reference the load at $z = 0$, and so the input is located at $z = -80$ m. In general we write $V_{in} = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$, where

$$V_0^- = \Gamma_L V_0^+ = \frac{80-50}{80+50}\,V_0^+ = \frac{3}{13}\,V_0^+$$

At $z = -80$ m we thus have

$$V_{in} = V_0^+\left[e^{j1.26} + \frac{3}{13}e^{-j1.26}\right]\ \ \Rightarrow\ \ V_0^+ = \frac{89.5 - j6.46}{e^{j1.26} + (3/13)e^{-j1.26}} = 42.7 - j100\ \text{V}$$

Now

$$V_{s,out} = V_0^+(1+\Gamma_L) = (42.7 - j100)(1 + 3/(13)) = 134\angle -1.17\,\text{rad} = \underline{52.6 - j123\ \text{V}}$$

As a check, we can evaluate the average power reaching the load:

$$P_{avg,L} = \frac{1}{2}\frac{|V_{s,out}|^2}{R_L} = \frac{1}{2}\frac{(134)^2}{80} = 112\ \text{W}$$

This must be the same power that occurs at the input impedance:

$$P_{avg,in} = \frac{1}{2}\text{Re}\left\{V_{in}I_{in}^*\right\} = \frac{1}{2}\text{Re}\left\{(89.5 - j6.46)(2.54 + j0.54)\right\} = 112\ \text{W}$$

where $I_{in} = V_{in}/Z_{in} = (89.5 - j6.46)/(33.17 - j9.57) = 2.54 + j0.54$.

13.16. A 300 ohm transmission line is 0.8 m long and is terminated with a short circuit. The line is operating in air with a wavelength of 0.3 m (incorrectly stated as 0.8 m in early printings) and is lossless.

a) If the input voltage amplitude is 10V, what is the maximum voltage amplitude at any point on the line? The net voltage anywhere on the line is the sum of the forward and backward wave voltages, and is written as $V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$. Since the line is short-circuited at the load end ($z = 0$), we have $V_0^- = -V_0^+$, and so

$$V(z) = V_0^+ \left(e^{-j\beta z} - e^{j\beta z}\right) = -2jV_0^+ \sin(j\beta z)$$

We now evaluate the voltage at the input, where $z = -0.8$m, and $\lambda = 0.3$m.

$$V_{in} = -2jV_0^+ \sin\left(\frac{2\pi(-0.8)}{0.3}\right) = -j1.73V_0^+$$

The magnitude of V_{in} is given as 10V, so we find $V_0^+ = 10/1.73 = 5.78$V. The maximum voltage amplitude on the line will be twice this value (where the sine function is unity), so $|V|_{max} = 2(5.78) = \underline{11.56\text{ V}}$.

b) What is the current amplitude in the short circuit? At the shorted end, the current will be

$$I_L = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0} = \frac{2V_0^+}{Z_0} = \frac{11.56}{300} = 0.039A = \underline{39\text{ mA}}$$

13.17. Determine the average power absorbed by each resistor in Fig. 13.27: The problem is made easier by first converting the current source/100 ohm resistor combination to its Thevenin equivalent. This is a $50\angle 0$ V voltage source in series with the 100 ohm resistor. The next step is to determine the input impedance of the 2.6λ length line, terminated by the 25 ohm resistor: We use $\beta l = (2\pi/\lambda)(2.6\lambda) = 16.33$ rad. This value, modulo 2π is (by subtracting 2π twice) 3.77 rad. Now

$$Z_{in} = 50\left[\frac{25\cos(3.77) + j50\sin(3.77)}{50\cos(3.77) + j25\sin(3.77)}\right] = 33.7 + j24.0$$

The equivalent circuit now consists of the series combination of 50 V source, 100 ohm resistor, and Z_{in}, as calculated above. The current in this circuit will be

$$I = \frac{50}{100 + 33.7 + j24.0} = 0.368\angle -.178$$

The power dissipated by the 25 ohm resistor is the same as the power dissipated by the real part of Z_{in}, or

$$P_{25} = P_{33.7} = \frac{1}{2}|I|^2 R = \frac{1}{2}(.368)^2(33.7) = \underline{2.28\text{ W}}$$

To find the power dissipated by the 100 ohm resistor, we need to return to the Norton configuration, with the original current source in parallel with the 100 ohm resistor, and in parallel with Z_{in}. The voltage across the 100 ohm resistor will be the same as that across Z_{in}, or
$V = IZ_{in} = (.368\angle -.178)(33.7 + j24.0) = 15.2\angle 0.44$. The power dissipated by the 100 ohm resistor is now

$$P_{100} = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}\frac{(15.2)^2}{100} = \underline{1.16\text{ W}}$$

13.18 The line shown in Fig. 13.28 is lossless. Find s on both sections 1 and 2: For section 2, we consider the propagation of one forward and one backward wave, comprising the superposition of all reflected waves from both ends of the section. The ratio of the backward to the forward wave amplitude is given by the reflection coefficient at the load, which is

$$\Gamma_L = \frac{50 - j100 - 50}{50 - j100 + 50} = \frac{-j}{1-j} = \frac{1}{2}(1-j)$$

Then $|\Gamma_L| = (1/2)\sqrt{(1-j)(1+j)} = 1/\sqrt{2}$. Finally

$$s_2 = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 1/\sqrt{2}}{1 - 1/\sqrt{2}} = \underline{5.83}$$

For section 1, we need the reflection coefficient at the junction (location of the 100 Ω resistor) seen by waves incident from section 1: We first need the input impedance of the .2λ length of section 2:

$$Z_{in2} = 50\left[\frac{(50 - j100)\cos(\beta_2 l) + j50\sin(\beta_2 l)}{50\cos(\beta_2 l) + j(50 - j100)\sin(\beta_2 l)}\right] = 50\left[\frac{(1-j2)(0.309) + j0.951}{0.309 + j(1-j2)(0.951)}\right]$$
$$= 8.63 + j3.82 = 9.44\angle 0.42 \text{ rad}$$

Now, this impedance is in parallel with the 100Ω resistor, leading to a net junction impedance found by

$$\frac{1}{Z_{inT}} = \frac{1}{100} + \frac{1}{8.63 + j3.82} \quad \Rightarrow \quad Z_{inT} = 8.06 + j3.23 = 8.69\angle 0.38 \text{ rad}$$

The reflection coefficient will be

$$\Gamma_j = \frac{Z_{inT} - 50}{Z_{inT} + 50} = -0.717 + j0.096 = 0.723\angle 3.0 \text{ rad}$$

and the standing wave ratio is $s_1 = (1 + 0.723)/(1 - 0.723) = \underline{6.22}$.

13.19. A lossless transmission line is 50 cm in length and operating at a frequency of 100 MHz. The line parameters are $L = 0.2$ μH/m and $C = 80$ pF/m. The line is terminated by a short circuit at $z = 0$, and there is a load, $Z_L = 50 + j20$ ohms across the line at location $z = -20$ cm. What average power is delivered to Z_L if the input voltage is $100\angle 0$ V? With the given capacitance and inductance, we find

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{2 \times 10^{-7}}{8 \times 10^{-11}}} = 50 \ \Omega$$

and

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2 \times 10^{-7})(9 \times 10^{-11})}} = 2.5 \times 10^8 \text{ m/s}$$

Now $\beta = \omega/v_p = (2\pi \times 10^8)/(2.5 \times 10^8) = 2.5$ rad/s. We then find the input impedance to the shorted line section of length 20 cm (putting this impedance at the location of Z_L, so we can combine them): We have $\beta l = (2.5)(0.2) = 0.50$, and so, using the input impedance formula with a zero load impedance, we find $Z_{in1} = j50\tan(0.50) = j27.4$ ohms.

13.19 (continued) Now, at the location of Z_L, the net impedance there is the parallel combination of Z_L and Z_{in1}: $Z_{net} = (50+j20)||(j27.4) = 7.93+j19.9$. We now transform this impedance to the line input, 30 cm to the left, obtaining (with $\beta l = (2.5)(.3) = 0.75$):

$$Z_{in2} = 50\left[\frac{(7.93+j19.9)\cos(.75)+j50\sin(.75)}{50\cos(.75)+j(7.93+j19.9)\sin(.75)}\right] = 35.9+j98.0 = 104.3\angle 1.22$$

The power delivered to Z_L is the same as the power delivered to Z_{in2}: The current magnitude is $|I| = (100)/(104.3) = 0.96$ A. So finally,

$$P = \frac{1}{2}|I|^2 R = \frac{1}{2}(0.96)^2(35.9) = \underline{16.5\text{ W}}$$

13.20. This problem was originally posed incorrectly. The corrected version should have an inductor in the input circuit instead of a capacitor. I will proceed with this replacement understood, and will change the wording as appropriate in parts *c* and *d*:

a) Determine s on the transmission line of Fig. 13.29. Note that the dielectric is air: The reflection coefficient at the load is

$$\Gamma_L = \frac{40+j30-50}{40+j30+50} = j0.333 = 0.333\angle 1.57\,\text{rad} \quad \text{Then } s = \frac{1+.333}{1-.333} = \underline{2.0}$$

b) Find the input impedance: With the length of the line at 2.7λ, we have $\beta l = (2\pi)(2.7) = 16.96$ rad. The input impedance is then

$$Z_{in} = 50\left[\frac{(40+j30)\cos(16.96)+j50\sin(16.96)}{50\cos(16.96)+j(40+j30)\sin(16.96)}\right] = 50\left[\frac{-1.236-j5.682}{1.308-j3.804}\right] = \underline{61.8-j37.5\ \Omega}$$

c) If $\omega L = 10\,\Omega$, find I_s: The source drives a total impedance given by $Z_{net} = 20+j\omega L+Z_{in} = 20+j10+61.8-j37.5 = 81.8-j27.5$. The current is now $I_s = 100/(81.8-j27.5) = \underline{1.10+j0.37\text{ A}}$.

d) What value of L will produce a maximum value for $|I_s|$ at $\omega = 1$ Grad/s? To achieve this, the imaginary part of the total impedance of part *c* must be reduced to zero (so we need an inductor). The inductor impedance must be equal to negative the imaginary part of the line input impedance, or $\omega L = 37.5$, so that $L = 37.5/\omega = \underline{37.5\text{ nH}}$. Continuing, for this value of L, calculate the average power:

e) supplied by the source: $P_s = (1/2)\text{Re}\{V_s I_s\} = (1/2)(100)^2/(81.8) = \underline{61.1\text{ W}}$.

f) delivered to $Z_L = 40+j30\,\Omega$: The power delivered to the load will be the same as the power delivered to the input impedance. We write

$$P_L = \frac{1}{2}\text{Re}\{Z_{in}\}|I_s|^2 = \frac{1}{2}(61.8)(1.22)^2 = \underline{46.1\text{ W}}$$

13.21. A lossless line having an air dielectric has a characteristic impedance of 400 Ω. The line is operating at 200 MHz and $Z_{in} = 200 - j200\ \Omega$. Use analytic methods or the Smith chart (or both) to find: (a) s; (b) Z_L if the line is 1 m long; (c) the distance from the load to the nearest voltage maximum: I will first use the analytic approach. Using normalized impedances, Eq. (13) becomes

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[\frac{z_L \cos(\beta L) + j \sin(\beta L)}{\cos(\beta L) + j z_L \sin(\beta L)}\right] = \left[\frac{z_L + j \tan(\beta L)}{1 + j z_L \tan(\beta L)}\right]$$

Solve for z_L:

$$z_L = \left[\frac{z_{in} - j \tan(\beta L)}{1 - j z_{in} \tan(\beta L)}\right]$$

where, with $\lambda = c/f = 3 \times 10^8/2 \times 10^8 = 1.50$ m, we find $\beta L = (2\pi)(1)/(1.50) = 4.19$, and so $\tan(\beta L) = 1.73$. Also, $z_{in} = (200 - j200)/400 = 0.5 - j0.5$. So

$$z_L = \frac{0.5 - j0.5 - j1.73}{1 - j(0.5 - j0.5)(1.73)} = 2.61 + j0.174$$

Finally, $Z_L = z_L(400) = \underline{1.04 \times 10^3 + j69.8\ \Omega}$. Next

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{6.42 \times 10^2 + j69.8}{1.44 \times 10^3 + j69.8} = .446 + j2.68 \times 10^{-2} = .447 \angle 6.0 \times 10^{-2}\ \text{rad}$$

Now

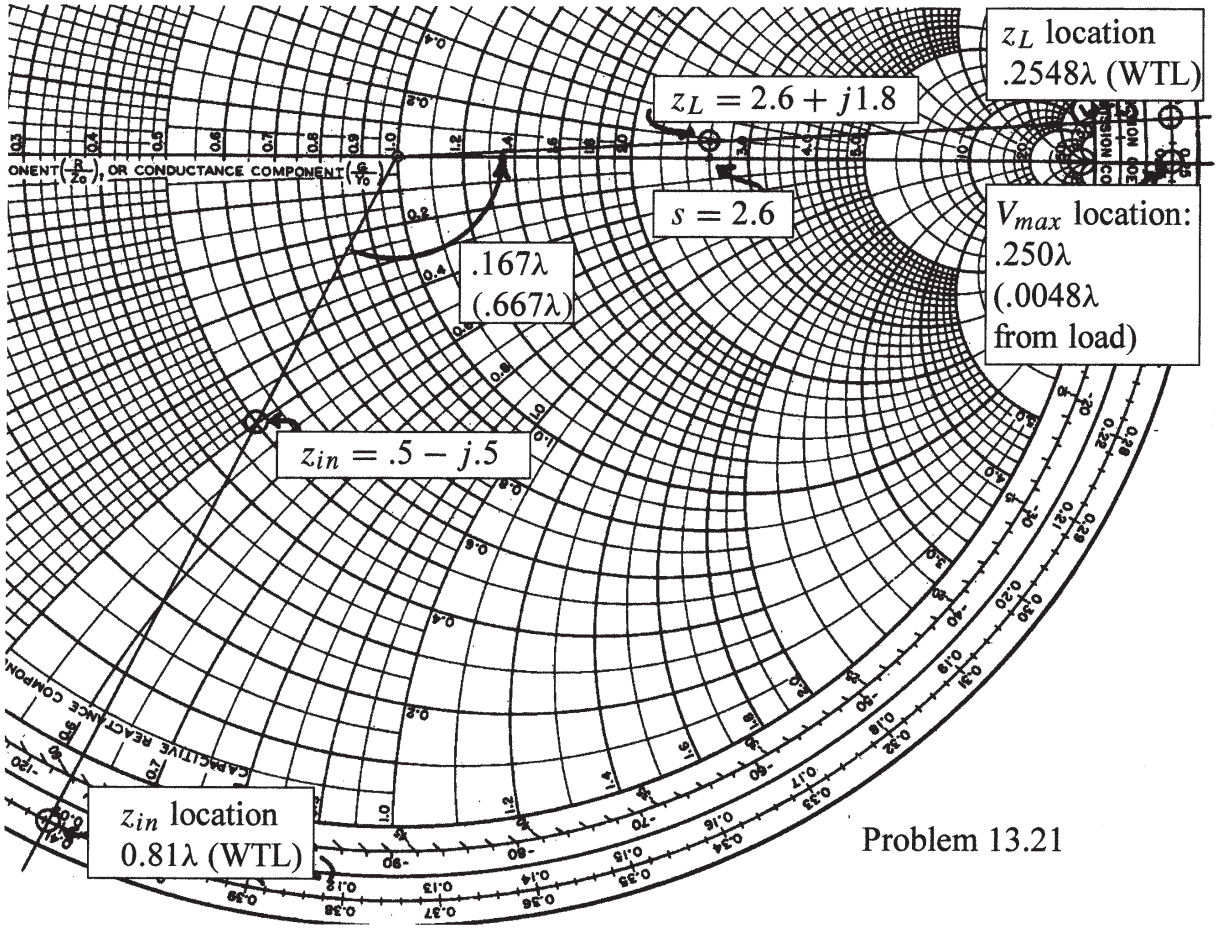
$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .447}{1 - .447} = \underline{2.62}$$

Finally

$$z_{max} = -\frac{\phi}{2\beta} = -\frac{\lambda\phi}{4\pi} = -\frac{(6.0 \times 10^{-2})(1.50)}{4\pi} = -7.2 \times 10^{-3}\ \text{m} = \underline{-7.2\ \text{mm}}$$

We next solve the problem using the Smith chart. Referring to the figure on the next page, we first locate and mark the normalized input impedance, $z_{in} = 0.5 - j0.5$. A line drawn from the origin through this point intersects the outer chart boundary at the position $0.0881\,\lambda$ on the wavelengths toward load (WTL) scale. With a wavelength of 1.5 m, the 1 meter line is 0.6667 wavelengths long. On the WTL scale, we add 0.6667λ, or equivalently, 0.1667λ (since 0.5λ is once around the chart), obtaining $(0.0881 + 0.1667)\lambda) = 0.2548\lambda$, which is the position of the load. A straight line is now drawn from the origin though the 0.2548λ position. A compass is then used to measure the distance between the origin and z_{in}. With this distance set, the compass is then used to scribe off the same distance from the origin to the load impedance, along the line between the origin and the 0.2548λ position. That point is the normalized load impedance, which is read to be $z_L = 2.6 + j0.18$. Thus $Z_L = z_L(400) = 1040 + j72$. This is in reasonable agreement with the analytic result of $1040 + j69.8$. The difference in imaginary parts arises from uncertainty in reading the chart in that region.

In transforming from the input to the load positions, we cross the $r > 1$ real axis of the chart at r=2.6. This is close to the value of the VSWR, as we found earlier. We also see that the $r > 1$ real axis (at which the first V_{max} occurs) is a distance of 0.0048λ (marked as $.005\lambda$ on the chart) in front of the load. The actual distance is $z_{max} = -0.0048(1.5)\,\text{m} = -0.0072\,\text{m} = -7.2\,\text{mm}$.



Problem 13.21

13.22. A lossless two-wire line has a characteristic impedance of 300 Ω and a capacitance of 15 pF/m. The load at $z = 0$ consists of a 600-Ω resistor in parallel with a 10-pF capacitor. If $\omega = 10^8$ rad/s and the line is 20m long, use the Smith chart to find a) $|\Gamma_L|$; b) s; c) Z_{in}. First, the wavelength on the line is found using $\lambda = 2\pi v_p/\omega$, where $v_p = 1/(CZ_0)$. Assuming higher accuracy in the given values than originally stated, we obtain

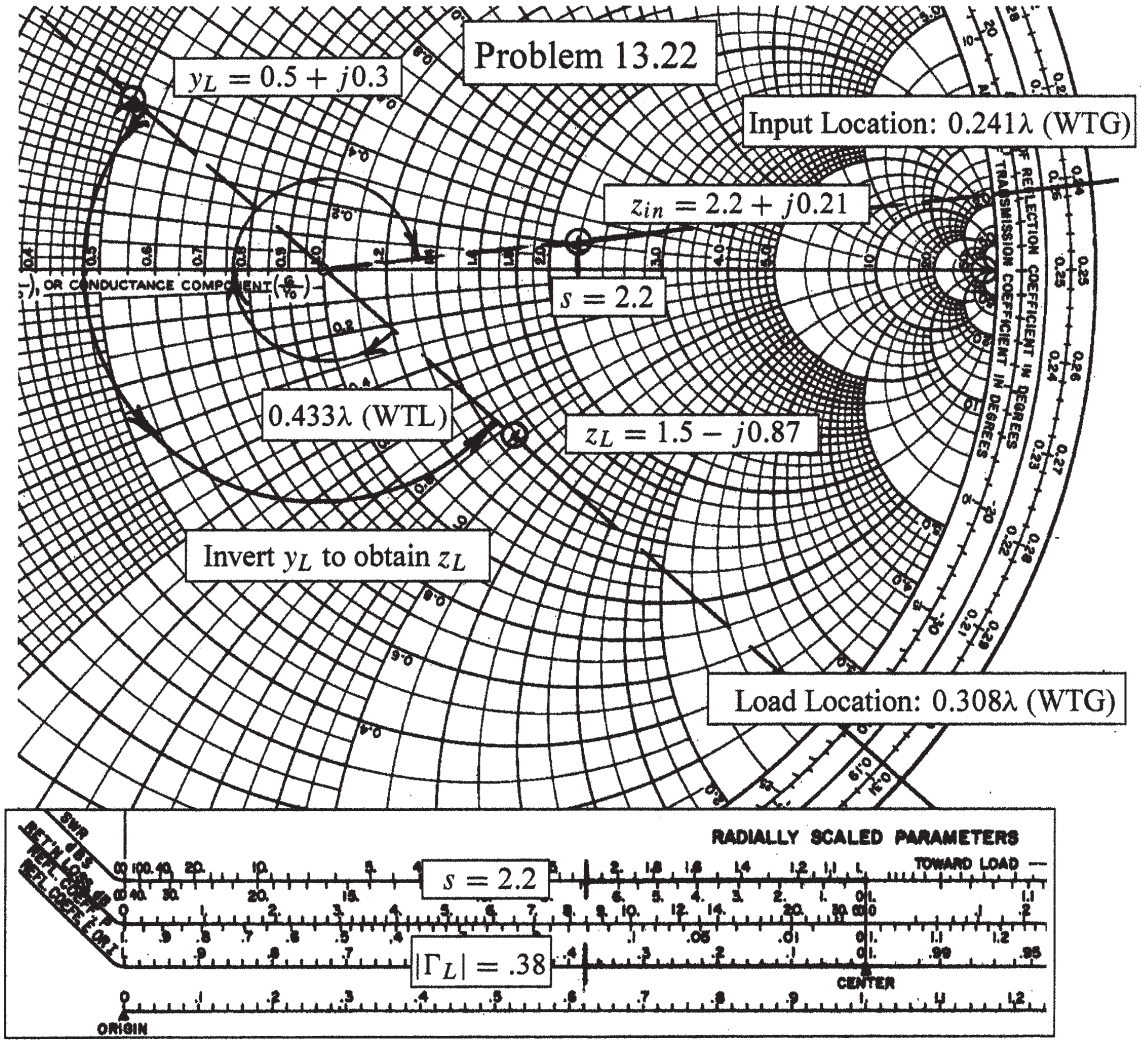
$$\lambda = \frac{2\pi}{\omega C Z_0} = \frac{2\pi}{(10^8)(15 \times 10^{-12})(300)} = 13.96\,\text{m}$$

The line length in wavelengths is therefore $20/13.96 = 1.433\lambda$. The normalized load admittance is now

$$y_L = Y_L Z_0 = Z_0 \left[\frac{1}{R_L} + j\omega C\right] = 300\left[\frac{1}{600} + j(10^8)(10^{-11})\right] = 0.50 + j0.30$$

The y_L value is plotted on the chart and labeled as y_L. Next, y_L is inverted to find z_L by transforming the point halfway around the chart, using the compass and a straight edge. The result, labeled z_L on the chart is read to be $z_L = 1.5 - j0.87$. This is close to the computed inverse of y_L, which is $1.47 - j0.88$. Scribing the compass arc length along the bottom scale for reflection coefficient yields $|\Gamma_L| = \underline{0.38}$. The VSWR is found by scribing the compass arc length either along the bottom SWR scale or along the positive real axis of the chart, both methods yielding $s = \underline{2.2}$.

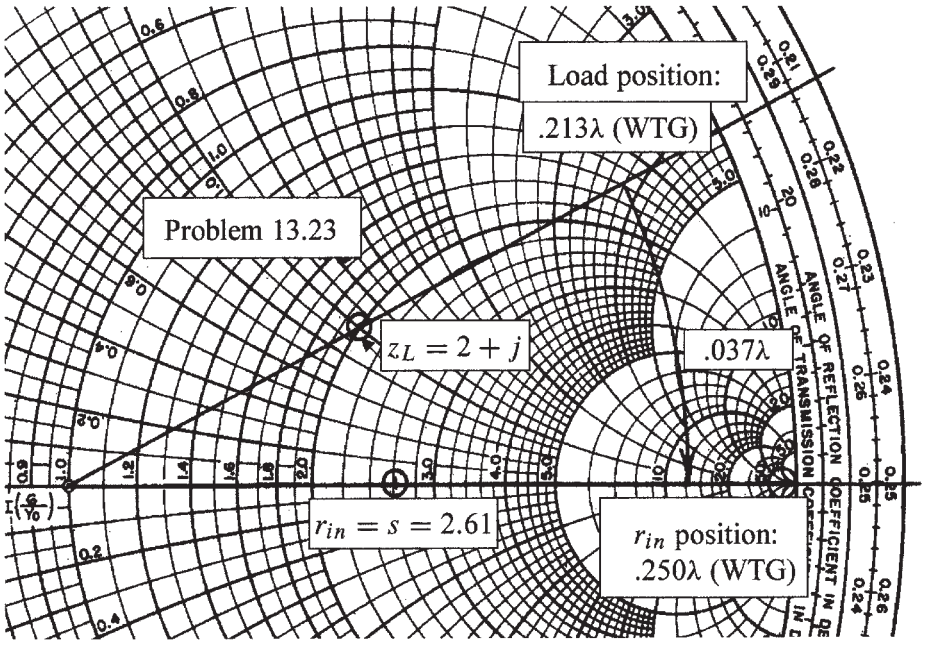
Now, the position of z_L is read on the outer edge of the chart as 0.308λ on the WTG scale. The point is now transformed through the line length distance of 1.433λ toward the generator (the net chart distance will be 0.433λ, since a full wavelength is two complete revolutions). The final reading on the WTG scale after the transformation is found through $(0.308 + 0.433 - 0.500)\lambda = 0.241\lambda$. Drawing a line between this mark on the WTG scale and the chart center, and scribing the compass arc length on this line, yields the normalized input impedance. This is read as $z_{in} = 2.2 + j0.21$ (the computed value found through the analytic solution is $z_{in} = 2.21 + j0.219$. The input impedance is now found by multiplying the chart reading by 300, or $Z_{in} = \underline{660 + j63\,\Omega}$.



13.23. The normalized load on a lossless transmission line is $z_L = 2 + j1$. Let $l = 20$ m (there was a missprint in the problem statement, since $\lambda = 20$ m should have been stated. I will specify answers in terms of wavelength). Make use of the Smith chart to find:

a) the shortest distance from the load to the point at which $z_{in} = r_{in} + j0$, where $r_{in} > 1$ (not greater than 0 as stated): Referring to the figure below, we start by marking the given z_L on the chart and drawing a line from the origin through this point to the outer boundary. On the WTG scale, we read the z_L location as 0.213λ. Moving from here toward the generator, we cross the positive Γ_R axis (at which the impedance is purely real and greater than 1) at 0.250λ. The distance is then $(0.250 - 0.213)\lambda = \underline{0.037\lambda}$ from the load. If we use $\lambda = 20$ m, the actual distance would be $20(0.037) = 0.74$ m.

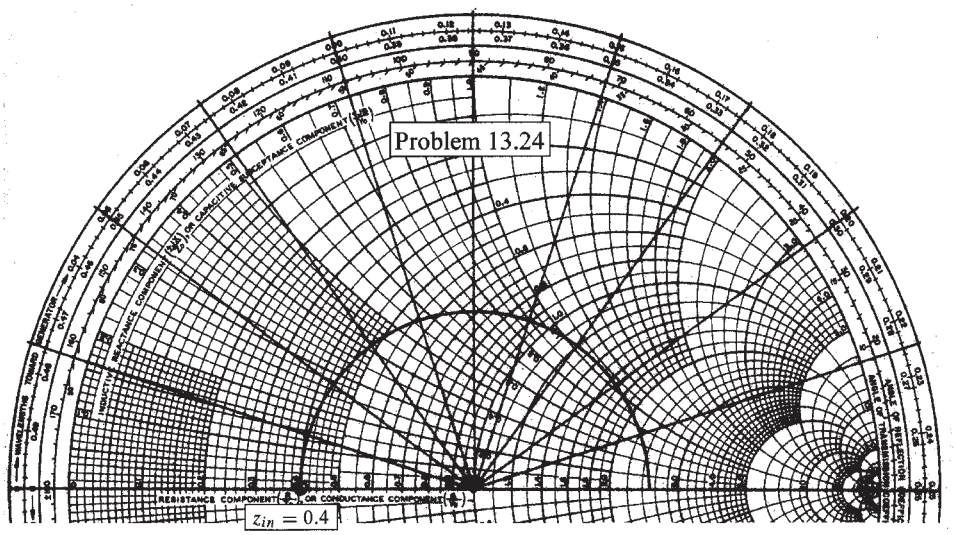
b) Find z_{in} at the point found in part a: Using a compass, we set its radius at the distance between the origin and z_L. We then scribe this distance along the real axis to find $z_{in} = r_{in} = \underline{2.61}$.



c) The line is cut at this point and the portion containing z_L is thrown away. A resistor $r = r_{in}$ of part a is connected across the line. What is s on the remainder of the line? This will be just s for the line as it was before. As we know, s will be the positive real axis value of the normalized impedance, or $s = \underline{2.61}$.

d) What is the shortest distance from this resistor to a point at which $z_{in} = 2 + j1$? This would return us to the original point, requiring a complete circle around the chart (one-half wavelength distance). The distance from the resistor will therefore be: $d = 0.500\,\lambda - 0.037\,\lambda = \underline{0.463\,\lambda}$. With $\lambda = 20$ m, the actual distance would be $20(0.463) = 9.26$ m.

13.24. With the aid of the Smith chart, plot a curve of $|Z_{in}|$ vs. l for the transmission line shown in Fig. 13.30. Cover the range $0 < l/\lambda < 0.25$. The required input impedance is that at the actual line input (to the left of the two 20Ω resistors. The input to the line section occurs just to the right of the 20Ω resistors, and the input impedance there we first find with the Smith chart. This impedance is in series with the two 20Ω resistors, so we add 40Ω to the calculated impedance from the Smith chart to find the net line input impedance. To begin, the 20Ω load resistor represents a normalized impedance of $z_l = 0.4$, which we mark on the chart (see below). Then, using a compass, draw a circle beginning at z_L and progressing clockwise to the positive real axis. The circle traces the locus of z_{in} values for line lengths over the range $0 < l < \lambda/4$.



On the chart, radial lines are drawn at positions corresponding to .025λ increments on the WTG scale. The intersections of the lines and the circle give a total of 11 z_{in} values. To these we add normalized impedance of $40/50 = 0.8$ to add the effect of the 40Ω resistors and obtain the normalized impedance at the line input. The magnitudes of these values are then found, and the results are multiplied by 50Ω. The table below summarizes the results.

l/λ	z_{inl} (to right of 40Ω)	$z_{in} = z_{inl} + 0.8$	$\|Z_{in}\| = 50\|z_{in}\|$
0	0.40	1.20	60
.025	0.41 + j.13	1.21 + j.13	61
.050	0.43 + j.27	1.23 + j.27	63
.075	0.48 + j.41	1.28 + j.41	67
.100	0.56 + j.57	1.36 + j.57	74
.125	0.68 + j.73	1.48 + j.73	83
.150	0.90 + j.90	1.70 + j.90	96
.175	1.20 + j1.05	2.00 + j1.05	113
.200	1.65 + j1.05	2.45 + j1.05	134
.225	2.2 + j.7	3.0 + j.7	154
.250	2.5	3.3	165

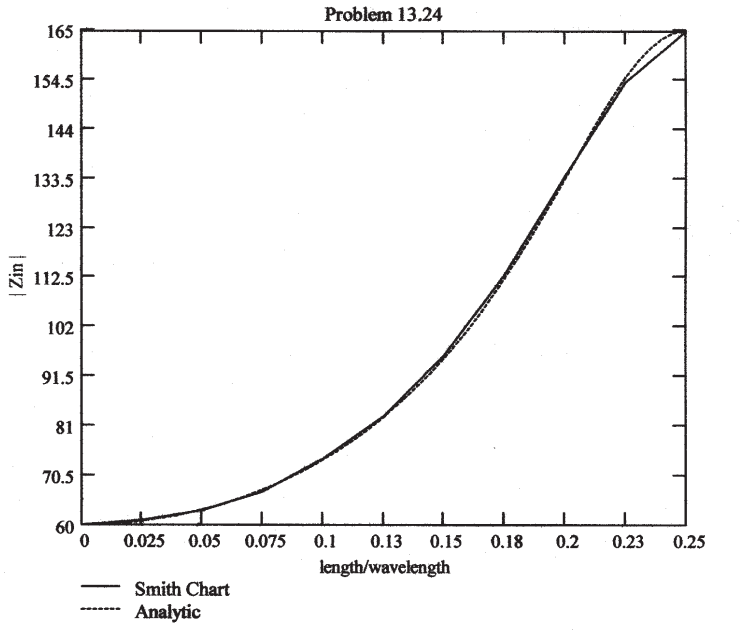
13.24. (continued) As a check, the line input input impedance can be found analytically through

$$Z_{in} = 40 + 50\left[\frac{20\cos(2\pi l/\lambda) + j50\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)}\right] = 50\left[\frac{60\cos(2\pi l/\lambda) + j66\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)}\right]$$

from which

$$|Z_{in}| = 50\left[\frac{36\cos^2(2\pi l/\lambda) + 43.6\sin^2(2\pi l/\lambda)}{25\cos^2(2\pi l/\lambda) + 4\sin^2(2\pi l/\lambda)}\right]^{1/2}$$

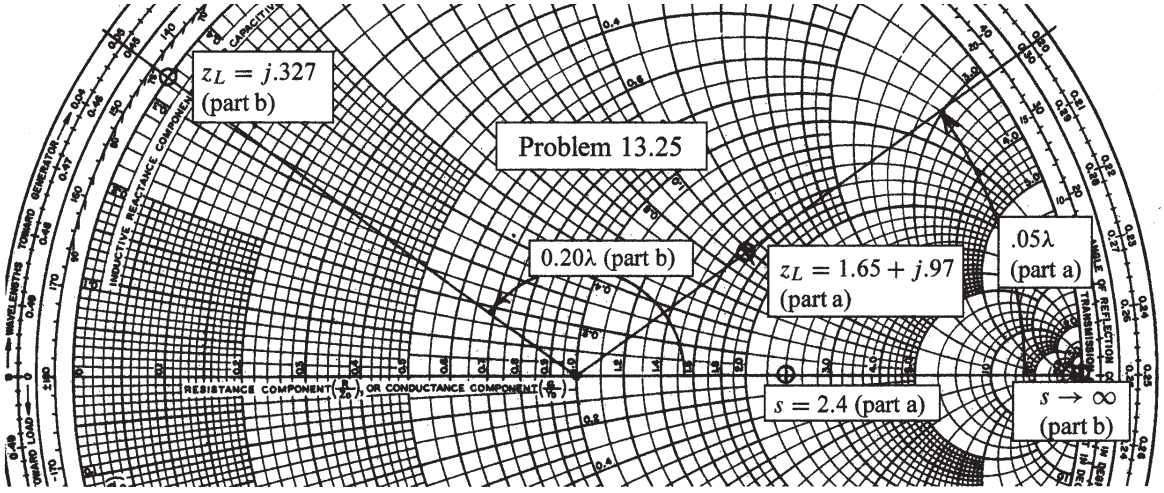
This function is plotted below along with the results obtained from the Smith chart. A fairly good comparison is obtained.



13.25. A 300-ohm transmission line is short-circuited at $z = 0$. A voltage maximum, $|V|_{max} = 10$ V, is found at $z = -25$ cm, and the minimum voltage, $|V|_{min} = 0$, is found at $z = -50$ cm. Use the Smith chart to find Z_L (with the short circuit replaced by the load) if the voltage readings are:

a) $|V|_{max} = 12$ V at $z = -5$ cm, and $vert V|_{min} = 5$ V: First, we know that the maximum and minimum voltages are spaced by $\lambda/4$. Since this distance is given as 25 cm, we see that $\lambda = 100$ cm = 1 m. Thus the maximum voltage location is $5/100 = 0.05\lambda$ in front of the load. The standing wave ratio is $s = |V|_{max}/|V|_{min} = 12/5 = 2.4$. We mark this on the positive real axis of the chart (see next page). The load position is now 0.05 wavelengths toward the load from the $|V|_{max}$ position, or at 0.30 λ on the WTL scale. A line is drawn from the origin through this point on the chart, as shown. We next set the compass to the distance between the origin and the $z = r = 2.4$ point on the real axis. We then scribe this same distance along the line drawn through the .30 λ position. The intersection is the value of z_L, which we read as $z_L = 1.65 + j.97$. The actual load impedance is then $Z_L = 300 z_L = \underline{495 + j290\ \Omega}$.

b) $|V|_{max} = 17$ V at $z = -20$ cm, and $|V|_{min} = 0$. In this case the standing wave ratio is infinite, which puts the starting point on the $r \to \infty$ point on the chart. The distance of 20 cm corresponds to $20/100 = 0.20\,\lambda$, placing the load position at $0.45\,\lambda$ on the WTL scale. A line is drawn from the origin through this location on the chart. An infinite standing wave ratio places us on the outer boundary of the chart, so we read $z_L = j0.327$ at the $0.45\,\lambda$ WTL position. Thus $Z_L = j300(0.327) \doteq \underline{j98\,\Omega}$.



13.26. A lossless 50Ω transmission line operates with a velocity that is $3/4c$. A load, $Z_L = 60 + j30\,\Omega$ is located at $z = 0$. Use the Smith chart to find:

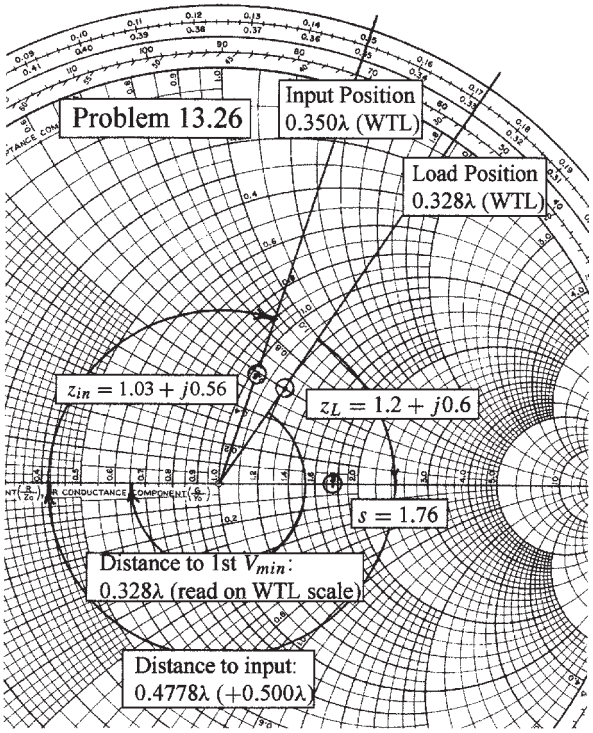
a) s: First we find the normalized load impedance, $z_L = (60 + j30)/50 = 1.2 + j0.6$, which is then marked on the chart (see below). Drawing a line from the chart center through this point yields its location at 0.328λ on the WTL scale. The distance from the origin to the load impedance point is now set on the compass; the standing wave ratio is then found by scribing this distance along the positive real axis, yielding $s = \underline{1.76}$, as shown. Alternately, use the s scale at the bottom of the chart, setting the compass point at the center, and scribing the distance on the scale to the left.

b) the distance from the load to the nearest voltage minimum if $f = 300$ MHz: This distance is found by transforming the load impedance clockwise around the chart until the negative real axis is reached. This distance in wavelengths is just the load position on the WTL scale, since the starting point for this scale is the negative real axis. So the distance is 0.328λ. The wavelength is

$$\lambda = \frac{v}{f} = \frac{(3/4)c}{300\text{MHz}} = \frac{3(3\times 10^8)}{4(3\times 10^8)} = 0.75\,\text{m}$$

So the actual distance to the first voltage minimum is $d_{min} = 0.328(0.75)\text{ m} = \underline{24.6\,\text{cm}}$.

c) the input impedance if $f = 200$ MHz and the input is at $z = -110$cm: The wavelength at this frequency is $\lambda = (3/4)(3\times 10^8)/(2\times 10^8) = 1.125$ m. The distance to the input in wavelengths is then $d_{in} = (1.10)/(1.125) = 0.9778\lambda$. Transforming the load through this distance toward the generator involves revolution once around the chart (0.500λ) plus the remainder of 0.4778λ, which leads to a final position of $0.1498\lambda \doteq 0.150\lambda$ on the WTG scale, or 0.350λ on the WTL scale. A line is drawn between this point and the chart center. Scribing the compass arc length through this line yields the normalized input impedance, read as $z_{in} = 1.03 + j0.56$. The actual input impedance is $Z_{in} = z_{in} \times 50 = \underline{51.5 + j28.0\,\Omega}$.

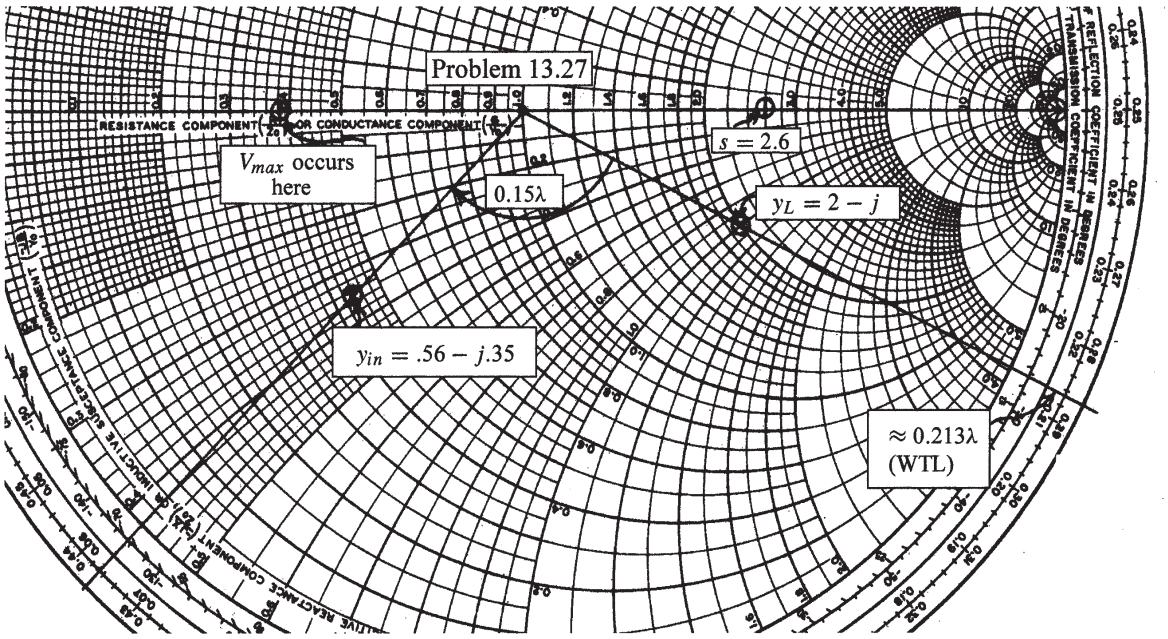


13.27. The characteristic admittance ($Y_0 = 1/Z_0$) of a lossless transmission line is 20 mS. The line is terminated in a load $Y_L = 40 - j20$ mS. Make use of the Smith chart to find:

a) s: We first find the normalized load admittance, which is $y_L = Y_L/Y_0 = 2 - j1$. This is plotted on the Smith chart below. We then set on the compass the distance between y_L and the origin. The same distance is then scribed along the positive real axis, and the value of s is read as $\underline{2.6}$.

b) Y_{in} if $l = 0.15\,\lambda$: First we draw a line from the origin through z_L and note its intersection with the WTG scale on the chart outer boundary. We note a reading on that scale of about 0.287 λ. To this we add 0.15 λ, obtaining about 0.437 λ, which we then mark on the chart (0.287 λ is not the precise value, but I have added 0.15 λ to that mark to obtain the point shown on the chart that is near to 0.437 λ. This "eyeballing" method increases the accuracy a little). A line drawn from the 0.437 λ position on the WTG scale to the origin passes through the input admittance. Using the compass, we scribe the distance found in part a across this line to find $y_{in} = 0.56 - j0.35$, or $Y_{in} = 20 y_{in} = \underline{11 - j7.0\text{ mS}}$.

c) the distance in wavelengths from Y_L to the nearest voltage maximum: On the admittance chart, the V_{max} position is on the negative Γ_r axis. This is at the zero position on the WTL scale. The load is at the approximate $\underline{0.213\,\lambda}$ point on the WTL scale, so this distance is the one we want.

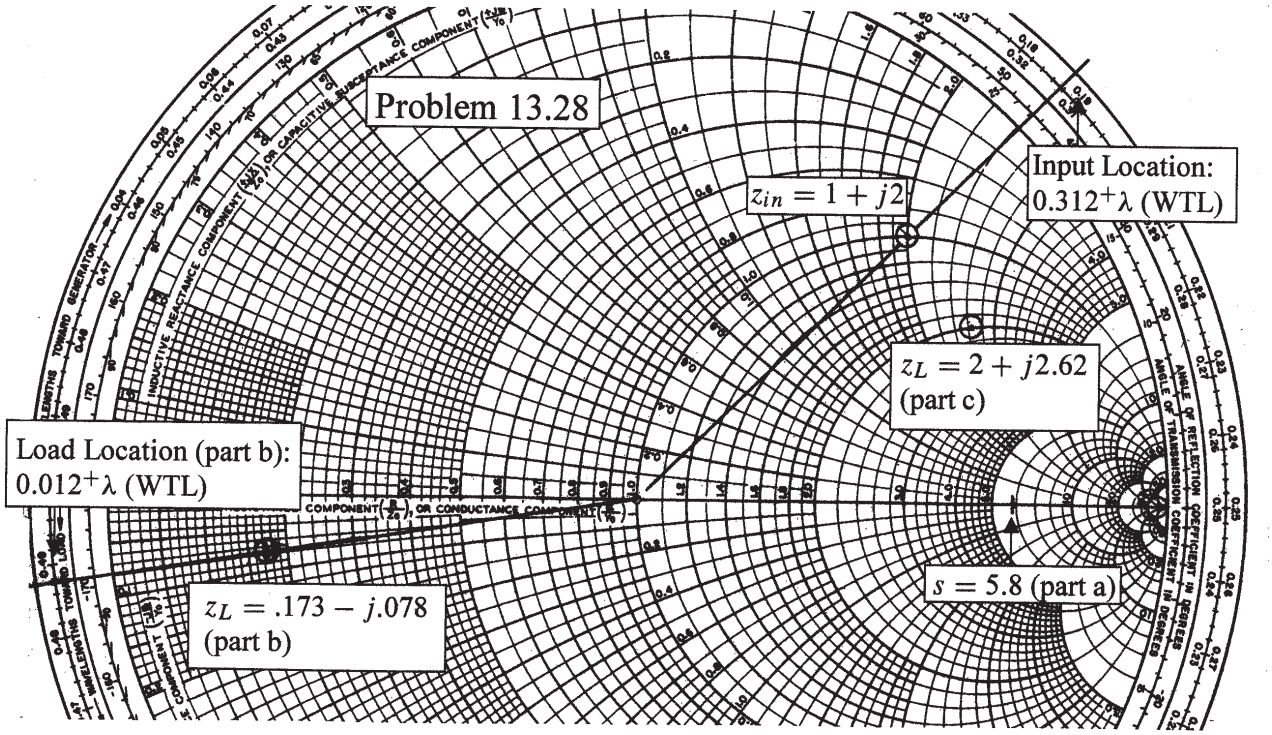


13.28. The wavelength on a certain lossless line is 10cm. If the normalized input impedance is $z_{in} = 1 + j2$, use the Smith chart to determine:

a) s: We begin by marking z_{in} on the chart (see below), and setting the compass at its distance from the origin. We then use the compass at that setting to scribe a mark on the positive real axis, noting the value there of $s = \underline{5.8}$.

b) z_L, if the length of the line is 12 cm: First, use a straight edge to draw a line from the origin through z_{in}, and through the outer scale. We read the input location as slightly more than 0.312λ on the WTL scale (this additional distance beyond the .312 mark is not measured, but is instead used to add a similar distance when the impedance is transformed). The line length of 12cm corresponds to 1.2 wavelengths. Thus, to transform to the load, we go counter-clockwise twice around the chart, plus 0.2λ, finally arriving at (again) slightly more than 0.012λ on the WTL scale. A line is drawn to the origin from that position, and the compass (with its previous setting) is scribed through the line. The intersection is the normalized load impedance, which we read as $z_L = \underline{0.173 - j0.078}$.

c) x_L, if $z_L = 2 + jx_L$, where $x_L > 0$. For this, use the compass at its original setting to scribe through the $r = 2$ circle in the upper half plane. At that point we read $x_L = \underline{2.62}$.



13.29. A standing wave ratio of 2.5 exists on a lossless 60 Ω line. Probe measurements locate a voltage minimum on the line whose location is marked by a small scratch on the line. When the load is replaced by a short circuit, the minima are 25 cm apart, and one minimum is located at a point 7 cm toward the source from the scratch. Find Z_L: We note first that the 25 cm separation between minima imply a wavelength of twice that, or $\lambda = 50$ cm. Suppose that the scratch locates the first voltage minimum. With the short in place, the first minimum occurs at the load, and the second at 25 cm in front of the load. The effect of replacing the short with the load is to move the minimum at 25 cm to a new location 7 cm toward the load, or at 18 cm. This is a possible location for the scratch, which would otherwise occur at multiples of a half-wavelength farther away from that point, toward the generator. Our assumed scratch position will be 18 cm or $18/50 = 0.36$ wavelengths from the load. Using the Smith chart (see below) we first draw a line from the origin through the 0.36λ point on the wavelengths toward load scale. We set the compass to the length corresponding to the $s = r = 2.5$ point on the chart, and then scribe this distance through the straight line. We read $z_L = 0.79 + j0.825$, from which $Z_L = 47.4 + j49.5\ \Omega$. As a check, I will do the problem analytically. First, we use

$$z_{min} = -18\,\text{cm} = -\frac{1}{2\beta}(\phi + \pi) \quad \Rightarrow \quad \phi = \left[\frac{4(18)}{50} - 1\right]\pi = 1.382\,\text{rad} = 79.2^\circ$$

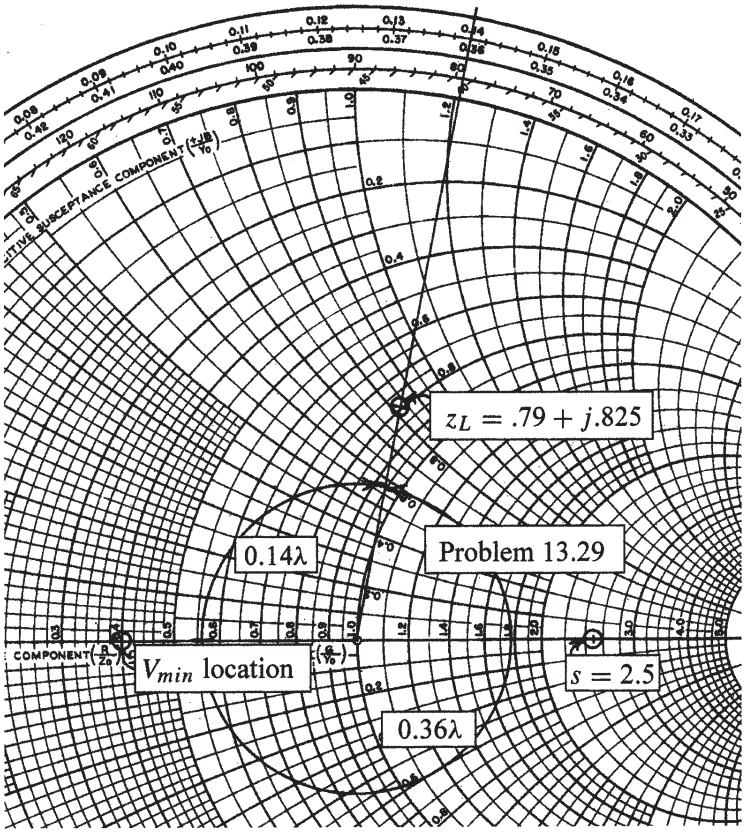
Now

$$|\Gamma_L| = \frac{s-1}{s+1} = \frac{2.5-1}{2.5+1} = 0.4286$$

and so $\Gamma_L = 0.4286\angle 1.382$. Using this, we find

$$z_L = \frac{1+\Gamma_L}{1-\Gamma_L} = 0.798 + j0.823$$

and thus $Z_L = z_L(60) = \underline{47.8 + j49.3\ \Omega}$.

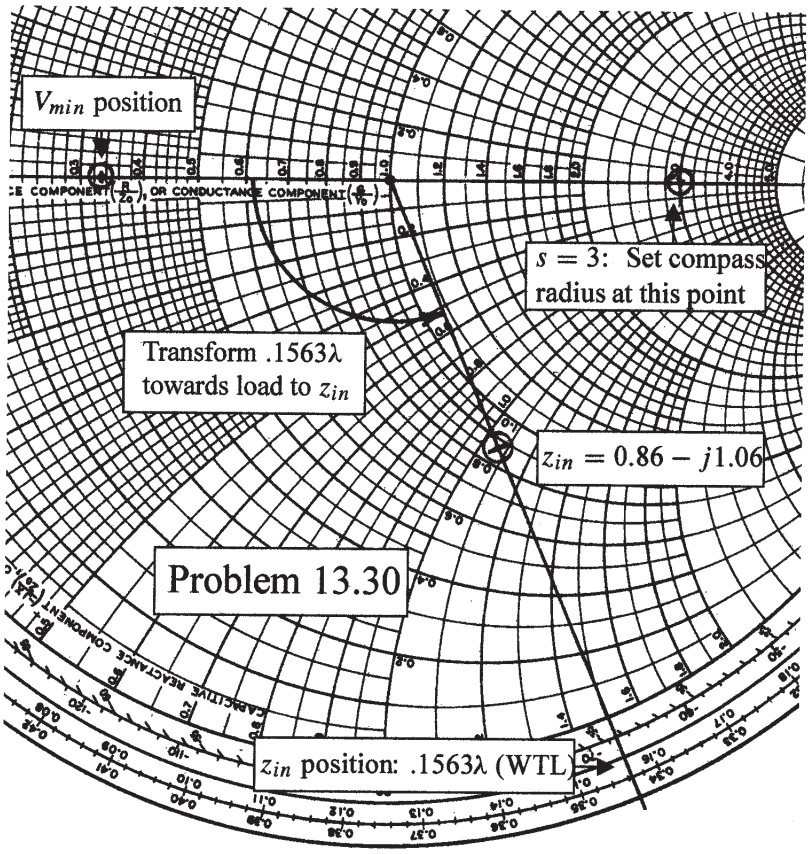


13.30. A 2-wire line, constructed of lossless wire of circular cross-section is gradually flared into a coupling loop that looks like an egg beater. At the point X, indicated by the arrow in Fig. 13.31, a short circuit is placed across the line. A probe is moved along the line and indicates that the first voltage minimum to the left of X is 16cm from X. With the short circuit removed, a voltage minimum is found 5cm to the left of X, and a voltage maximum is located that is 3 times voltage of the minimum. Use the Smith chart to determine:

a) f: No Smith chart is needed to find f, since we know that the first voltage minimum in front of a short circuit is one-half wavelength away. Therefore, $\lambda = 2(16) = 32$cm, and (assuming an air-filled line), $f = c/\lambda = 3 \times 10^8/0.32 = \underline{0.938\,\text{GHz}}$.

b) s: Again, no Smith chart is needed, since s is the ratio of the maximum to the minimum voltage amplitudes. Since we are given that $V_{max} = 3V_{min}$, we find $s = \underline{3}$.

c) the normalized input impedance of the egg beater as seen looking the right at point X: Now we need the chart. From the figure below, $s = 3$ is marked on the positive real axis, which determines the compass radius setting. This point is then transformed, using the compass, to the negative real axis, which corresponds to the location of a voltage minimum. Since the first V_{min} is 5cm in front of X, this corresponds to $(5/32)\lambda = 0.1563\lambda$ to the left of X. On the chart, we now move this distance from the V_{min} location toward the load, using the WTL scale. A line is drawn from the origin through the 0.1563λ mark on the WTL scale, and the compass is used to scribe the original radius through this line. The intersection is the normalized input impedance, which is read as $z_{in} = \underline{0.86 - j1.06}$.



13.31. In order to compare the relative sharpness of the maxima and minima of a standing wave, assume a load $z_L = 4 + j0$ is located at $z = 0$. Let $|V|_{min} = 1$ and $\lambda = 1$ m. Determine the width of the

a) minimum, where $|V| < 1.1$: We begin with the general phasor voltage in the line:

$$V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

With $z_L = 4 + j0$, we recognize the real part as the standing wave ratio. Since the load impedance is real, the reflection coefficient is also real, and so we write

$$\Gamma = |\Gamma| = \frac{s-1}{s+1} = \frac{4-1}{4+1} = 0.6$$

The voltage magnitude is then

$$\begin{aligned} |V(z)| &= \sqrt{V(z)V^*(z)} = V^+\left[(e^{-j\beta z} + \Gamma e^{j\beta z})(e^{j\beta z} + \Gamma e^{-j\beta z})\right]^{1/2} \\ &= V^+\left[1 + 2\Gamma\cos(2\beta z) + \Gamma^2\right]^{1/2} \end{aligned}$$

Note that with $\cos(2\beta z) = \pm 1$, we obtain $|V| = V^+(1 \pm \Gamma)$ as expected. With $s = 4$ and with $|V|_{min} = 1$, we find $|V|_{max} = 4$. Then with $\Gamma = 0.6$, it follows that $V^+ = 2.5$. The net expression for $|V(z)|$ is then

$$V(z) = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)}$$

To find the width in z of the voltage minimum, defined as $|V| < 1.1$, we set $|V(z)| = 1.1$ and solve for z: We find

$$\left(\frac{1.1}{2.5}\right)^2 = 1.36 + 1.2\cos(2\beta z) \quad \Rightarrow \quad 2\beta z = \cos^{-1}(-0.9726)$$

Thus $2\beta z = 2.904$. At this stage, we note the the $|V|_{min}$ point will occur at $2\beta z = \pi$. We therefore compute the range, Δz, over which $|V| < 1.1$ through the equation:

$$2\beta(\Delta z) = 2(\pi - 2.904) \quad \Rightarrow \quad \Delta z = \frac{\pi - 2.904}{2\pi/1} = 0.0378\,\text{m} = \underline{3.8\,\text{cm}}$$

where $\lambda = 1$ m has been used.

b) Determine the width of the maximum, where $|V| > 4/1.1$: We use the same equation for $|V(z)|$, which in this case reads:

$$4/1.1 = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)} \quad \Rightarrow \quad \cos(2\beta z) = 0.6298$$

Since the maximum corresponds to $2\beta z = 0$, we find the range through

$$2\beta\Delta z = 2\cos^{-1}(0.6298) \quad \Rightarrow \quad \Delta z = \frac{0.8896}{2\pi/1} = 0.142\,\text{m} = \underline{14.2\,\text{cm}}$$

13.32. A lossless line is operating with $Z_0 = 40\,\Omega$, $f = 20$ MHz, and $\beta = 7.5\pi$ rad/m. With a short circuit replacing the load, a minimum is found at a point on the line marked by a small spot of puce paint. With the load installed, it is found that $s = 1.5$ and a voltage minimum is located 1m toward the source from the puce dot.

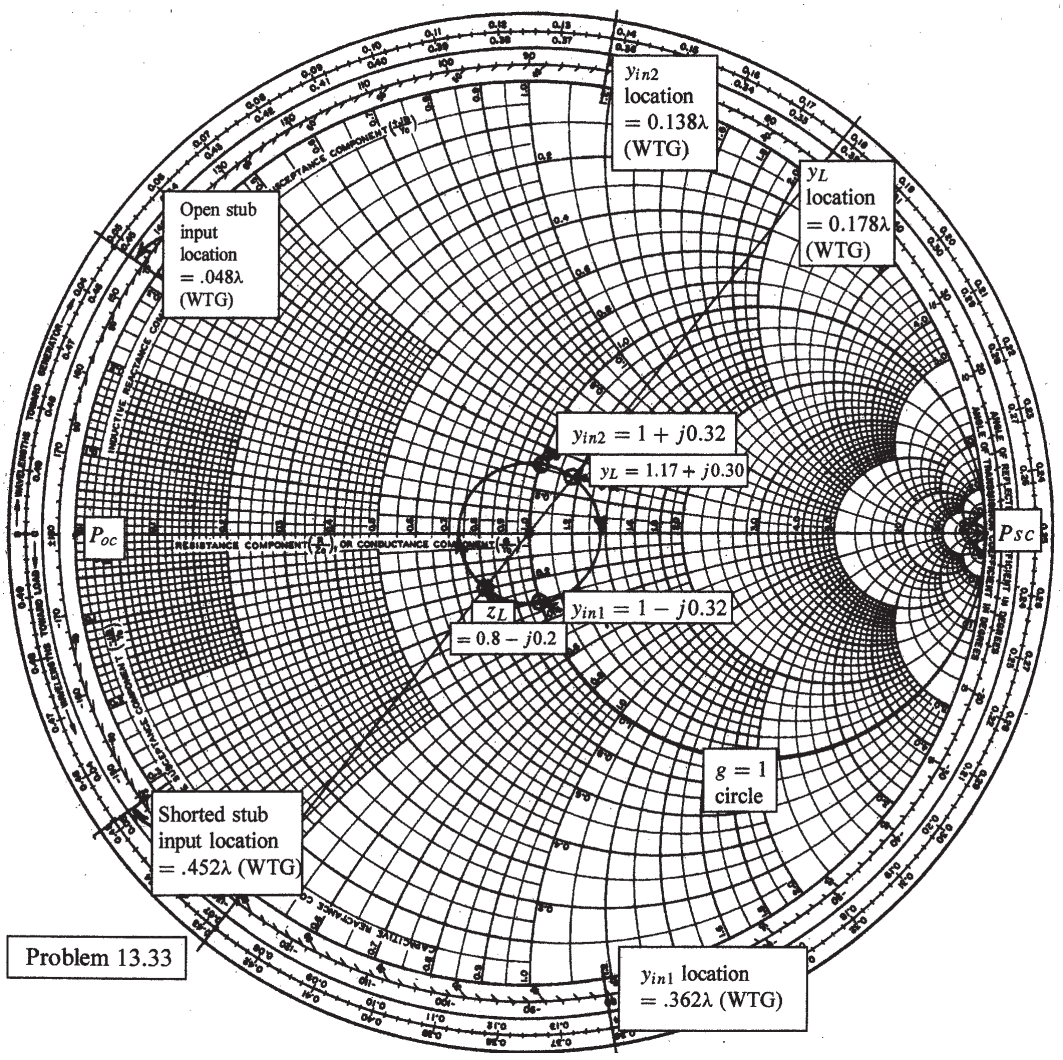
a) Find Z_L: First, the wavelength is given by $\lambda = 2\pi/\beta = 2/7.5 = 0.2667$m. The 1m distance is therefore 3.75λ. With the short installed, the V_{min} positions will be at multiples of $\lambda/2$ to the left of the short. Therefore, with the actual load installed, the V_{min} position as stated would be $3.75\lambda + n\lambda/2$, *which means that a maximum voltage occurs at the load*. This being the case, the normalized load impedance will lie on the positive real axis of the Smith chart, and will be equal to the standing wave ratio. Therefore, $Z_L = 40(1.5) = \underline{60\,\Omega}$.

b) What load would produce $s = 1.5$ with $|V|_{max}$ at the paint spot? With $|V|_{max}$ at the paint spot and with the spot an integer multiple of $\lambda/2$ to the left of the load, $|V|_{max}$ must also occur at the load. The answer is therefore the same as part a, or $Z_L = \underline{60\,\Omega}$.

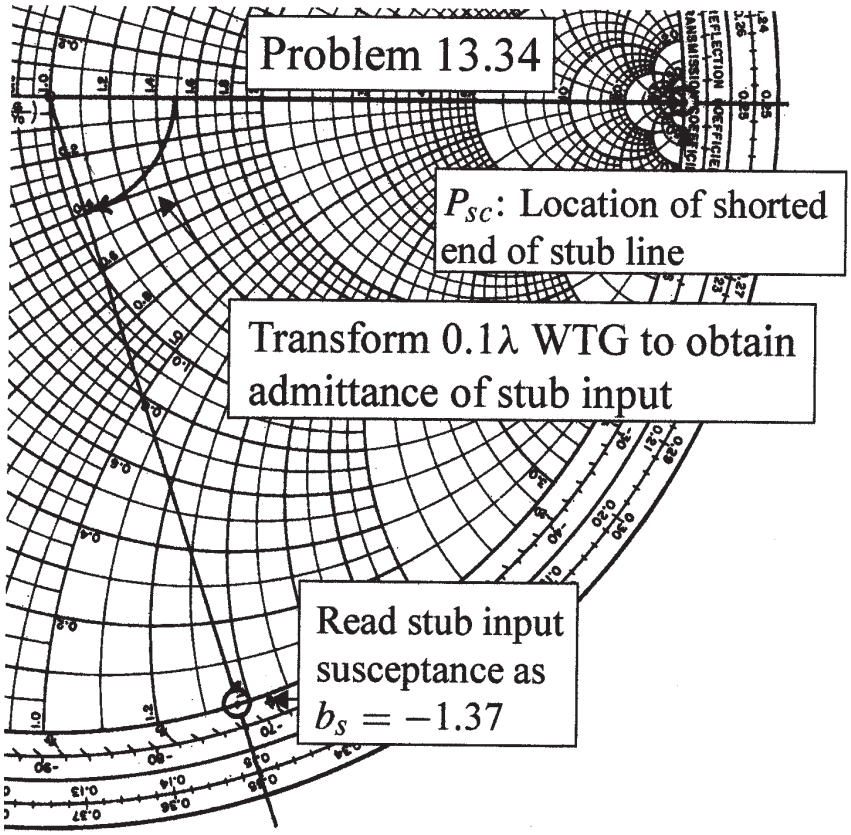
13.33. In Fig. 13.14, let $Z_L = 40 - j10\,\Omega$, $Z_0 = 50\,\Omega$, $f = 800$ MHz, and $v = c$.

a) Find the shortest length, d_1, of a short-circuited stub, and the shortest distance d that it may be located from the load to provide a perfect match on the main line to the left of the stub: The Smith chart construction is shown on the next page. First we find $z_L = (40 - j10)/50 = 0.8 - j0.2$ and plot it on the chart. Next, we find $y_L = 1/z_L$ by transforming this point halfway around the chart, where we read $y_L = 1.17 + j0.30$. This point is to be transformed to a location at which the real part of the normalized admittance is unity. The $g = 1$ circle is highlighted on the chart; y_L transforms to two locations on it: $y_{in1} = 1 - j0.32$ and $y_{in2} = 1 + j0.32$. The stub is connected at either of these two points. The stub input admittance must cancel the imaginary part of the line admittance at that point. If y_{in2} is chosen, the stub must have input admittance of $-j0.32$. This point is marked on the outer circle and occurs at $0.452\,\lambda$ on the WTG scale. The length of the stub is found by computing the distance between its input, found above, and the short-circuit position (stub load end), marked as P_{sc}. This distance is $d_1 = (0.452 - 0.250)\lambda = 0.202\,\lambda$. With $f = 800$ MHz and $v = c$, the wavelength is $\lambda = (3 \times 10^8)/(8 \times 10^8) = 0.375$ m. The distance is thus $d_1 = (0.202)(0.375) = 0.758\,\text{m} = \underline{7.6\,\text{cm}}$. This is the shortest of the two possible stub lengths, since if we had used y_{in1}, we would have needed a stub input admittance of $+j0.32$, which would have required a longer stub length to realize. The length of the main line between its load and the stub attachment point is found on the chart by measuring the distance between y_L and y_{in2}, in moving clockwise (toward generator). This distance will be $d = [0.500 - (0.178 - 0.138)]\,\lambda = 0.46\,\lambda$. The actual length is then $d = (0.46)(0.375) = 0.173\text{m} = \underline{17.3\,\text{cm}}$.

13.33b) Repeat for an open-circuited stub: In this case, everything is the same, except for the load-end position of the stub, which now occurs at the P_{oc} point on the chart. To use the shortest possible stub, we need to use $y_{in1} = 1 - j0.32$, requiring $y_s = +j0.32$. We find the stub length by moving from P_{oc} to the point at which the admittance is $j0.32$. This occurs at $0.048\,\lambda$ on the WTG scale, which thus determines the required stub length. Now $d_1 = (0.048)(0.375) = 0.18\,\text{m} = \underline{1.8\,\text{cm}}$. The attachment point is found by transforming y_L to y_{in1}, where the former point is located at $0.178\,\lambda$ on the WTG scale, and the latter is at $0.362\,\lambda$ on the same scale. The distance is then $d = (0.362 - 0.178)\lambda = 0.184\lambda$. The actual length is $d = (0.184)(0.375) = 0.069\,\text{m} = \underline{6.9\,\text{cm}}$.



13.34. The lossless line shown in Fig. 13.32 is operating with $\lambda = 100$cm. If $d_1 = 10$cm, $d = 25$cm, and the line is matched to the left of the stub, what is Z_L? For the line to be matched, it is required that the sum of the normalized input admittances of the shorted stub and the main line at the point where the stub is connected be unity. So the input susceptances of the two lines must cancel. To find the stub input susceptance, use the Smith chart to transform the short circuit point 0.1λ toward the generator, and read the input value as $b_s = -1.37$ (note that the stub length is one-tenth of a wavelength). The main line input admittance must now be $y_{in} = 1 + j1.37$. This line is one-quarter wavelength long, so the normalized load impedance is equal to the normalized input admittance. Thus $z_L = 1 + j1.37$, so that $Z_L = 300 z_L = \underline{300 + j411\ \Omega}$.

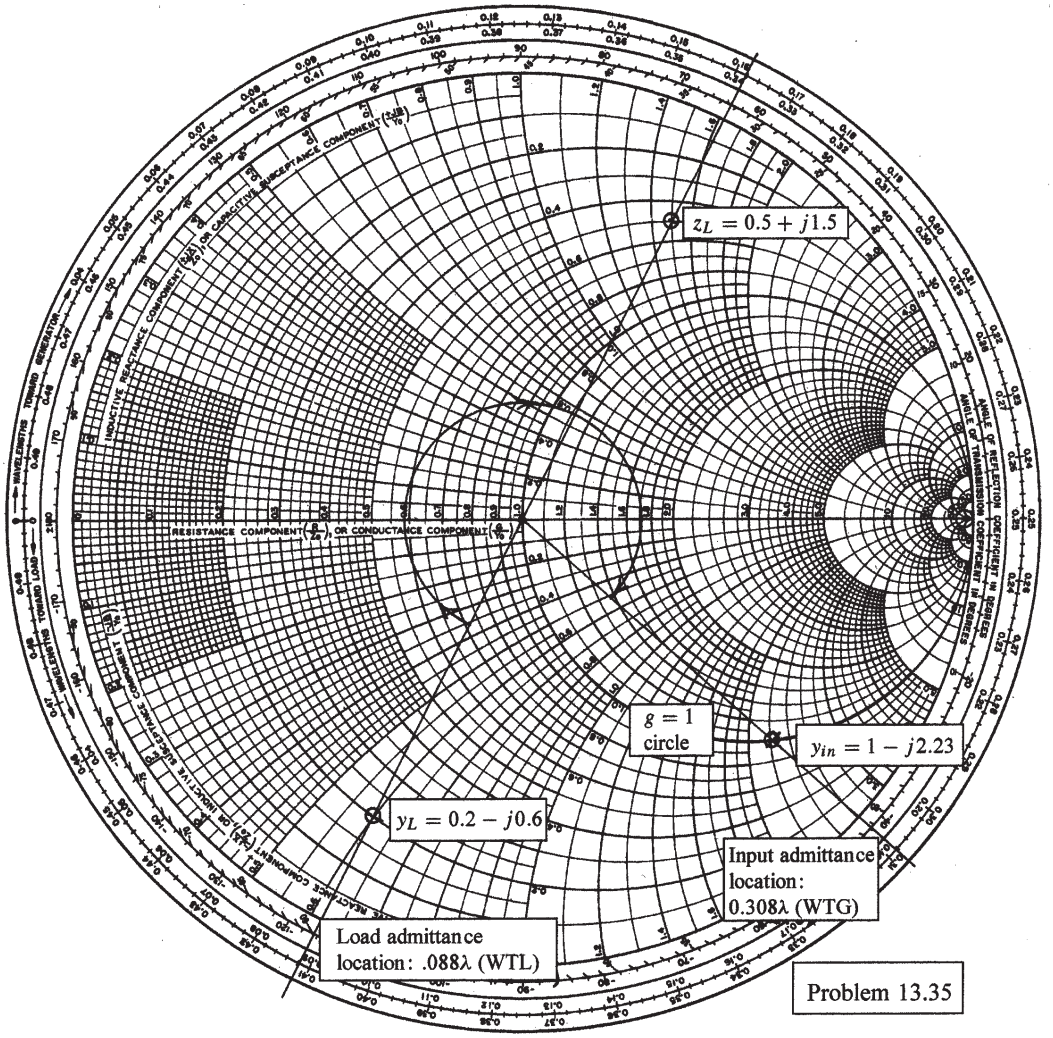


13.35. A load, $Z_L = 25 + j75\ \Omega$, is located at $z = 0$ on a lossless two-wire line for which $Z_0 = 50\ \Omega$ and $v = c$.

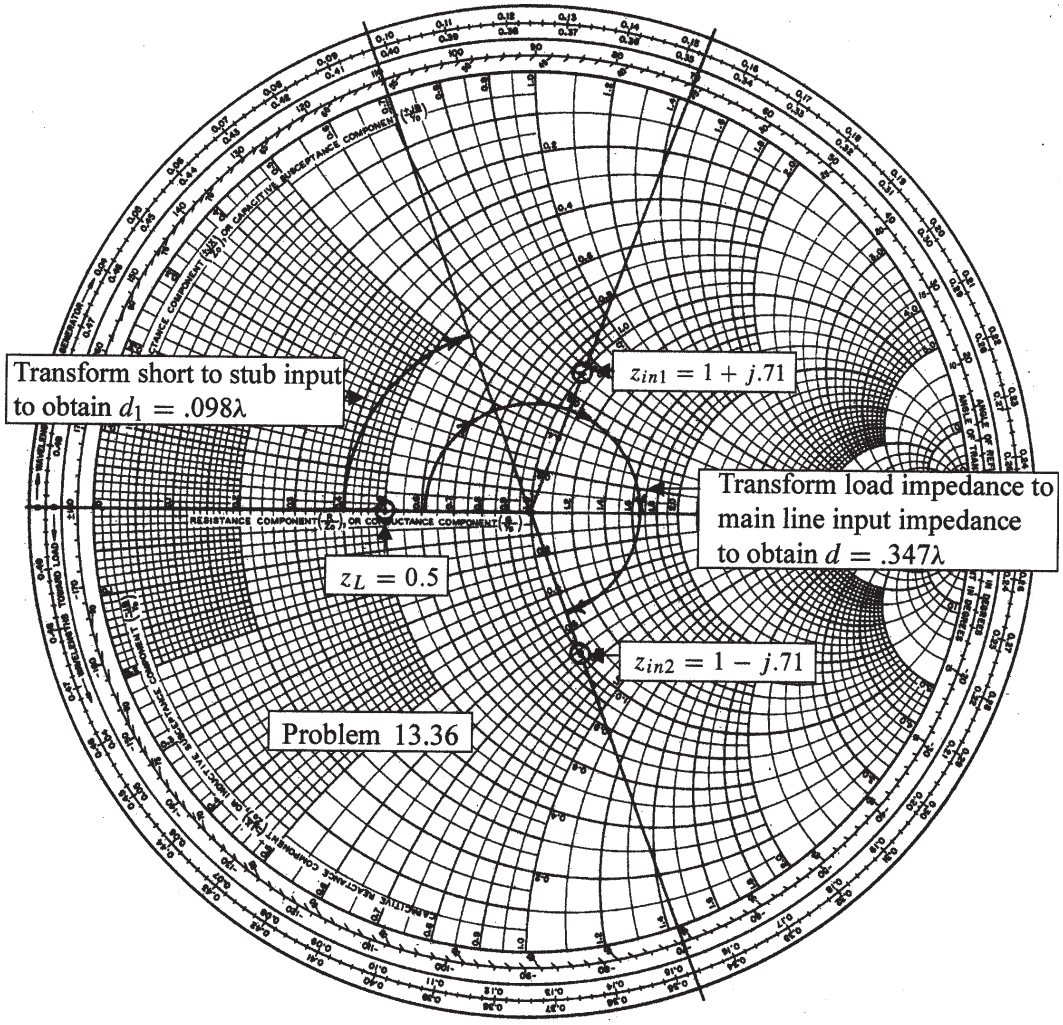
a) If $f = 300$ MHz, find the shortest distance d ($z = -d$) at which the input impedance has a real part equal to $1/Z_0$ and a negative imaginary part: The Smith chart construction is shown below. We begin by calculating $z_L = (25 + j75)/50 = 0.5 + j1.5$, which we then locate on the chart. Next, this point is transformed by rotation halfway around the chart to find $y_L = 1/z_L = 0.20 - j0.60$, which is located at 0.088 λ on the WTL scale. This point is then transformed toward the generator until it intersects the $g = 1$ circle (shown highlighted) with a negative imaginary part. This occurs at point $y_{in} = 1.0 - j2.23$, located at 0.308 λ on the WTG scale. The total distance between load and input is then $d = (0.088 + 0.308)\lambda = 0.396\lambda$. At 300 MHz, and with $v = c$, the wavelength is $\lambda = 1$ m. Thus the distance is $d = 0.396\,\text{m} = \underline{39.6\,\text{cm}}$.

b) What value of capacitance C should be connected across the line at that point to provide unity standing wave ratio on the remaining portion of the line? To cancel the input normalized susceptance of -2.23, we need a capacitive normalized susceptance of +2.23. We therefore write

$$\omega C = \frac{2.23}{Z_0} \quad \Rightarrow \quad C = \frac{2.23}{(50)(2\pi \times 3 \times 10^8)} = 2.4 \times 10^{-11}\ \text{F} = \underline{24\ \text{pF}}$$



13.36. The two-wire lines shown in Fig. 13.33 are all lossless and have $Z_0 = 200\,\Omega$. Find d and the shortest possible value for d_1 to provide a matched load if $\lambda = 100$cm. In this case, we have a series combination of the loaded line section and the shorted stub, so we use impedances and the Smith chart as an impedance diagram. The requirement for matching is that the total normalized impedance at the junction (consisting of the sum of the input impedances to the stub and main loaded section) is unity. First, we find $z_L = 100/200 = 0.5$ and mark this on the chart (see below). We then transform this point toward the generator until we reach the $r = 1$ circle. This happens at two possible points, indicated as $z_{in1} = 1 + j.71$ and $z_{in2} = 1 - j.71$. The stub input impedance must cancel the imaginary part of the loaded section input impedance, or $z_{ins} = \pm j.71$. The shortest stub length that accomplishes this is found by transforming the short circuit point on the chart to the point $z_{ins} = +j0.71$, which yields a stub length of $d_1 = .098\lambda = \underline{9.8\,\text{cm}}$. The length of the loaded section is then found by transforming $z_L = 0.5$ to the point $z_{in2} = 1 - j.71$, so that $z_{ins} + z_{in2} = 1$, as required. This transformation distance is $d = 0.347\lambda = \underline{37.7\,\text{cm}}$.



13.37. In the transmission line of Fig. 13.17, $R_L = Z_0 = 50\ \Omega$. Determine and plot the voltage at the load resistor and the current in the battery as functions of time by constructing appropriate voltage and current reflection diagrams: Referring to the figure, closing the switch launches a voltage wave whose value is given by Eq. (50):

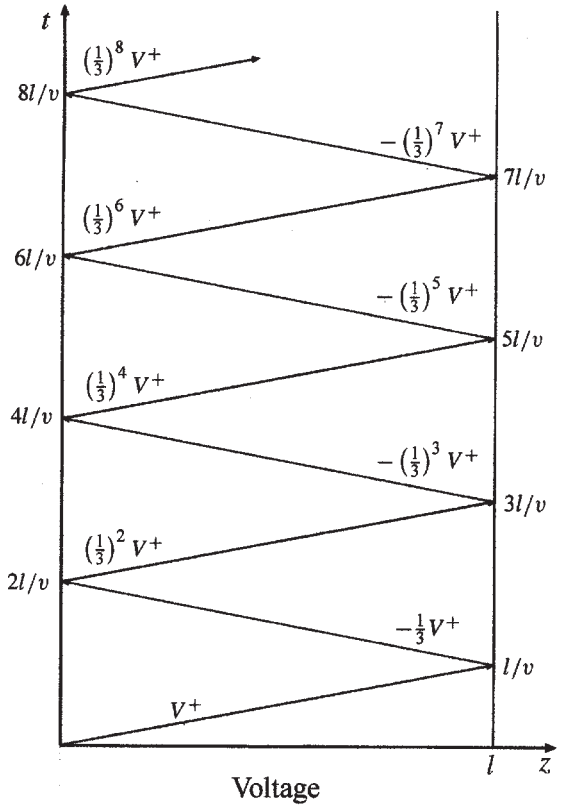
$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0} = \frac{50}{75} V_0 = \frac{2}{3} V_0$$

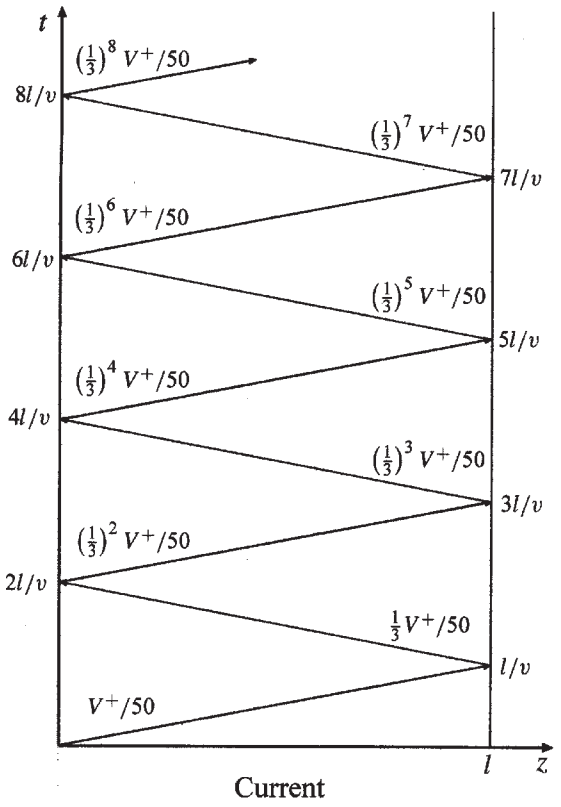
We note that $\Gamma_L = 0$, since the load impedance is matched to that of the line. So the voltage wave traverses the line and does not reflect. The voltage reflection diagram would be that shown in Fig. 13.18a, except that no waves are present after time $t = l/v$. Likewise, the current reflection diagram is that of Fig. 13.19a, except, again, no waves exist after $t = l/v$. The voltage at the load will be just $V_1^+ = (2/3)V_0$ for times beyond l/v. The current through the battery is found through

$$I_1^+ = \frac{V_1^+}{Z_0} = \frac{V_0}{75}\ \text{A}$$

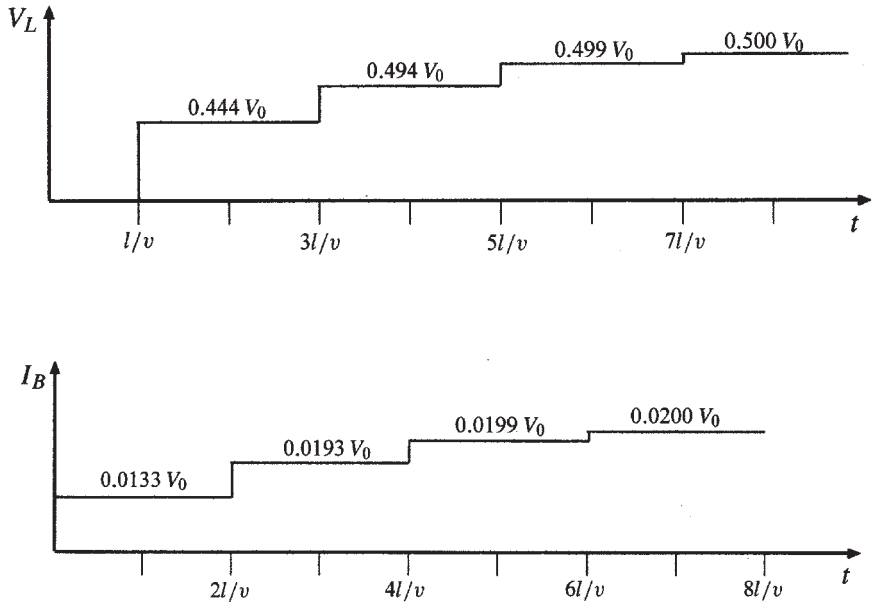
This current initiates at $t = 0$, and continues indefinitely.

13.38. Repeat Problem 37, with $Z_0 = 50\Omega$, and $R_L = R_g = 25\Omega$. Carry out the analysis for the time period $0 < t < 8l/v$. At the generator end, we have $\Gamma_g = -1/3$, as before. The difference is at the load end, where $\Gamma_L = -1/3$, whereas in Problem 37, the load was matched. The initial wave, as in the last problem, is of magnitude $V^+ = (2/3)V_0$. Using these values, voltage and current reflection diagrams are constructed, and are shown below:





13.38. (continued) From the diagrams, voltage and current plots are constructed. First, the load voltage is found by adding voltages along the right side of the voltage diagram at the indicated times. Second, the current through the battery is found by adding currents along the left side of the current reflection diagram. Both plots are shown below, where currents and voltages are expressed to three significant figures. The steady state values, $V_L = 0.5\text{V}$ and $I_B = 0.02\text{A}$, are expected as $t \to \infty$.



13.39. In the transmission line of Fig. 13.17, $Z_0 = 50\,\Omega$ and $R_L = R_g = 25\,\Omega$. The switch is closed at $t = 0$ and *is opened again* at time $t = l/4v$, thus creating a rectangular voltage *pulse* in the line. Construct an appropriate voltage reflection diagram for this case and use it to make a plot of the voltage at the load resistor as a function of time for $0 < t < 8l/v$ (note that the effect of opening the switch is to initiate a second voltage wave, whose value is such that it leaves a net current of zero in its wake): The value of the initial voltage wave, formed by closing the switch, will be

$$V^+ = \frac{Z_0}{R_g + Z_0} V_0 = \frac{50}{25 + 50} V_0 = \frac{2}{3} V_0$$

On opening the switch, a second wave, $V^{+\prime}$, is generated which leaves a net current behind it of zero. This means that $V^{+\prime} = -V^+ = -(2/3)V_0$. Note also that when the switch is opened, the reflection coefficient at the generator end of the line becomes unity. The reflection coefficient at the load end is $\Gamma_L = (25 - 50)/(25 + 50) = -(1/3)$. The reflection diagram is now constructed in the usual manner, and is shown on the next page. The path of the second wave as it reflects from either end is shown in dashed lines, and is a replica of the first wave path, displaced later in time by $l/(4v)$.a All values for the second wave after each reflection are equal but of opposite sign to the immediately preceding first wave values. The load voltage as a function of time is found by accumulating voltage values as they are read moving up along the right hand boundary of the chart. The resulting function, plotted just below the reflection diagram, is found to be a sequence of pulses that alternate signs. The pulse amplitudes are calculated as follows:

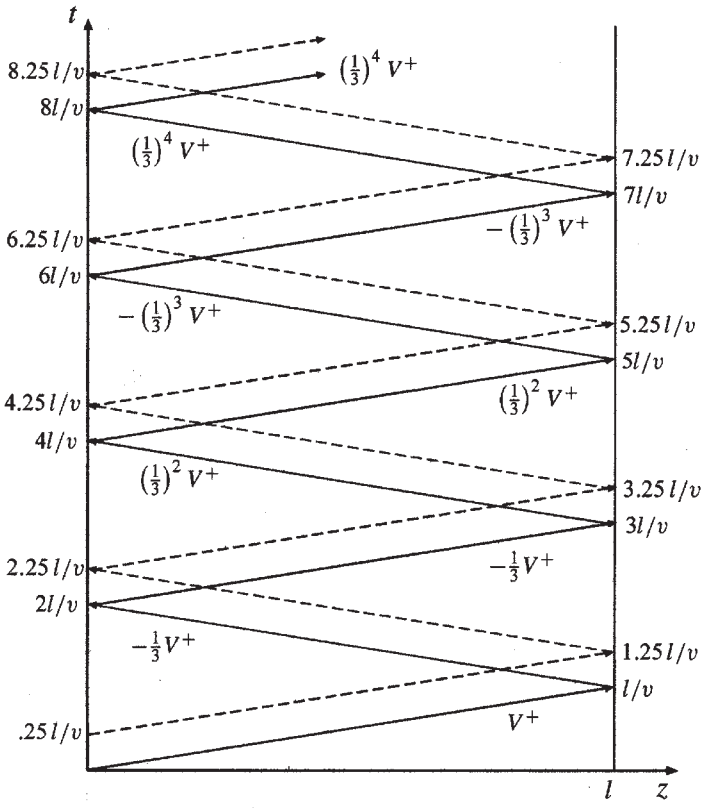
13.39. (continued)

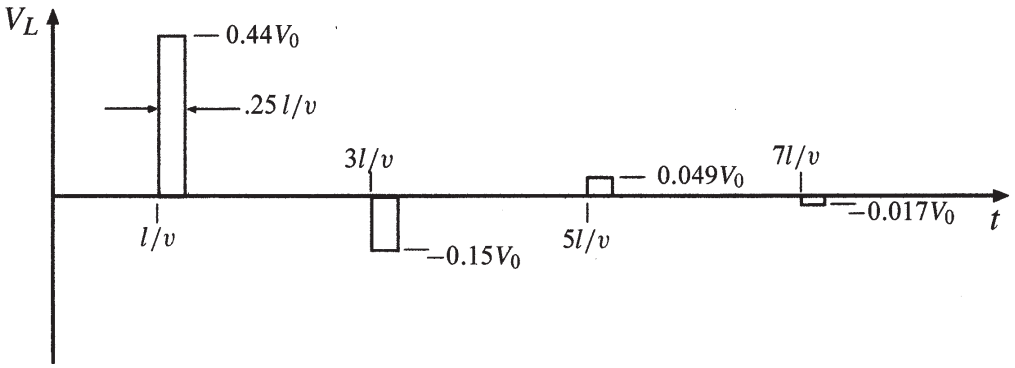
$$\frac{l}{v} < t < \frac{5l}{4v}: \quad V_1 = \left(1 - \frac{1}{3}\right) V^+ = 0.44\, V_0$$

$$\frac{3l}{v} < t < \frac{13l}{4v}: \quad V_2 = -\frac{1}{3}\left(1 - \frac{1}{3}\right) V^+ = -0.15\, V_0$$

$$\frac{5l}{v} < t < \frac{21l}{4v}: \quad V_3 = \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right) V^+ = 0.049\, V_0$$

$$\frac{7l}{v} < t < \frac{29l}{4v}: \quad V_4 = -\left(\frac{1}{3}\right)^3 \left(1 - \frac{1}{3}\right) V^+ = -0.017\, V_0$$

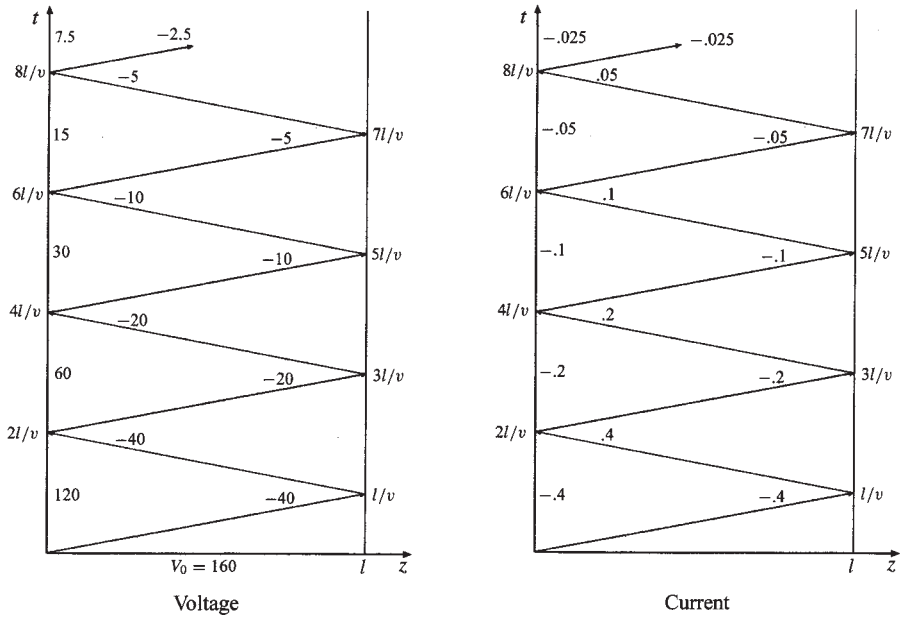


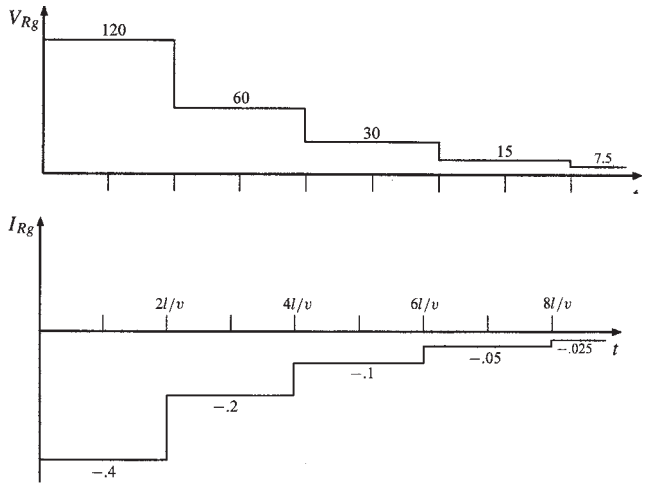


13.40. In the charged line of Fig. 13.22, the characteristic impedance is $Z_0 = 100\Omega$, and $R_g = 300\Omega$. The line is charged to initial voltage $V_0 = 160$ V, and the switch is closed at $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round trips). This problem accompanies Example 13.6 as the other special case of the basic charged line problem, in which now $R_g > Z_0$. On closing the switch, the initial voltage wave is

$$V^+ = -V_0 \frac{Z_0}{R_g + Z_0} = -160 \frac{100}{400} = -40\,\text{V}$$

Now, with $\Gamma_g = 1/2$ and $\Gamma_L = 1$, the voltage and current reflection diagrams are constructed as shown below. Plots of the voltage and current at the resistor are then found by accumulating values from the left sides of the two charts, producing the plots as shown.



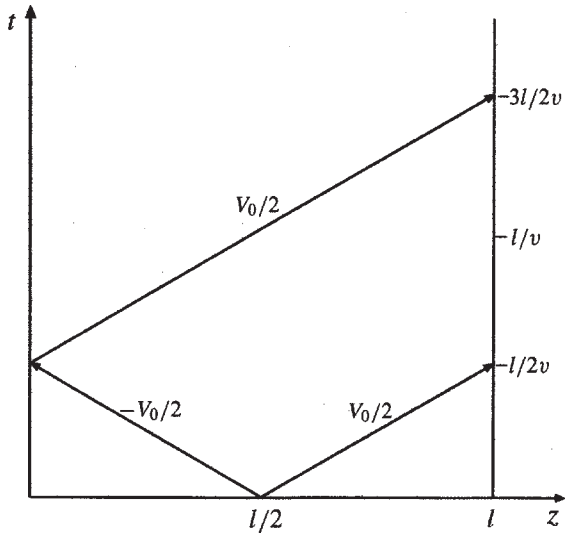


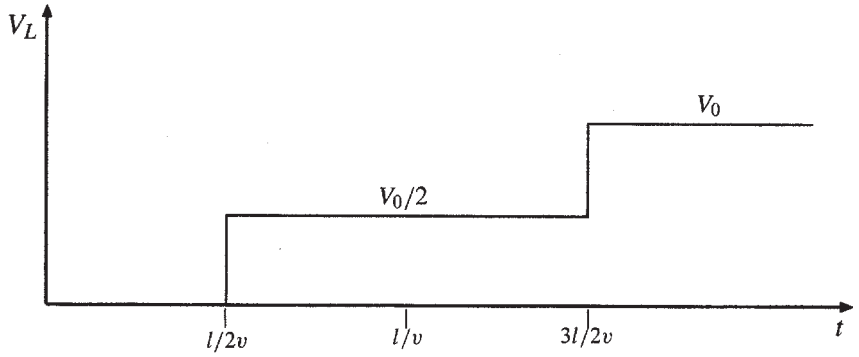
13.41. In the transmission line of Fig. 13.34, the switch is located *midway* down the line, and is closed at $t = 0$. Construct a voltage reflection diagram for this case, where $R_L = Z_0$. Plot the load resistor voltage as a function of time: With the left half of the line charged to V_0, closing the switch initiates (at the switch location) *two* voltage waves: The first is of value $-V_0/2$ and propagates toward the left; the second is of value $V_0/2$ and propagates toward the right. The backward wave reflects at the battery with $\Gamma_g = -1$. No reflection occurs at the load end, since the load is matched to the line. The reflection diagram and load voltage plot are shown below. The results are summarized as follows:

$$0 < t < \frac{l}{2v}: \quad V_L = 0$$

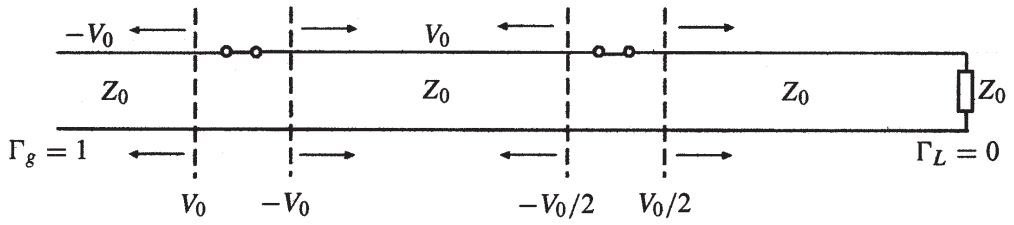
$$\frac{l}{2v} < t < \frac{3l}{2v}: \quad V_L = \frac{V_0}{2}$$

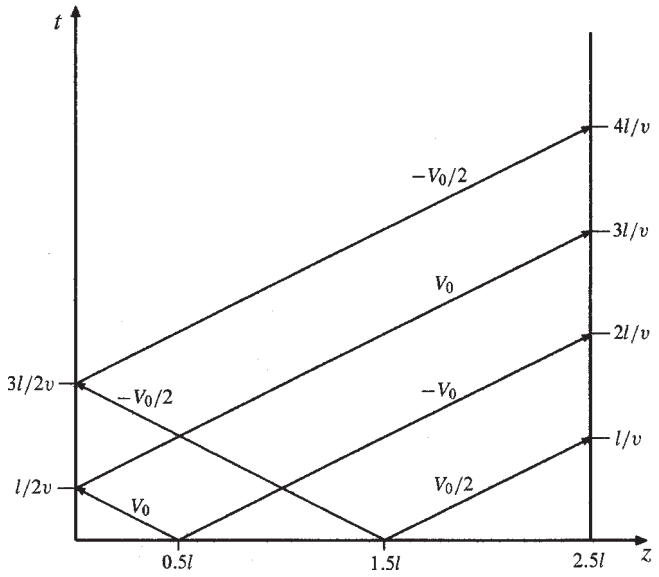
$$t > \frac{3l}{2v}: \quad V_L = V_0$$

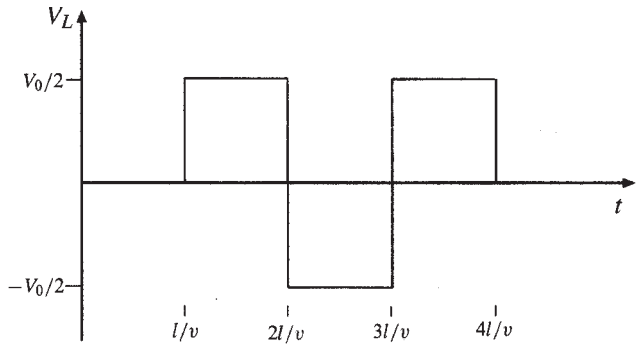




13.42. A simple *frozen wave generator* is shown in Fig. 13.35. Both switches are closed simultaneously at $t = 0$. Construct an appropriate voltage reflection diagram for the case in which $R_L = Z_0$. Determine and plot the load voltage as a function of time: Closing the switches sets up a total of four voltage waves as shown in the diagram below. Note that the first and second waves from the left are of magnitude V_0, since in fact we are superimposing voltage waves from the $-V_0$ and $+V_0$ charged sections acting alone. The reflection diagram is drawn and is used to construct the load voltage with time by accumulating voltages up the right hand vertical axis.







CHAPTER 14

14.1. A parallel-plate waveguide is known to have a cutoff wavelength for the $m = 1$ TE and TM modes of $\lambda_{c1} = 0.4$ cm. The guide is operated at wavelength $\lambda = 1$ mm. How many modes propagate? The cutoff wavelength for mode m is $\lambda_{cm} = 2nd/m$, where n is the refractive index of the guide interior. For the first mode, we are given

$$\lambda_{c1} = \frac{2nd}{1} = 0.4 \text{ cm} \quad \Rightarrow \quad d = \frac{0.4}{2n} = \frac{0.2}{n} \text{ cm}$$

Now, for mode m to propagate, we require

$$\lambda \leq \frac{2nd}{m} = \frac{0.4}{m} \quad \Rightarrow \quad m \leq \frac{0.4}{\lambda} = \frac{0.4}{0.1} = 4$$

So, accounting for 2 modes (TE and TM) for each value of m, and the single TEM mode, we will have a total of $\underline{9}$ modes.

14.2. A parallel-plate guide is to be constructed for operation in the TEM mode only over the frequency range $0 < f < 3$ GHz. The dielectric between plates is to be teflon ($\epsilon'_R = 2.1$). Determine the maximum allowable plate separation, d: We require that $f < f_{c1}$, which, using (7), becomes

$$f < \frac{c}{2nd} \quad \Rightarrow \quad d_{max} = \frac{c}{2nf_{max}} = \frac{3 \times 10^8}{2\sqrt{2.1}\,(3 \times 10^9)} = \underline{3.45 \text{ cm}}$$

14.3. A lossless parallel-plate waveguide is known to propagate the $m = 2$ TE and TM modes at frequencies as low as 10GHz. If the plate separation is 1 cm, determine the dielectric constant of the medium between plates: Use

$$f_{c2} = \frac{c}{nd} = \frac{3 \times 10^{10}}{n(1)} = 10^{10} \Rightarrow n = 3 \text{ or } \epsilon_R = \underline{9}$$

14.4. A $d = 1$ cm parallel-plate guide is made with glass ($n = 1.45$) between plates. If the operating frequency is 32 GHz, which modes will propagate? For a propagating mode, we require $f > f_{cm}$ Using (7) and the given values, we write

$$f > \frac{mc}{2nd} \quad \Rightarrow \quad m < \frac{2fnd}{c} = \frac{2(32 \times 10^9)(1.45)(.01)}{3 \times 10^8} = 3.09$$

The maximum allowed m in this case is thus 3, and the propagating modes will be $\underline{\text{TM}_1}$, $\underline{\text{TE}_1}$, $\underline{\text{TM}_2}$, $\underline{\text{TE}_2}$, $\underline{\text{TM}_3}$, and $\underline{\text{TE}_3}$.

14.5. For the guide of Problem 14.4, and at the 32 GHz frequency, determine the difference between the group delays of the highest order mode (TE or TM) and the TEM mode. Assume a propagation distance of 10 cm: From Problem 14.4, we found $m_{max} = 3$. The group velocity of a TE or TM mode for $m = 3$ is

$$v_{g3} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c3}}{f}\right)^2} \text{ where } f_{c3} = \frac{3(3 \times 10^{10})}{2(1.45)(1)} = 3.1 \times 10^{10} = 31 \text{ GHz}$$

14.5. (continued) Thus

$$v_{g3} = \frac{3\times 10^{10}}{1.45}\sqrt{1-\left(\frac{31}{32}\right)^2} = 5.13\times 10^9 \text{ cm/s}$$

For the TEM mode (assuming no material dispersion) $v_{g,TEM} = c/n = 3\times 10^{10}/1.45 = 2.07\times 10^{10}$ cm/s. The group delay difference is now

$$\Delta t_g = z\left(\frac{1}{v_{g3}} - \frac{1}{v_{g,TEM}}\right) = 10\left(\frac{1}{5.13\times 10^9} - \frac{1}{2.07\times 10^{10}}\right) = \underline{1.5 \text{ ns}}$$

14.6. The cutoff frequency of the $m = 1$ TE and TM modes in a parallel-plate guide is known to be $f_{c1} = 7.5$ GHz. The guide is used at wavelength $\lambda = 1.5$ cm. Find the group velocity of the $m = 2$ TE and TM modes. First we know that $f_{c2} = 2f_{c1} = 15$ GHz. Then $f = c/\lambda = 3\times 10^8/.015 = 20$ GHz. Now, using (23),

$$v_{g2} = \frac{c}{n}\sqrt{1-\left(\frac{f_{c2}}{f}\right)^2} = \frac{c}{n}\sqrt{1-\left(\frac{15}{20}\right)^2} = \underline{2\times 10^8/n \text{ m/s}}$$

n was not specified in the problem.

14.7. A parallel-plate guide is partially filled with two lossless dielectrics (Fig. 14.23) where $\epsilon'_{R1} = 4.0$, $\epsilon'_{R2} = 2.1$, and $d = 1$ cm. At a certain frequency, it is found that the TM_1 mode propagates through the guide without suffering any reflective loss at the dielectric interface.

a) Find this frequency: The ray angle is such that the wave is incident on the interface at Brewster's angle. In this case

$$\theta_B = \tan^{-1}\sqrt{\frac{2.1}{4.0}} = 35.9^\circ$$

The ray angle is thus $\theta = 90 - 35.9 = 54.1^\circ$. The cutoff frequency for the $m = 1$ mode is

$$f_{c1} = \frac{c}{2d\sqrt{\epsilon'_{R1}}} = \frac{3\times 10^{10}}{2(1)(2)} = 7.5\,\text{GHz}$$

The frequency is thus $f = f_{c1}/\cos\theta = 7.5/\cos(54.1^\circ) = \underline{12.8\,\text{GHz}}$.

b) Is the guide operating at a single TM mode at the frequency found in part *a*? The cutoff frequency for the next higher mode, TM_2 is $f_{c2} = 2f_{c1} = 15$ GHz. The 12.8 GHz operating frequency is below this, so TM_2 will not propagate. So the answer is <u>yes</u>.

14.8. In the guide of Problem 14.7, it is found that $m = 1$ modes propagating from left to right totally reflect at the interface, so that no power is transmitted into the region of dielectric constant ϵ'_{R2}.

a) Determine the range of frequencies over which this will occur: For total reflection, the ray angle measured from the normal to the interface must be greater than or equal to the critical angle, θ_c, where $\sin\theta_c = (\epsilon'_{R2}/\epsilon'_{R1})^{1/2}$. The *minimum* mode ray angle is then $\theta_{1\,min} = 90^\circ - \theta_c$. Now, using (5), we write

$$90^\circ - \theta_c = \cos^{-1}\left(\frac{\pi}{k_{min}d}\right) = \cos^{-1}\left(\frac{\pi c}{2\pi f_{min}d\sqrt{4}}\right) = \cos^{-1}\left(\frac{c}{4df_{min}}\right)$$

14.8a. (continued)

Now

$$\cos(90-\theta_c) = \sin\theta_c = \sqrt{\frac{\epsilon'_{R2}}{\epsilon'_{R1}}} = \frac{c}{4df_{min}}$$

Therefore $f_{min} = c/(2\sqrt{2.1}d) = (3\times10^8)/(2\sqrt{2.1}(.01)) = 10.35\,\text{GHz}$. The frequency range is thus $\underline{f > 10.35\,\text{GHz}}$.

b) Does your part *a* answer in any way relate to the cutoff frequency for $m = 1$ modes in any region? We note that $f_{min} = c/(2\sqrt{2.1}d) = f_{c1}$ in guide 2. To summarize, as frequency is lowered, the ray angle in guide 1 decreases, which leads to the incident angle at the interface increasing to eventually reach and surpass the critical angle. At the critical angle, the refracted angle in guide 2 is 90°, which corresponds to a zero degree ray angle in that guide. This defines the cutoff condition in guide 2. So it would make sense that $f_{min} = f_{c1}$(guide 2).

14.9. A rectangular waveguide has dimensions $a = 6$ cm and $b = 4$ cm.

a) Over what range of frequencies will the guide operate single mode? The cutoff frequency for mode mp is, using Eq. (54):

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where n is the refractive index of the guide interior. We require that the frequency lie between the cutoff frequencies of the TE_{10} and TE_{01} modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3\times10^8}{2n(.06)} = \frac{2.5\times10^9}{n}$$

$$f_{c01} = \frac{c}{2nb} = \frac{3\times10^8}{2n(.04)} = \frac{3.75\times10^9}{n}$$

Thus, the range of frequencies over which single mode operation will occur is

$$\underline{\frac{2.5}{n}\,\text{GHz} < f < \frac{3.75}{n}\,\text{GHz}}$$

b) Over what frequency range will the guide support *both* TE_{10} and TE_{01} modes and no others? We note first that f must be greater than f_{c01} to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be f_{c11}, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{.06}\right)^2 + \left(\frac{1}{.04}\right)^2} = \frac{30c}{2n} = \frac{4.5\times10^9}{n}$$

The allowed frequency range is then

$$\underline{\frac{3.75}{n}\,\text{GHz} < f < \frac{4.5}{n}\,\text{GHz}}$$

14.10. Two rectangular waveguides are joined end-to-end. The guides have identical dimensions, where $a = 2b$. One guide is air-filled; the other is filled with a lossless dielectric characterized by ϵ'_R.

a) Determine the maximum allowable value of ϵ'_R such that single mode operation can be simultaneously ensured in *both* guides at some frequency: Since $a = 2b$, the cutoff frequency for any mode in either guide is written using (54):

$$f_{cmp} = \sqrt{\left(\frac{mc}{4nb}\right)^2 + \left(\frac{pc}{2nb}\right)^2}$$

where $n = 1$ in guide 1 and $n = \sqrt{\epsilon'_R}$ in guide 2. We see that, with $a = 2b$, the next modes (having the next higher cutoff frequency) above TE_{10} with be TE_{20} and TE_{01}. We also see that in general, $f_{cmp}(\text{guide 2}) < f_{cmp}(\text{guide 1})$. To assure single mode operation in both guides, the operating frequency must be above cutoff for TE_{10} in both guides, and below cutoff for the next mode in both guides. The allowed frequency range is therefore $f_{c10}(\text{guide 1}) < f < f_{c20}(\text{guide 2})$. This leads to $c/(2a) < f < c/(a\sqrt{\epsilon'_R})$. For this range to be viable, it is required that $\underline{\epsilon'_R < 4}$.

b) Write an expression for the frequency range over which single mode operation will occur in both guides; your answer should be in terms of ϵ'_R, guide dimensions as needed, and other known constants: This was already found in part *a*:

$$\underline{\frac{c}{2a} < f < \frac{c}{\sqrt{\epsilon'_R}\, a}}$$

where $\epsilon'_R < 4$.

14.11. An air-filled rectangular waveguide is to be constructed for single-mode operation at 15 GHz. Specify the guide dimensions, a and b, such that the design frequency is 10/while being 10% lower than the cutoff frequency for the next higher-order mode: For an air-filled guide, we have

$$f_{c,mp} = \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{pc}{2b}\right)^2}$$

For TE_{10} we have $f_{c10} = c/2a$, while for the next mode (TE_{01}), $f_{c01} = c/2b$. Our requirements state that $f = 1.1 f_{c10} = 0.9 f_{c01}$. So $f_{c10} = 15/1.1 = 13.6$ GHz and $f_{c01} = 15/0.9 = 16.7$ GHz. The guide dimensions will be

$$a = \frac{c}{2f_{c10}} = \frac{3 \times 10^{10}}{2(13.6 \times 10^9)} = \underline{1.1\,\text{cm}} \;\text{ and }\; b = \frac{c}{2f_{c01}} = \frac{3 \times 10^{10}}{2(16.7 \times 10^9)} = \underline{0.90\,\text{cm}}$$

14.12. Using the relation $P_{av} = (1/2)\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$, and Eqs. (44) through (46), show that the average power density in the TE_{10} mode in a rectangular waveguide is given by

$$P_{av} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \qquad \text{W/m}^2$$

(note that the sin term is erroneously to the first power in the original problem statement). Inspecting (44) through (46), we see that (46) includes a factor of j, and so would lead to an imaginary part of the

total power when the cross product with E_y is taken. Therefore, the real power in this case is found through the cross product of (44) with the complex conjugate of (45), or

$$P_{av} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_{ys} \times \mathbf{H}_{xs}^*\right\} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \qquad \text{W/m}^2$$

14.13. Integrate the result of Problem 14.12 over the guide cross-section $0 < x < a$, $0 < y < b$, to show that the power in Watts transmitted down the guide is given as

$$P = \frac{\beta_{10} ab}{4\omega\mu} E_0^2 = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \text{ W}$$

where $\eta = \sqrt{\mu/\epsilon}$ (note misprint in problem statement), and θ_{10} is the wave angle associated with the TE_{10} mode. Interpret. First, the integration:

$$P = \int_0^b \int_0^a \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\, \mathbf{a}_z \cdot \mathbf{a}_z \, dx\, dy = \frac{\beta_{10} ab}{4\omega\mu} E_0^2$$

Next, from (20), we have $\beta_{10} = \omega\sqrt{\mu\epsilon}\sin\theta_{10}$, which, on substitution, leads to

$$P = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \text{ W} \ \text{ with } \eta = \sqrt{\frac{\mu}{\epsilon}}$$

The $\sin\theta_{10}$ dependence demonstrates the principle of group velocity as energy velocity (or power). This was considered in the discussion leading to Eq. (23).

14.14. Show that the group dispersion parameter, $d^2\beta/d\omega^2$, for given mode in a parallel-plate or rectangular waveguide is given by

$$\frac{d^2\beta}{d\omega^2} = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

where ω_c is the radian cutoff frequency for the mode in question (note that the first derivative form was already found, resulting in Eq. (23)). First, taking the reciprocal of (23), we find

$$\frac{d\beta}{d\omega} = \frac{n}{c}\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1/2}$$

Taking the derivative of this equation with respect to ω leads to

$$\frac{d^2\beta}{d\omega^2} = \frac{n}{c}\left(-\frac{1}{2}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}\left(\frac{2\omega_c^2}{\omega^3}\right) = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

14.15. Consider a transform-limited pulse of center frequency $f = 10$ GHz and of full-width $2T = 1.0$ ns. The pulse propagates in a lossless single mode rectangular guide which is air-filled and in which the 10 GHz operating frequency is 1.1 times the cutoff frequency of the TE_{10} mode. Using the result of Problem 14.14, determine the length of the guide over which the pulse broadens to twice its initial width: The broadened pulse will have width given by $T' = \sqrt{T^2 + (\Delta\tau)^2}$, where $\Delta\tau = \beta_2 L/T$ for a transform limited pulse (assumed gaussian). β_2 is the Problem 14.14 result evaluated at the operating frequency, or

$$\beta_2 = \frac{d^2\beta}{d\omega^2}\Big|_{\omega=10\,\text{GHz}} = -\frac{1}{(2\pi \times 10^{10})(3 \times 10^8)}\left(\frac{1}{1.1}\right)^2\left[1 - \left(\frac{1}{1.1}\right)^2\right]^{-3/2}$$

$$= 6.1 \times 10^{-19}\ \text{s}^2/\text{m} = 0.61\ \text{ns}^2/\text{m}$$

Now $\Delta\tau = 0.61L/0.5 = 1.2L$ ns. For the pulse width to double, we have $T' = 1$ ns, and

$$\sqrt{(.05)^2 + (1.2L)^2} = 1 \ \Rightarrow \ L = 0.72\,\text{m} = \underline{72\,\text{cm}}$$

14.15. (continued)

What simple step can be taken to reduce the amount of pulse broadening in this guide, while maintaining the same initial pulse width? It can be seen that β_2 can be reduced by increasing the operating frequency relative to the cutoff frequency; i.e., operate as far above cutoff as possible, without allowing the next higher-order modes to propagate.

14.16. A symmetric dielectric slab waveguide has a slab thickness $d = 10\ \mu$m, with $n_1 = 1.48$ and $n_2 = 1.45$. If the operating wavelength is $\lambda = 1.3\ \mu$m, what modes will propagate? We use the condition expressed through (77): $k_0 d\sqrt{n_1^2 - n_2^2} \geq (m-1)\pi$. Since $k_0 = 2\pi/\lambda$, the condition becomes

$$\frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} \geq (m-1) \quad \Rightarrow \quad \frac{2(10)}{1.3}\sqrt{(1.48)^2 - (1.45)^2} = 4.56 \geq m - 1$$

Therefore, $m_{max} = 5$, and we have TE and TM modes for which $m = 1, 2, 3, 4, 5$ propagating (ten total).

14.17. A symmetric slab waveguide is known to support only a single pair of TE and TM modes at wavelength $\lambda = 1.55\ \mu$m. If the slab thickness is 5 μm, what is the maximum value of n_1 if $n_2 = 3.3$ (assume 3.30)? Using (78) we have

$$\frac{2\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} < \pi \quad \Rightarrow \quad n_1 < \sqrt{\frac{\lambda}{2d} + n_2^2} = \sqrt{\frac{1.55}{2(5)} + (3.30)^2} = \underline{3.32}$$

14.18. $n_1 = 1.50$, $n_2 = 1.45$, and $d = 10\ \mu$m in a symmetric slab waveguide (note that the index values were reversed in the original problem statement).

a) What is the phase velocity of the $m = 1$ TE or TM mode at cutoff? At cutoff, the mode propagates in the slab at the critical angle, which means that the phase velocity will be equal to that of a plane wave in the upper or lower media of index n_2. Phase velocity will therefore be $v_p(\text{cutoff}) = c/n_2 = 3 \times 10^8/1.45 = \underline{2.07 \times 10^8\ \text{m/s}}$.

b) What is the phase velocity of the $m = 2$ TE or TM modes at cutoff? The reasoning of part *a* applies to all modes, so the answer is the same, or $\underline{2.07 \times 10^8\ \text{m/s}}$.

14.19. An *asymmetric* slab waveguide is shown in Fig. 14.24. In this case, the regions above and below the slab have unequal refractive indices, where $n_1 > n_3 > n_2$ (note error in problem statement).

a) Write, in terms of the appropriate indices, an expression for the minimum possible wave angle, θ_1, that a guided mode may have: The wave angle must be equal to or greater than the critical angle of total reflection at *both* interfaces. The minimum wave angle is thus determined by the *greater* of the two critical angles. Since $n_3 > n_2$, we find $\theta_{min} = \theta_{c,13} = \underline{\sin^{-1}(n_3/n_1)}$.

b) Write an expression for the maximum phase velocity a guided mode may have in this structure, using given or known parameters: We have $v_{p,max} = \omega/\beta_{min}$, where $\beta_{min} = n_1 k_0 \sin\theta_{1,min} = n_1 k_0 n_3/n_1 = n_3 k_0$. Thus $v_{p,max} = \omega/(n_3 k_0) = \underline{c/n_3}$.

14.20. A step index optical fiber is known to be single mode at wavelengths $\lambda > 1.2\ \mu$m. Another fiber is to be fabricated from the same materials, but is to be single mode at wavelengths $\lambda > 0.63\ \mu$m. By what percentage must the core radius of the new fiber differ from the old one, and should it be larger or smaller? We use the cutoff condition, given by (80):

$$\lambda > \frac{2\pi a}{2.405}\sqrt{n_1^2 - n_2^2}$$

14.20. (continued) With λ reduced, the core radius, a, must also be reduced by the same fraction. Therefore, the percentage *reduction* required in the core radius will be

$$\% = \frac{1.2 - .63}{1.2} \times 100 = \underline{47.5\%}$$

14.21. A short dipole carrying current $I_0 \cos \omega t$ in the $\mathbf{a}_z$ direction is located at the origin in free space.

a) If $\beta = 1$ rad/m, $r = 2$ m, $\theta = 45^\circ$, $\phi = 0$, and $t = 0$, give a unit vector in rectangular components that shows the instantaneous direction of **E**: In spherical coordinates, the components of **E** are given by (82) and (83):

$$E_r = \frac{I_0 d\eta}{2\pi} \cos\theta e^{-j2\pi r/\lambda} \left(\frac{1}{r^2} + \frac{\lambda}{j2\pi r^3} \right)$$

$$E_\theta = \frac{I_0 d\eta}{4\pi} \sin\theta e^{-j2\pi r/\lambda} \left(j\frac{2\pi}{\lambda r} + \frac{1}{r^2} + \frac{\lambda}{j2\pi r^3} \right)$$

Since we want a unit vector at $t = 0$, we need only the relative amplitudes of the two components, but we need the absolute phases. Since $\theta = 45^\circ$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Also, with $\beta = 1 = 2\pi/\lambda$, it follows that $\lambda = 2\pi$ m. The above two equations can be simplified by these substitutions, while dropping all amplitude terms that are common to both. Obtain

$$A_r = \left(\frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

$$A_\theta = \frac{1}{2} \left(j\frac{1}{r} + \frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

Now with $r = 2$ m, we obtain

$$A_r = \left(\frac{1}{4} - j\frac{1}{8} \right) e^{-j2} = \frac{1}{4}(1.12) e^{-j26.6^\circ} e^{-j2}$$

$$A_\theta = \left(j\frac{1}{4} + \frac{1}{8} - j\frac{1}{16} \right) e^{-j2} = \frac{1}{4}(0.90) e^{j56.3^\circ} e^{-j2}$$

The total vector is now $\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta$. We can normalize the vector by first finding the magnitude:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}^*} = \frac{1}{4}\sqrt{(1.12)^2 + (0.90)^2} = 0.359$$

Dividing the field vector by this magnitude and converting 2 rad to 114.6°, we write the normalized vector as

$$\mathbf{A}_{Ns} = 0.780 e^{-j141.2^\circ} \mathbf{a}_r + 0.627 e^{-58.3^\circ} \mathbf{a}_\theta$$

In real instantaneous form, this becomes

$$\mathbf{A}_N(t) = \mathrm{Re}\left(\mathbf{A}_{Ns} e^{j\omega t} \right) = 0.780 \cos(\omega t - 141.2^\circ) \mathbf{a}_r + 0.627 \cos(\omega t - 58.3^\circ) \mathbf{a}_\theta$$

We evaluate this at $t = 0$ to find

$$\mathbf{A}_N(0) = 0.780 \cos(141.2^\circ) \mathbf{a}_r + 0.627 \cos(58.3^\circ) \mathbf{a}_\theta = -0.608 \mathbf{a}_r + 0.330 \mathbf{a}_\theta$$

14.21a. (continued)

Dividing by the magnitude, $\sqrt{(0.608)^2+(0.330)^2}=0.692$, we obtain the unit vector at $t=0$: $\mathbf{a}_N(0)=-0.879\mathbf{a}_r+0.477\mathbf{a}_\theta$. We next convert this to cartesian components:

$$a_{Nx}=\mathbf{a}_N(0)\cdot\mathbf{a}_x=-0.879\sin\theta\cos\phi+0.477\cos\theta\cos\phi=\frac{1}{\sqrt{2}}(-0.879+0.477)=-0.284$$

$$a_{Ny}=\mathbf{a}_N(0)\cdot\mathbf{a}_y=-0.879\sin\theta\sin\phi+0.477\cos\theta\sin\phi=0\ \text{ since }\phi=0$$

$$a_{Nz}=\mathbf{a}_N(0)\cdot\mathbf{a}_z=-0.879\cos\theta-0.477\sin\theta=\frac{1}{\sqrt{2}}(-0.879-0.477)=-0.959$$

The final result is then

$$\mathbf{a}_N(0)=\underline{-0.284\mathbf{a}_x-0.959\mathbf{a}_z}$$

b) What fraction of the total average power is radiated in the belt, $80^\circ<\theta<100^\circ$? We use the far-zone phasor fields, (84) and (85), and first find the average power density:

$$P_{avg}=\frac{1}{2}\text{Re}[E_{\theta s}H_{\phi s}^*]=\frac{I_0^2d^2\eta}{8\lambda^2r^2}\sin^2\theta\ \text{ W/m}^2$$

We integrate this over the given belt, an at radius r:

$$P_{belt}=\int_0^{2\pi}\int_{80^\circ}^{100^\circ}\frac{I_0^2d^2\eta}{8\lambda^2r^2}\sin^2\theta\,r^2\sin\theta\,d\theta\,d\phi=\frac{\pi I_0^2d^2\eta}{4\lambda^2}\int_{80^\circ}^{100^\circ}\sin^3\theta\,d\theta$$

Evaluating the integral, we find

$$P_{belt}=\frac{\pi I_0^2d^2\eta}{4\lambda^2}\left[-\frac{1}{3}\cos\theta\left(\sin^2\theta+2\right)\right]_{80}^{100}=(0.344)\frac{\pi I_0^2d^2\eta}{4\lambda^2}$$

The total power is found by performing the same integral over θ, where $0<\theta<180^\circ$. Doing this, it is found that

$$P_{tot}=(1.333)\frac{\pi I_0^2d^2\eta}{4\lambda^2}$$

The fraction of the total power in the belt is then $f=0.344/1.333=\underline{0.258}$.

14.22. Prepare a curve, r vs. θ in polar coordinates, showing the locus in the $\phi=0$ plane where:

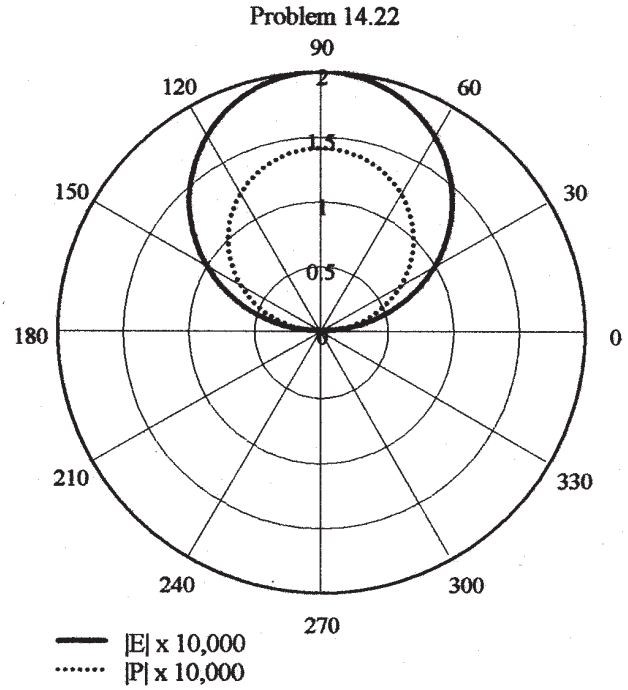
a) the radiation field $|E_{\theta s}|$ is one-half of its value at $r=10^4$ m, $\theta=\pi/2$: Assuming the far field approximation, we use (84) to set up the equation:

$$|E_{\theta s}|=\frac{I_0d\eta}{2\lambda r}\sin\theta=\frac{1}{2}\times\frac{I_0d\eta}{2\times10^4\lambda}\ \Rightarrow\ r=2\times10^4\sin\theta$$

b) the average radiated power density, $P_{r,av}$, is one-half of its value at $r=10^4$ m, $\theta=\pi/2$. To find the average power, we use (84) and (85) in

$$P_{r,av}=\frac{1}{2}\text{Re}\{E_{\theta s}H_{\phi s}^*\}=\frac{1}{2}\frac{I_0^2d^2\eta}{4\lambda^2r^2}\sin^2\theta=\frac{1}{2}\times\frac{1}{2}\frac{I_0^2d^2\eta}{4\lambda^2(10^8)}\ \Rightarrow\ r=\sqrt{2}\times10^4\sin\theta$$

14.22. (continued) The polar plots for field ($r = 2 \times 10^4 \sin\theta$) and power ($r = \sqrt{2} \times 10^4 \sin\theta$) are shown below. Both are circles.



14.23. Two short antennas at the origin in free space carry identical currents of $5\cos\omega t$ A, one in the $\mathbf{a}_z$ direction, one in the $\mathbf{a}_y$ direction. Let $\lambda = 2\pi$ m and $d = 0.1$ m. Find $\mathbf{E}_s$ at the distant point:

a) $(x = 0, y = 1000, z = 0)$: This point lies along the axial direction of the $\mathbf{a}_y$ antenna, so its contribution to the field will be zero. This leaves the $\mathbf{a}_z$ antenna, and since $\theta = 90^\circ$, only the $E_{\theta s}$ component will be present (as (82) and (83) show). Since we are in the far zone, (84) applies. We use $\theta = 90^\circ$, $d = 0.1$, $\lambda = 2\pi$, $\eta = \eta_0 = 120\pi$, and $r = 1000$ to write:

$$\mathbf{E}_s = E_{\theta s}\mathbf{a}_\theta = j\frac{I_0 d\eta}{2\lambda r}\sin\theta e^{-j2\pi r/\lambda}\,\mathbf{a}_\theta = j\frac{5(0.1)(120\pi)}{4\pi(1000)}e^{-j1000}\,\mathbf{a}_\theta$$
$$= j(1.5\times10^{-2})e^{-j1000}\,\mathbf{a}_\theta = \underline{-j(1.5\times10^{-2})e^{-j1000}\,\mathbf{a}_z \text{ V/m}}$$

b) $(0, 0, 1000)$: Along the z axis, only the $\mathbf{a}_y$ antenna will contribute to the field. Since the distance is the same, we can apply the part a result, modified such the the field direction is in $-\mathbf{a}_y$: $\mathbf{E}_s = \underline{-j(1.5\times10^{-2})e^{-j1000}\,\mathbf{a}_y \text{ V/m}}$

c) $(1000, 0, 0)$: Here, both antennas will contribute. Applying the results of parts a and b, we find $\mathbf{E}_s = \underline{-j(1.5\times10^{-2})(\mathbf{a}_y + \mathbf{a}_z)}$.

d) Find $\mathbf{E}$ at $(1000, 0, 0)$ at $t = 0$: This is found through

$$\mathbf{E}(t) = \text{Re}\left(\mathbf{E}_s e^{j\omega t}\right) = (1.5\times10^{-2})\sin(\omega t - 1000)(\mathbf{a}_y + \mathbf{a}_z)$$

Evaluating at $t = 0$, we find
$\mathbf{E}(0) = (1.5\times10^{-2})[-\sin(1000)](\mathbf{a}_y + \mathbf{a}_z) = \underline{-(1.24\times10^{-2})(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}}$.

e) Find $|\mathbf{E}|$ at $(1000, 0, 0)$ at $t = 0$: Taking the magnitude of the part d result, we find $|\mathbf{E}| = \underline{1.75\times10^{-2} \text{ V/m}}$.

14.24. A short current element has $d = 0.03\lambda$. Calculate the radiation resistance for each of the following current distributions:

a) uniform: In this case, (86) applies directly and we find

$$R_{rad} = 80\pi^2\left(\frac{d}{\lambda}\right)^2 = 80\pi^2(.03)^2 = \underline{0.711\,\Omega}$$

b) linear, $I(z) = I_0(0.5d - |z|)/0.5d$: Here, the average current is $0.5I_0$, and so the average power drops by a factor of 0.25. The radiation resistance therefore is down to one-fourth the value found in part a, or $R_{rad} = (0.25)(0.711) = \underline{0.178\,\Omega}$.

c) step, I_0 for $0 < |z| < 0.25d$ and $0.5I_0$ for $0.25d < |z| < 0.5d$: In this case the average current on the wire is $0.75I_0$. The radiated power (and radiation resistance) are down to a factor of $(0.75)^2$ times their values for a uniform current, and so $R_{rad} = (0.75)^2(0.711) = \underline{0.400\,\Omega}$.

14.25. A dipole antenna in free space has a linear current distribution. If the length is 0.02λ, what value of I_0 is required to:

a) provide a radiation-field amplitude of 100 mV/m at a distance of one mile, at $\theta = 90^\circ$: With a linear current distribution, the peak current, I_0, occurs at the center of the dipole; current decreases linearly to zero at the two ends. The average current is thus $I_0/2$, and we use Eq. (84) to write:

$$|E_\theta| = \frac{I_0 d\eta_0}{4\lambda r}\sin(90^\circ) = \frac{I_0(0.02)(120\pi)}{(4)(5280)(12)(0.0254)} = 0.1 \;\;\Rightarrow\;\; I_0 = \underline{85.4\text{ A}}$$

b) radiate a total power of 1 watt? We use

$$P_{avg} = \left(\frac{1}{4}\right)\left(\frac{1}{2}I_0^2 R_{rad}\right)$$

where the radiation resistance is given by Eq. (86), and where the factor of 1/4 arises from the average current of $I_0/2$: We obtain $P_{avg} = 10\pi^2 I_0^2(0.02)^2 = 1 \;\;\Rightarrow\;\; I_0 = \underline{5.03\text{ A}}$.

14.26. A monopole antenna in free space, extending vertically over a perfectly conducting plane, has a linear current distribution. If the length of the antenna is 0.01λ, what value of I_0 is required to

a) provide a radiation field amplitude of 100 mV/m at a distance of 1 mi, at $\theta = 90^\circ$: The image antenna below the plane provides a radiation pattern that is identical to a dipole antenna of length 0.02λ. The radiation field is thus given by (84) in free space, where $\theta = 90^\circ$, and with an additional factor of $1/2$ included to account for the linear current distribution:

$$|E_\theta| = \frac{1}{2}\frac{I_0 d\eta_0}{2\lambda r} \;\;\Rightarrow\;\; I_0 = \frac{4r|E_\theta|}{(d/\lambda)\eta_0} = \frac{4(5289)(12\times.0254)(100\times10^{-3})}{(.02)(377)} = \underline{85.4\text{ A}}$$

b) radiate a total power of 1W: For the monopole over the conducting plane, power is radiated only over the upper half-space. This reduces the radiation resistance of the equivalent dipole antenna by a factor of one-half. Additionally, the linear current distribution reduces the radiation resistance of a dipole having uniform current by a factor of one-fourth. Therefore, R_{rad} is one-eighth the value obtained from (86), or $R_{rad} = 10\pi^2(d/\lambda)^2$. The current magnitude is now

$$I_0 = \left[\frac{2P_{av}}{R_{rad}}\right]^{1/2} = \left[\frac{2(1)}{10\pi^2(d/\lambda)^2}\right]^{1/2} = \frac{\sqrt{2}}{\sqrt{10}\,\pi(.02)} = \underline{7.1\text{ A}}$$

14.27. The radiation field of a certain short vertical current element is $E_{\theta s} = (20/r)\sin\theta\, e^{-j10\pi r}$ V/m if it is located at the origin in free space.

a) Find $E_{\theta s}$ at $P(r = 100, \theta = 90^\circ, \phi = 30^\circ)$: Substituting these values into the given formula, find

$$E_{\theta s} = \frac{20}{100}\sin(90^\circ)e^{-j10\pi(100)} = \underline{0.2e^{-j1000\pi}\ \text{V/m}}$$

b) Find $E_{\theta s}$ at P if the vertical element is located at $A(0.1, 90^\circ, 90^\circ)$: This places the element on the y axis at $y = 0.1$. As a result of moving the antenna from the origin to $y = 0.1$, the change in distance to point P is negligible when considering the change in field *amplitude*, but is not when considering the change in *phase*. Consider lines drawn from the origin to P and from $y = 0.1$ to P. These lines can be considered essentially parallel, and so the difference in their lengths is $l \doteq 0.1\sin(30^\circ)$, with the line from $y = 0.1$ being shorter by this amount. The construction and arguments are similar to those used in the discussion of the electric dipole in Sec. 4.7. The electric field is now the result of part *a*, modified by including a shorter distance, r, in the phase term only. We show this as an additional phase factor:

$$E_{\theta s} = 0.2e^{-j1000\pi}e^{j10\pi(0.1\sin 30} = \underline{0.2e^{-j1000\pi}e^{j0.5\pi}\ \text{V/m}}$$

c) Find $E_{\theta s}$ at P if identical elements are located at $A(0.1, 90^\circ, 90^\circ)$ and $B(0.1, 90^\circ, 270^\circ)$: The original element of part *b* is still in place, but a new one has been added at $y = -0.1$. Again, constructing a line between B and P, we find, using the same arguments as in part *b*, that the length of this line is approximately $0.1\sin(30^\circ)$ *longer* than the distance from the origin to P. The part *b* result is thus modified to include the contribution from the second element, whose field will add to that of the first:

$$E_{\theta s} = 0.2e^{-j1000\pi}\left(e^{j0.5\pi} + e^{-j0.5\pi}\right) = 0.2e^{-j1000\pi}2\cos(0.5\pi) = \underline{0}$$

The two fields are out of phase at P under the approximations we have used.